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SOME RIGIDITY THEOREMS FOR FINSLER MANIFOLDS OF SECTIONAL FLAG CURVATURE

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Abstract. In this paper we study some rigidity properties for Finsler manifolds of sectional flag curvature. We prove that any Landsberg manifold of non-zero sectional flag curvature and any closed Finsler manifold of negative sectional flag curvature must be Riemannian.

1. Introduction

The flag curvature, a natural extension of the sectional curvature in Riemannian geometry, plays the central role in Finsler geometry. Generally, the flag curvature depends not only on the section but also on the flagpole. A Finsler metric is *of scalar flag curvature* if its flag curvature depends only on the flagpole. Contrast to it, Professor Zhongmin Shen suggests a parallel notion: a metric is *of sectional flag curvature* if its flag curvature depends only on the section (see also [2]).

In this paper we shall study the rigidity properties for Finsler metrics of sectional flag curvature. First we recall that in 1975 Numata proved that any Landsberg manifold ($\dim \geq 3$) of nonzero scalar flag curvature must be Riemannian [7], the following theorem can be viewed as the analogous result for sectional flag curvature.

Theorem 1.1. *Any Landsberg manifold of nonzero sectional flag curvature must be Riemannian.*

For general Finsler metrics, the most important rigidity result is the Akbar-Zadeh's theorem: any closed Finsler manifold of negative constant flag curvature must be Riemannian [1]. Our second result can be viewed as the generalization of Akbar-Zadeh's theorem.

Theorem 1.2. *Any closed Finsler manifold of negative sectional flag curvature must be Riemannian.*

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2. Preliminaries

In this section, we give a brief description of basic quantities and fundamental formulas in Finsler geometry, for more details one is referred to see [5]. Throughout this paper, we shall use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their range.

Let (M, F) be a Finsler n-manifold with Finsler metric $F\colon TM \to [0, \infty)$. Let $(x, y) = (x^i, y^i)$ be the local coordinates on TM, and $\pi\colon \widetilde{TM} = TM\backslash 0 \to M$ the natural projection. Unlike in the Riemannian case, most Finsler quantities are functions of TM rather than M. Some fundamental quantities and relations:

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \qquad \text{(positive definite fundamental tensor)}$$

$$C_{ijk}(x,y) := \frac{1}{4}\frac{\partial^3 F^2(x,y)}{\partial y^i \partial y^j \partial y^k}, \qquad \text{(Cartan tensor)}$$

$$\begin{aligned}
(g^{ij}) &:= (g_{ij})^{-1}, \quad C^i_{jk} = g^{il}C_{ljk}, \\
\gamma^k_{ij} &:= \frac{1}{2}g^{km}\left(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right), \\
N^i_j &:= \gamma^i_{jk}y^k - C^i_{jk}\gamma^k_{rs}y^r y^s .
\end{aligned}$$

According to [4], the pulled-back bundle π^*TM admits a unique linear connection, called the *Chern connection*. Its connection forms are characterized by the structure equations:

- Torsion freeness:

$$d\omega^i = \omega^j \wedge \omega^i_j, \tag{2.1}$$

- Almost g-compatibility:

$$dg_{ij} = g_{ik}\omega^k_j + g_{kj}\omega^k_i + 2C_{ijk}\omega^{n+k},$$

where

$$\omega^i := dx^i, \quad \omega^{n+k} := dy^k + y^j\omega^k_j . \tag{2.2}$$

It is easy to know that torsion freeness is equivalent to the absence of dy^k terms in ω^i_j; namely,

$$\omega^i_j = \Gamma^i_{jk}dx^k,$$

together with the symmetry

$$\Gamma^i_{jk} = \Gamma^i_{kj} .$$

The *first Chern curvature tensor* $R_j{}^i{}_{kl}$ and the *second Chern curvature tensor* $P_j{}^i{}_{kl}$ are defined by the following structure equation:

$$d\omega^i_j = \omega^k_j \wedge \omega^i_k + \frac{1}{2}R_j{}^i{}_{kl}\omega^k \wedge \omega^l + P_j{}^i{}_{kl}\omega^k \wedge \omega^{n+l}, \tag{2.3}$$

where $R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}$. The local expressions of $R_j{}^i{}_{kl}$ and $P_j{}^i{}_{kl}$ are

$$R_j{}^i{}_{kl} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls}\,,$$

and

$$P_j{}^i{}_{kl} = -\frac{\partial \Gamma^i_{jk}}{\partial y^l}\,,$$

respectively, where

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}\,.$$

From (2.2) and (2.3) we have

$$d\omega^{n+i} = \omega^{n+j} \wedge \omega^i_j + \frac{1}{2} R^i{}_{kl}\omega^k \wedge \omega^l - L^i{}_{kl}\omega^k \wedge \omega^{n+l}\,, \tag{2.4}$$

where

$$R^i{}_{kl} := y^j R_j{}^i{}_{kl}\,, \quad L^i{}_{kl} := -y^j P_j{}^i{}_{kl}\,.$$

L^i_{jk} is called the *Landsberg curvature*, and (M, F) is called a *Landsberg manifold* if $L^i_{jk} = 0$. Let $L_{ijk} = g_{il}L^l_{jk}$, then both C_{ijk} and L_{ijk} are symmetric on all their indices, and by Euler's Lemma we have

$$y^i C_{ijk} = y^i L_{ijk} = 0\,. \tag{2.5}$$

Let $R_{ijkl} := g_{js}R_i{}^s{}_{kl}, R^i{}_j := y^k R^i{}_{jk}$ and $R_{ij} := g_{ik}R^k{}_j$, then

$$R_{ij} = R_{ji}\,. \tag{2.6}$$

Let $\mathbf{g}_y := g_{ij}(x,y)dx^i \otimes dx^j, \mathbf{C}_y := C_{ijk}(x,y)dx^i \otimes dx^j \otimes dx^k$, and $\mathbf{R}_y := R_{ij}(x,y)dx^i \otimes dx^j$, they are all symmetric. For a tangent plane $P \subset T_xM$ containing y, let

$$\mathbf{K}(P,y) = \mathbf{K}(y;u) := \frac{\mathbf{R}_y\,(u,u)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - [\mathbf{g}_y(y,u)]^2}\,, \tag{2.7}$$

where $u \in P$ such that $P = span\{y,u\}$. $\mathbf{K}$ is called the *flag curvature*. In general $\mathbf{K}(P,y)$ depends both on the section P and the flagpole y. We say that (M,F) is *of sectional flag curvature* if $\mathbf{K}(P,y)$ depends only on the section P. For tensors on slit tangent bundle $\widetilde{TM}$, one can define the horizontal covariant derivative and the vertical covariant derivative. For example, if $T = T^i_j dx^j \otimes \frac{\partial}{\partial x^i}$, then *the horizontal covariant derivative* $T^i_{j|k}$ and *the vertical covariant derivative* $T^i_{j\cdot k}$ are related by

$$T^i_{j|k}\omega^k + T^i_{j\cdot k}\omega^{n+k} := dT^i_j + T^k_j\omega^i_k - T^i_k\omega^k_j\,,$$

and thus

$$T^i_{j|k} = \frac{\delta T^i_j}{\delta x^k} - T^i_s\Gamma^s_{jk} + T^s_j\Gamma^i_{sk}\,,$$

$$T^i_{j\cdot k} = \frac{\partial T^i_j}{\partial y^k}\,.$$

In term of horizontal covariant, the *geodesic differentiation* $\dot{T}$ of T is defined by $\dot{T}^i_j = T^i_{j|k} y^k$. The horizontal and vertical covariant derivative satisfies the product rule, and

$$g_{ij|k} = 0, \quad y^i_{|k} = 0\,. \tag{2.8}$$

In term of geodesic differentiation the Landsberg curvature and the Cartan tensor are related by $L_{ijk} = C_{ijk|l} y^l = \dot{C}_{ijk}$.

3. The Proof of Theorems

In this section we shall complete the proof of Theorems 1.1 and 1.2. For this purpose, Let us first prove some lemmas.

Lemma 3.1. *Let $\mathbf{R}_{\cdot y} := \mathbf{R}_{ij\cdot k} dx^i \otimes dx^j \otimes dx^k$. If (M, F) is of sectional flag curvature, then for any $y, u \in T_xM$ with $y \neq 0$, the following holds:*

$$\mathbf{R}_{\cdot y}(u, u, u) = 2\mathbf{K}(y; u) F^2(y) \mathbf{C}_y(u, u, u)\,. \tag{3.1}$$

Proof. . Express y and u as $y = y^i \frac{\partial}{\partial x^i}$ and $u = u^i \frac{\partial}{\partial x^i}$, respectively, then (2.7) can be rewritten as

$$R_{ij}(x, y) u^i u^j = \mathbf{K}(y; u)\big(g_{ij}(x, y) y^i y^j g_{kl}(x, y) u^k u^l - \big(g_{ij}(x, y) y^i u^j\big)^2\big)\,. \tag{3.2}$$

Let $y(t) = y + tu$, then $y(0) = y, y'(0) = u$. Since (M, F) is of sectional flag curvature, and $\operatorname{span}\{y, u\} = \operatorname{span}\{y(t), u\}$ for any t, we conclude that $\mathbf{K}\big(y(t); u\big)$ is constant for any t. Hence, replace y by $y(t)$ in (3.2) and calculate the derivative with respect to t on the two sides at $t = 0$, and use (2.5), one can reach at (3.1) easily. $\square$

For $y \in T_xM\backslash\{0\}$, let $\{y^\perp\} = \{u \in T_xM : \mathbf{g}_y(y, u) = 0\}$, $S_x = \{(y, u) : y, u \in T_xM, u \in \{y^\perp\}, F(y) = \mathbf{g}_y(u, u) = 1\}$, and $S = \bigcup\limits_{x\in M} S_x$. Note that S_x is always closed for any $x \in M$, and S is also closed if M is closed. We have

Lemma 3.2. *Let $y_0, u_0 \in T_xM$ be two vectors such that*

$$\mathbf{C}_{y_0}(u_0, u_0, u_0) = \max_{(y,u)\in S_x} \mathbf{C}_y(u, u, u)\,, \tag{3.3}$$

then $\mathbf{C}_{y_0}(u_0, u_0, v_0) = 0$ for any $v_0 \in \{y_0^\perp\}$ with $\mathbf{g}_{y_0}(u_0, v_0) = 0$. Consequently, $\mathbf{C}_{y_0}(u_0, u_0, v) = \mathbf{g}_{y_0}(v, u_0) \cdot \mathbf{C}_{y_0}(u_0, u_0, u_0)$ for any $v \in T_xM$.

Proof. Let y_0, u_0 be two vectors such that (3.3) holds, and $v_0 \in \{y_0^\perp\}$ with $\mathbf{g}_{y_0}(u_0, v_0) = 0$. Without loss of generality, we may assume that $\mathbf{g}_{y_0}(v_0, v_0) = 1$. Let $u(t) = u_0 \cos t + v_0 \sin t$, then $u(0) = u_0, u'(0) = v_0$, and $\big(y_0, u(t)\big) \in S_x$ for any t. It is clear that the function $f(t) = \mathbf{C}_{y_0}\big(u(t), u(t), u(t)\big)$ attains its maximum at $t = 0$, and thus

$$0 = \frac{df}{dt}\Big|_{t=0} = 3\mathbf{C}_{y_0}(u_0, u_0, v_0)\,.$$

Notice that $\mathbf{C}_{y_0}(y_0, \cdot, \cdot) = 0$, one has $\mathbf{C}_{y_0}(u_0, u_0, v) = \mathbf{g}_{y_0}(v, u_0) \cdot \mathbf{C}_{y_0}(u_0, u_0, u_0)$ for any $v \in T_xM$. $\square$

In order to prove Theorems 1.1 and 1.2, we need the following fundamental identity for Cartan tensor [3, 6]:

$$
\begin{aligned}
C_{ijk|p|q}y^p y^q + C_{ijm}R^m_{\ k} = & -\frac{1}{3}g_{im}R^m_{\ k\cdot j} - \frac{1}{3}g_{jm}R^m_{\ k\cdot i} \\
& -\frac{1}{6}g_{im}R^m_{\ j\cdot k} - \frac{1}{6}g_{jm}R^m_{\ i\cdot k}\,.
\end{aligned} \tag{3.4}
$$

(3.4) can be rewritten as

$$
\begin{aligned}
\ddot{C}_{ijk} = & \frac{1}{3}(C_{ijm}R^m_{\ k} + C_{jkm}R^m_{\ i} + C_{kim}R^m_{\ j}) \\
& -\frac{1}{3}(R_{ij\cdot k} + R_{jk\cdot i} + R_{ki\cdot j})\,,
\end{aligned} \tag{3.5}
$$

and consequently,

$$
\ddot{C}_{ijk}u^i u^j u^k = \mathbf{C}_y\Big(u, u, u^k R^m_{\ k}\frac{\partial}{\partial x^m}\Big) - \mathbf{R}_{\cdot y}(u,u,u)\,, \quad \forall u = u^i\frac{\partial}{\partial x^i}\,. \tag{3.6}
$$

Now we are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Suppose that (M, F) be a Landsberg manifold of nonzero sectional flag curvature. For fixed $x \in M$, let $y_0, u_0 \in T_xM$ be two vectors such that (3.3) holds. Then by Lemma 3.2, we have

$$
\begin{aligned}
\mathbf{C}_{y_0}\Big(u_0,, u_0, u_0^k R^m_{\ k}\frac{\partial}{\partial x^m}\Big) &= \mathbf{g}_{y_0}\Big(u_0^k R^m_{\ k}\frac{\partial}{\partial x^m}, u_0\Big)\cdot \mathbf{C}_{y_0}(u_0,, u_0, u_0) \\
&= \mathbf{R}_{y_0}(u_0, u_0)\cdot \mathbf{C}_{y_0}(u_0,, u_0, u_0)\,.
\end{aligned} \tag{3.7}
$$

Since (M, F) is Landsberg, $\ddot{C}_{ijk} = \dot{L}_{ijk} = 0$, which together with (2.7), (3.1), (3.6) and (3.7) yields

$$
\begin{aligned}
0 &= \mathbf{R}_{y_0}(u_0, u_0)\cdot \mathbf{C}_{y_0}(u_0,, u_0, u_0) - \mathbf{R}_{\cdot y_0}(u_0, u_0, u_0) \\
&= -\mathbf{K}(y_0; u_0)\cdot \mathbf{C}_{y_0}(u_0,, u_0, u_0)\,.
\end{aligned}
$$

Notice that $\mathbf{K}(y_0; u_0) \neq 0$, we get $\mathbf{C}_{y_0}(u_0,, u_0, u_0) = 0$, namely, $\max\limits_{(y,u)\in S_x}\mathbf{C}_y(u,u,u) = 0$. As $x \in M$ is arbitrary, we finally obtain $\mathbf{C}_y(u,u,u) = 0$ for any $(y,u) \in S$, Hence (M, F) is Riemannian. □

Proof of Theorem 1.2. Let (M, F) be a closed Finsler manifold of negative sectional curvature. Since M is closed, so is S, and there exist two vectors $y_0, u_0 \in T_{x_0}M$ such that $\mathbf{C}_{y_0}(u_0, u_0, u_0) = \max\limits_{(y,u)\in S}\mathbf{C}_y(u,u,u)$. Let $c\colon (-\epsilon, \epsilon) \to M$ be the normal geodesic with the initial condition $c(0) = x_0, \dot{c}(0) = y_0$, and $U = U(t)$ be the parallel vector field along c such that $U(0) = u_0$. Then $\big(\dot{c}(t), U(t)\big) \in S_{c(t)}$ for any $t \in (-\epsilon, \epsilon)$, and the function $f(t) = \mathbf{C}_{\dot{c}(t)}\big(U(t), U(t), U(t)\big)$ attains its maximum at $t = 0$. By maximum principle and (3.6) we have

$$
0 \geq \frac{d^2 f}{dt^2}\Big|_{t=0} = \mathbf{C}_{y_0}\Big(u_0, u_0, u_0^k R^m_{\ k}(x_0, y_0)\frac{\partial}{\partial x^m}\Big) - \mathbf{R}_{\cdot y_0}(u_0, u_0, u_0)\,,
$$

which together with Lemma 3.1 and (3.7) yields

$$
0 \geq -\mathbf{K}(y_0; u_0)\cdot \mathbf{C}_{y_0}(u_0,, u_0, u_0)\,.
$$

Since $\mathbf{K}(y_0; u_0) < 0$, we conclude that $\mathbf{C}_{y_0}(u_0, , u_0, u_0) = 0$, i.e., $\max_{(y,u)\in S} \mathbf{C}_y(u, u, u) = 0$. Consequently, (M, F) is Riemannian, and the theorem is proved. $\square$

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