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CONFORMALLY FLAT LORENTZIAN THREE-SPACES WITH VARIOUS PROPERTIES OF SYMMETRY AND HOMOGENEITY

Giovanni Calvaruso

Abstract. We study conformally flat Lorentzian three-manifolds which are either semi-symmetric or pseudo-symmetric. Their complete classification is obtained under hypotheses of local homogeneity and curvature homogeneity. Moreover, examples which are not curvature homogeneous are described.

1. Introduction

Conformally flat manifolds are a classical field of investigation in Riemannian geometry. In particular, conformally flat Riemannian manifolds carrying different properties of symmetry have been intensively studied. The locally symmetric ones were classified by Ryan [20], who proved the following

Theorem 1.1 ([20]). *Let M be an n-dimensional conformally flat Riemannian space with parallel Ricci tensor. Then M has as its simply connected Riemannian covering one of the following spaces:*

$$\mathbb{R}^n, \quad \mathbb{S}^n(k), \quad \mathbb{H}^n(-k), \quad \mathbb{R}\times\mathbb{S}^{n-1}(k), \quad \mathbb{R}\times\mathbb{H}^{n-1}(-k), \quad \mathbb{S}^p(k)\times\mathbb{H}^{n-p}(-k),$$

where by $\mathbb{S}^n(k)$ we denote a Euclidean n-sphere with constant curvature $k>0$, and by $\mathbb{H}^n(-k)$ we denote an n-dimensional simply connected, connected space with constant curvature $-k<0$.

Semi-symmetric spaces are a well-known and natural generalization of locally symmetric spaces. A pseudo-Riemannian manifold (M,g) is said to be *semi-symmetric* if its curvature tensor R satisfies

$$R(X,Y)\cdot R=0\,, \tag{1.1}$$

for all vector fields X, Y on M, where $R(X,Y)$ acts as a derivation on R. Equation (1.1) is the integrability condition of the equation $\nabla R=0$, which determines locally symmetric spaces. Riemannian manifolds satisfying (1.1) are called "semi-symmetric"since their curvature tensor at any point is the same as the curvature tensor of a symmetric space (which may change with the point). So, locally symmetric spaces are obviously semi-symmetric, but the converse is not true, as

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was proved by Takagi [25]. All proper semi-symmetric spaces were classified by Szabó ([23], [24]). In any dimension greater than two there exist examples of semi-symmetric spaces which are not locally symmetric. We may refer to [1] for a survey. The classification of conformally flat semi-symmetric Riemannian manifolds was given in [6] (see also [21] for the case of complete manifolds):

Theorem 1.2 ([6])**.** *A conformally flat semi-symmetric Riemannian space* (*of dimension* $n > 2$) *is either locally symmetric or locally irreducible and isometric to a semi-symmetric real cone.*

A *pseudo-symmetric space* is a pseudo-Riemannian manifold (M, g) whose curvature tensor R satisfies

$$R(X, Y) \cdot R = f(X \wedge Y) \cdot R\,, \tag{1.2}$$

for all vector fields X and Y on M, where f is a smooth function on M and

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y\,.$$

When $f = c$ is a real constant, (M, g) is said to be pseudo-symmetric of constant type. Clearly, semi-symmetric spaces are pseudo-symmetric spaces of constant type with $c = 0$. Riemannian Conformally flat pseudo-symmetric spaces of constant type were studied in [16] and [7]. A three-dimensional *Riemannian* manifold is pseudo-symmetric if and only if *two of its Ricci eigenvalues coincide* [14].

Recently, several geometrical properties determined by the curvature have been investigated in pseudo-Riemannian settings, in particular for Lorentzian manifolds. Very often, the same property allows more examples in the Lorentzian case than in the Riemannian one. This is essentially due to the different behaviour of self-adjoint operators in the two cases. In fact, a self-adjoint operator in the Riemannian case is always diagonalizable, but in the Lorentzian case it may assume four different canonical forms [19], so allowing many interesting phenomena which do not have a Riemannian counterpart. A typical and particularly interesting example is given by properties related to the Ricci operator Q of a pseudo-Riemannian manifold (M, g). It is well known that in the case of a conformally flat manifold, the Ricci operator completely determines the curvature.

Conformally flat semi-symmetric pseudo-Riemannian manifolds, of dimension $n \geq 4$, were classified in [17], under the hypothesis of completeness, so generalizing the Riemannian result proved in [21]. Moreover, while locally symmetric spaces listed in Theorem 1.1 are the only conformally flat locally homogeneous Riemannian manifolds [26] (and, more in general, the only *curvature homogeneous* ones [12]), there exist conformally flat homogeneous Lorentzian three-spaces which are not symmetric, and they were independently classified in [8] and [18].

The results of [17] leave open the problem of studying conformally flat semi-symmetric Lorentzian three-spaces and, more generally, the pseudo-symmetric ones. In this paper, we completely classify conformally flat semi- and pseudo-symmetric Lorentzian three-spaces under hypotheses of local homogeneity and curvature homogeneity and we exhibit some examples which are not curvature homogeneous. We briefly recall that a pseudo-Riemannian manifold (M, g) is *curvature homogeneous up to order k* if, for any points $p, q \in M$, there exists a linear isometry

$\phi\colon T_pM \to T_qM$ such that $\phi * (\nabla^i R(q)) = \nabla^i R(p)$ for all $i \leq k$ ([3],[22]). If $k = 0$, then (M, g) is simply called a curvature homogeneous space.

The paper is organized in the following way. In Section 2 we shall recall some basic facts concerning conformal flatness and three-dimensional Lorentzian manifolds, as well as the characterization of semi-symmetric and pseudo-symmetric Lorentzian three-spaces. In Section 3 we describe in short the classification of conformally flat locally symmetric Lorentzian three-spaces. In Section 4, 5 and 6 we shall classify conformally flat semi- and pseudo-symmetric Lorentzian three-spaces which are homogeneous, curvature homogeneous up to order 1 and curvature homogeneous, respectively. In Section 7 we shall describe some examples which are not curvature homogeneous. We shall also exhibit a family of n-dimensional conformally flat semi-symmetric Lorentzian spaces which do not fit into the classification of [17], showing how the hypothesis of completeness is essential to obtain that classification result.

2. Preliminaries

Let (M, g) be a pseudo-Riemannian manifold of dimension $n > 2$, ∇ its Levi-Civita connection and $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ its curvature tensor. By ϱ, Q and τ we denote respectively the Ricci tensor, the Ricci operator associated to ϱ through g and the scalar curvature of (M, g). As it is well known, the curvature tensor of a conformally flat space is given by

$$R(X,Y)Z = \frac{1}{n-2}(QX \wedge Y + X \wedge QY)Z - \frac{\tau}{(n-1)(n-2)}(X \wedge Y)Z\,. \tag{2.1}$$

Moreover, (2.1)characterizes conformally flat spaces of dimension $n \geq 4$, while it is trivially satisfied by any three-dimensional manifold. Conversely, the condition

$$(\nabla_X \varrho)(Y,Z) - (\nabla_Y \varrho)(X,Z) = \frac{1}{2}\big(g((\nabla_X \tau)Y, Z) - g((\nabla_Y \tau)X, Z)\big)\,, \tag{2.2}$$

which characterizes conformally flat three-spaces, is trivially satisfied by any conformally flat manifold of dimension $n \geq 4$. Note that whenever τ is constant (in particular, when (M, g) is curvature homogeneous), (2.2) reduces to equation

$$(\nabla_X \varrho)(Y,Z) = (\nabla_Y \varrho)(X,Z)\,, \tag{2.3}$$

expressing the fact that ϱ is a *Codazzi* tensor.

We now recall some basic facts about the Ricci operator of three-dimensional Lorentzian manifolds. At any point p of a Lorentzian manifold (M, g), the Ricci operator Q_p is self-adjoint. Contrarily to the Riemannian case, where Q_p is always diagonalizable, in Lorentzian settings four different cases can occur [19], known as *Segre types*. The possible cases are the following:

(1) *Segre type* $\{11, 1\}$: Q_p is symmetric and so, diagonalizable. The comma is used to separate the spacelike and timelike eigenvectors. In the degenerate case, at least two of the Ricci eigenvalues coincide.

(2) *Segre type* $\{1z\bar{z}\}$: Q_p has one real and two complex conjugate eigenvalues.

(3) *Segre type* $\{21\}$: Q_p has two real eigenvalues (coinciding in the degenerate case), one of which has multiplicity two and each associated to a one-dimensional eigenspace.

(4) *Segre type* $\{3\}$: Q_p has three equal eigenvalues, associated to a one-dimensional eigenspace.

In particular, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, such that Q_p takes one of the following forms:

$$\begin{aligned} &\text{Segre type } \{11,1\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \text{Segre type } \{1z\bar{z}\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \\ &\text{Segre type } \{21\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & \varepsilon \\ 0 & -\varepsilon & b-2\varepsilon \end{pmatrix}, \quad \text{Segre type } \{3\}: \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}. \end{aligned} \tag{2.4}$$

When (M, g) is curvature homogeneous, starting from $\{(e_i)_p\}$, we can use the linear isometries from T_pM into the tangent spaces at any other point, to construct a pseudo-orthonormal frame field $\{e_i\}$, such that the components of Q with respect to $\{e_i\}$ remain constant along M. Hence, in this case Q has the same Segre type at any point and has constant eigenvalues.

Applying either (1.1) or (1.2) to vectors belonging to the pseudo-orthonormal bases $\{(e_i)_p\}$ for which (2.4) holds, it is easily seen that both these conditions are incompatible with the Ricci operator being of Segre types either $\{1z\bar{z}\}$ or $\{3\}$. In fact, at any point p of a semi-symmetric Lorentzian three-space (M, g), the Ricci operator Q_p turns out to be of some special degenerate Segre types $\{11,1\}$ or $\{21\}$, while pseudo-symmetry just forces Q_p to be of these degenerate Segre types (we can also refer to [15] and [10] for more details). Finally, because of its tensorial character, if equation (1.1) (respectively, (1.2)) is satisfied at each point p of a Lorentzian three-manifold (M, g), then (M, g) is semi-symmetric (respectively, pseudo-symmetric). Thus, we have the following.

Theorem 2.1. *A Lorentzian three-space (M, g) is semi-symmetric if and only if at any point p of (M, g), one of the following conditions holds for the Ricci operator Q_p:*

(i) *Q_p is of degenerate Segre type $\{11,1\}$ with three equal eigenvalues.*
(ii) *Q_p is of degenerate Segre type $\{11,1\}$ with eigenvalues $\{0, \kappa, \kappa\}$.*
(iii) *Q_p is of degenerate Segre type $\{21\}$ with triple eigenvalue 0, that is, is two-step nilpotent.*

Theorem 2.2. *A Lorentzian three-space (M, g) is pseudo-symmetric if and only if at any point p of (M, g), the Ricci operator Q_p is*

(i) *either of degenerate Segre type $\{11,1\}$, or*
(ii) *of degenerate Segre type $\{21\}$.*

3. Conformally flat locally symmetric Lorentzian three-spaces

As proved in [9], only some Segre types are possible for the Ricci operator Q of a symmetric Lorentzian three-space (M, g). More precisely, one of the following three cases must occur.

(a) Q has three equal Ricci eigenvalues. Then, the sectional curvature of (M, g) is constant, and (M, g) is locally isometric to one of the *Lorentzian space forms* $\mathbb{S}^3_1$, $\mathbb{R}^3_1$ or $\mathbb{H}^3_1$, of positive, null and negative constant curvature, respectively [19].

(b) Q is of degenerate Segre type $\{11, 1\}$ with eigenvalues $\{0, \kappa, \kappa\}$. In this case, (M, g) is locally reducible and locally isometric to either $\mathbb{R} \times \mathbb{S}^2_1$, $\mathbb{R} \times \mathbb{H}^2_1$, $\mathbb{S}^2 \times \mathbb{R}_1$ or $\mathbb{H}^2 \times \mathbb{R}_1$.

(c) Q is of degenerate Segre type $\{21\}$ with $a = b + \varepsilon = 0$ as triple eigenvalue, that is, is two-step nilpotent. Then, (M, g) necessarily admits a parallel null vector field [9]. A local description of three-dimensional symmetric spaces with a parallel null vector field was given in [13], while a global model $(\tilde{N}, \tilde{g})$ for Lorentzian symmetric three-spaces admitting a parallel null vector field was described in [11]. $(\tilde{N}, \tilde{g})$ admits a global pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with e_3 timelike, a real constant k and a smooth function B, satisfying

$$[e_1, e_2] = [e_1, e_3] = -ke_1 - B(e_2 - e_3)\,, \qquad [e_2, e_3] = 0 \tag{3.1}$$

and

$$e_1(B) = -k^2 - \eta\,, \qquad e_2(B) - e_3(B) = 0\,. \tag{3.2}$$

Now, any locally symmetric Lorentzian three-space is conformally flat, because its Ricci tensor is parallel and so, satisfies (2.3). Therefore, we have at once the following

Theorem 3.1. *A three-dimensional conformally flat Lorentzian locally symmetric space (M, g) is locally isometric to either*

(i) *a Lorentzian space form $\mathbb{S}^3_1$, $\mathbb{R}^3_1$ or $\mathbb{H}^3_1$,*

(ii) *a direct product $\mathbb{R} \times \mathbb{S}^2_1$, $\mathbb{R} \times \mathbb{H}^2_1$, $\mathbb{S}^2 \times \mathbb{R}_1$ or $\mathbb{H}^2 \times \mathbb{R}_1$, or*

(iii) *$(\tilde{N}, \tilde{g})$ described by* (3.1)–(3.2).

Clearly, case (iii) of Theorem 3.1 does not have any relation with the Riemannian classification result given in Theorem 1.1.

4. Locally homogeneous examples

Throughout the paper, a semi-symmetric (respectively, pseudo-symmetric) space will be called *proper* when it is not locally symmetric (respectively, not semi-symmetric).

We now classify proper semi-symmetric locally homogeneous examples. By the main result of [9], a locally homogeneous complete Lorentzian three-manifold (M, g), which is not locally symmetric, is isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric. The curvature of three-dimensional

Lorentzian Lie groups was completely described in [8], and the following result was proved.

Theorem 4.1 ([8])**.** *Let (G,g) be a three-dimensional connected, simply connected Lorentzian Lie group and $\mathfrak{g}$ its Lie algebra. The Ricci tensor of (G,g) satisfies* (2.3) *if and only if one of the following cases occurs:*

a) (G,g) *is symmetric;*

b) $G=E(1,1)$ (*the group of rigid motions on the Minkowski two-space*) *and $\mathfrak{g}$ admits a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 timelike, such that*

$$[e_1,e_2]=\alpha e_1\,,\quad [e_1,e_3]=-\alpha e_1\,,\quad [e_2,e_3]=\alpha(e_2+e_3)\,,\quad \alpha\neq 0\,; \tag{4.1}$$

c) *G is the universal covering of $O(1,2)$ or $SL(2,\mathbb{R})$ and $\mathfrak{g}$ admits a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 timelike, such that*

$$\begin{aligned} &[e_1,e_2]=\pm\sqrt{3}\beta e_2-\beta e_3\,,\quad [e_1,e_3]=-\beta e_2\mp\sqrt{3}\beta e_3\,,\\ &[e_2,e_3]=-2\beta e_1\,,\quad \beta\neq 0\,; \end{aligned} \tag{4.2}$$

d) *G is non-unimodular and $\mathfrak{g}$ admits a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 timelike, such that*

$$\begin{aligned} &[e_1,e_2]=-\alpha e_1-\beta(e_2+e_3)\,,\quad [e_1,e_3]=\alpha e_1+\beta(e_2+e_3)\,,\\ &[e_2,e_3]=\delta(e_2+e_3)\,,\quad \textit{with } \alpha\delta(\alpha-\delta)\neq 0. \end{aligned} \tag{4.3}$$

Knowing the Lie brackets $[e_i,e_j]$ for a pseudo-orthonormal frame field on a Lorentzian manifold (M,g), one can describe its Levi Civita connection by means of the *Koszul formula*

$$2g(\nabla_{e_i}e_j,e_k)=g\big([e_i,e_j],e_k\big)-g\big([e_j,e_k],e_i\big)+g\big([e_k,e_i],e_j\big) \tag{4.4}$$

and then calculate the curvature and the Ricci operator of (M,g). In the case of Lorentzian Lie groups listed above, by direct calculation starting from (4.1)–(4.3) we can deduce that with respect to the pseudo-orthonormal bases of the Lie algebras described in Theorem 4.1, the Ricci operators in cases b) and d) are given by

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & -2\alpha^2 & 2\alpha^2\\ 0 & -2\alpha^2 & 2\alpha^2 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 & 0 & 0\\ 0 & \alpha\delta-\alpha^2 & -(\alpha\delta-\alpha^2)\\ 0 & \alpha\delta-\alpha^2 & -(\alpha\delta-\alpha^2) \end{pmatrix}, \tag{4.5}$$

respectively, while in case c) of Theorem 4.1, the Ricci operator is

$$\begin{pmatrix} -8\beta^2 & 0 & 0\\ 0 & 4\beta^2 & \mp 4\sqrt{3}\beta^2\\ 0 & \pm 4\sqrt{3}\beta^2 & 4\beta^2 \end{pmatrix}. \tag{4.6}$$

In the case described by (4.5), it is easy to check that the Ricci operators are of degenerate Segre type $\{21\}$ with triple eigenvalue 0 (that is, two-step nilpotent). On the other hand, in the case described by (4.6), the Ricci eigenvalues are the solutions of

$$(-8\beta^2-\lambda)[(4\beta^2-\lambda)^2+48\beta^2]=0\,.$$

Hence, the Ricci operator has two complex conjugates eigenvalues and so, it is of Segre type $\{1z\bar{z}\}$. Thus, by Theorems 2.1 and 2.2 we have the following

Theorem 4.2. *A complete Lorentzian three-manifold (M, g) is a conformally flat proper semi-symmetric locally homogeneous space if and only if it is locally isometric to*

(i) *either $E(1,1)$ with Lie algebra* (4.1)*, or*

(ii) *a non-unimodular Lorentzian Lie group with Lie algebra* (4.3)*.*

Moreover, there are not conformally flat pseudo-symmetric locally homogeneous Lorentzian three-spaces which are not semi-symmetric.

5. Curvature homogeneous up to order one examples

A Lorentzian three-manifold, curvature homogeneous up to order two, is locally homogeneous ([4],[9]). Lorentzian three-spaces curvature homogeneous of order one were completely described in [4]. There exist exactly two classes of nonhomogeneous curvature homogeneous of order one Lorentzian three-manifolds, to which we refer as (M_1, g) and (M_2, g). We report here their description.

1) (M_1, g) is a three-dimensional Lorentzian manifold admitting (at least locally) a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, two constants C and D and a function θ, such that

$$\begin{aligned} &[e_1, e_2] = -(\theta + D)e_2 + \eta(C - \theta)e_3\,, \\ &[e_1, e_3] = \eta(C + \theta)e_2 + (\theta - D)e_3\,, \qquad \eta = \pm 1\,, \\ &[e_2, e_3] = 0\,, \end{aligned} \tag{5.1}$$

and

$$e_1(\theta) = \eta - 2(C + D)\theta\,, \qquad (e_2 + \eta\, e_3)(\theta) = 0\,. \tag{5.2}$$

In particular, (M_1, g) is locally homogeneous if and only if either θ is constant or $C = D = 0$. Using (5.1), (5.2) and the Koszul formula (4.4), we easily conclude that with respect to $\{e_i\}$, the Ricci operator of (M_1, g) is given by

$$\begin{pmatrix} -2D^2 & 0 & 0 \\ 0 & -2D^2 - \eta & 1 \\ 0 & 1 & -2D^2 + \eta \end{pmatrix} \tag{5.3}$$

2) (M_2, g) is a three-dimensional Lorentzian manifold admitting (at least locally) a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, a constant G and a function ψ, such that

$$\begin{aligned} &[e_1, e_2] = -e_2 - (G + 2)e_3\,, \\ &[e_1, e_3] = -Ge_2 + e_3\,, \\ &[e_2, e_3] = 2(G + 1)e_1 - \psi e_2 - \psi e_3\,, \end{aligned} \tag{5.4}$$

and

$$a = -2(G + 1)^2\,, \qquad b = -(e_2 + e_3)(\psi)\,, \qquad e_1(\psi) = (G + 1)\psi\,. \tag{5.5}$$

In particular, (M_2, g) is locally homogeneous if and only if ψ is constant, and is never locally symmetric. The Ricci operator of (M_2, g) with respect to $\{e_i\}$ is given by

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \tag{5.6}$$

By (5.3) it easily follows that the Ricci operator of (M_1, g) is of degenerate Segre type $\{21\}$, with triple Ricci eigenvalue $\lambda = -2D^2$. On the other hand, it is evident by (5.6) that (M_2, g) has a Ricci operator of degenerate Segre type $\{11, 1\}$, with Ricci eigenvalues $\{a, b, b\}$. Henceforth, by Theorems 2.1 and 2.2, we have that all curvature homogeneous up to order one Lorentzian three-spaces (M_1, g) and (M_2, g) are pseudo-symmetric (of constant type). In particular, (M_1, g) is semi-symmetric if and only if $D = 0$, while (M_2, g) is semi-symmetric if and only if $G = -1$.

As proved in [5], excluding the locally symmetric cases, (M_1, g) satisfies (2.3) (and so, is conformally flat) if and only if $D = -2C \neq 0$, while (M_2, g) is never conformally flat. Therefore, we have the following

Theorem 5.1. *A nonhomogeneous Lorentzian three-manifold (M, g), curvature homogeneous up to order one, is a conformally flat proper pseudo-symmetric space if and only if it is locally isometric to (M_1, g) with $D = -2C \neq 0$. Moreover, there are not nonhomogeneous curvature homogeneous up to order one Lorentzian three-spaces which are conformally flat and proper semi-symmetric.*

6. Curvature homogeneous examples

By Theorem 2.2, the Ricci operator at each point of a pseudo-symmetric Lorentzian three-space is of degenerate Segre type either $\{11, 1\}$ or $\{21\}$. We now discuss these possibilities separately for curvature homogeneous Lorentzian three-manifolds.

Segre type $\{11, 1\}$. Let (M, g) be a three-dimensional curvature homogeneous Lorentzian manifold with Ricci operator of Segre type $\{11, 1\}$ and $\{e_1, e_2, e_3\}$ a pseudo-orthonormal frame field for which the Ricci operator is of diagonal form, with constant eigenvalues a, b, c. Following [9], we put

$$\nabla_{e_i} e_j = \sum_k \varepsilon_j b^i_{jk} e_k \,, \tag{6.1}$$

where $\varepsilon_j = g(e_j, e_j) = \pm 1$ for all j. Then, the functions b^i_{jk} determine completely the Levi Civita connection of (M, g), and conversely. Note that from $\nabla g = 0$ it follows at once $b^i_{kj} = -b^i_{jk}$, for all i, j, k. In particular, $b^i_{jj} = 0$. If all connection functions b^i_{jk} are constant, then (M, g) is locally isometric to a three-dimensional Lorentzian Lie group [9]. In particular, (M, g) is locally homogeneous.

Since the components of the Ricci tensor with respect to $\{e_i\}$ are constant, we have

$$\nabla_i \varrho_{jk} = -\varepsilon_j \varepsilon_k (q_j - q_k) b^i_{jk} \,, \tag{6.2}$$

for all indices i, j, k, where $q_1 = a$, $q_2 = b$, $q_3 = c$ are the Ricci eigenvalues. In particular, $\nabla_i \varrho_{jj} = 0$ for all i, j. By (6.2) it follows that (M, g) satisfies (2.3) if and only if

$$\begin{aligned} &(b-a)b^1_{12} = (b-a)b^2_{12} = 0\,, \quad (a-c)b^1_{13} = (a-c)b^3_{13} = 0\,, \\ &(b-c)b^2_{23} = (b-c)b^3_{23} = 0\,, \quad (b-c)b^1_{23} = (a-c)b^2_{13} = (b-a)b^3_{12}\,. \end{aligned} \tag{6.3}$$

Assume now that the Ricci operator of (M, g) is of degenerate Segre type. If $a = b \neq c$ then, by (6.3) we easily find

$$b^1_{13} = b^3_{13} = b^2_{23} = b^3_{23} = b^1_{23} = b^2_{13} = 0\,. \tag{6.4}$$

Since $a = b$ and (6.4) holds, by (6.2) we conclude that $\nabla_i \varrho_{jk} = 0$ for all i, j, k. So, (M, g) is locally symmetric.

If $a \neq b = c$, we proceed in exactly the same way, concluding that (M, g) is locally symmetric (in particular, locally homogeneous). Therefore, we proved the following

Theorem 6.1. *There are not curvature homogeneous proper pseudo-symmetric* (*in particular, semi-symmetric*) *Lorentzian three-spaces with Ricci operator of Segre type* $\{11, 1\}$.

Degenerate Segre type $\{21\}$. We now consider a three-dimensional curvature homogeneous Lorentzian manifold (M, g) whose Ricci operator is of degenerate Segre type $\{21\}$, which, by Theorem 2.2, is necessarily pseudo-symmetric. In order to make use of the results of [3], starting from the pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ for which (2.4) holds, we now build a "null"basis, putting

$$E_1 = e_1\,, \quad E_2 = \frac{1}{\sqrt{2}}(e_2 + \eta e_3)\,, \quad E_3 = \frac{1}{\sqrt{2}}(e_2 - \eta e_3)\,. \tag{6.5}$$

Then, the components of the metric tensor are given by

$$g(E_1, E_1) = g(E_2, E_3) = 1\,, \qquad g(E_i, E_j) = 0 \quad \text{otherwise,} \tag{6.6}$$

and the ones of the Ricci tensor are

$$\begin{aligned} &\varrho(E_1, E_1) = \varrho(E_2, E_3) = b + \eta\,, \quad \varrho(E_3, E_3) = -2\eta\,, \\ &\varrho(E_i, E_j) = 0 \quad \text{otherwise.} \end{aligned} \tag{6.7}$$

As it was shown in [3], the null frame field $\{E_i\}$, which is defined up to a null transformation preserving the eigenspace of $a = b + \eta$, can always be chosen in such a way that the Levi Civita connection is completely determined by

$$\begin{aligned} &[E_1, E_2] = -2FE_1\,, \\ &[E_1, E_3] = -AE_1 - GE_2 - (C + H)E_3\,, \\ &[E_2, E_3] = (H - C)E_1 - IE_2 - FE_3\,, \end{aligned} \tag{6.8}$$

where A, C, F, G, H, I are six smooth functions. In order to have a Ricci operator of the form (6.7), these functions must satisfy the following system of partial differential equations:

$$
\left\{
\begin{array}{l}
E_2(F) - 3F^2 = 0\,, \\[2mm]
E_1(C) - E_2(A) + C^2 + AF + \dfrac{b+\eta}{2} = 0\,, \\[2mm]
E_1(H) - 2E_3(F) + H^2 + 2FI + 2AF + \dfrac{b+\eta}{2} = 0\,, \\[2mm]
E_1(F) - E_2(C) + 4CF = 0\,, \\[2mm]
E_2(H) + 2CF - 2FH = 0\,, \\[2mm]
E_1(G) - E_3(A) + A^2 + 3CG + GH - AI - 2\eta = 0\,, \\[2mm]
E_1(I) - E_2(G) + FG + CI + HI = 0\,, \\[2mm]
E_2(G) - E_3(C) + AC - AH + 2FG = 0\,, \\[2mm]
E_2(I) - E_3(F) + C^2 - 2CH + 2FI - \dfrac{b+\eta}{2} = 0\,.
\end{array}
\right.
\tag{6.9}
$$

We now compute the components of $\nabla\varrho$. Taking into account (6.7) and (6.8), we find that the possibly nonvanishing components of $\nabla\varrho$ with respect to $\{E_i\}$ are given by

$$
\begin{array}{lll}
\nabla_1\varrho_{33} = -4\eta C\,, & \nabla_2\varrho_{33} = -4\eta F\,, & \nabla_3\varrho_{33} = -4\eta I\,, \\
\nabla_1\varrho_{13} = -4\eta F\,, & & \nabla_3\varrho_{13} = 2\eta H\,.
\end{array}
\tag{6.10}
$$

By 6.10 it easily follows that (M,g) satisfies (2.3) (and so, is conformally flat) if and only if

$$H + 2C = F = 0\,. \tag{6.11}$$

Because of (6.11), equations (6.9) now reduce to

$$
\left\{
\begin{array}{l}
E_1(C) = 2C^2 + \dfrac{b+\eta}{4} = 0\,, \\[2mm]
E_2(A) = 3C^2 + \dfrac{3(b+\eta)}{4} = 0\,, \\[2mm]
E_2(C) = 0\,, \\[2mm]
E_1(G) - E_3(A) = AI - A^2 - CG + 2\eta\,, \\[2mm]
E_1(I) - E_2(G) = CI\,, \\[2mm]
E_2(G) - E_3(C) = -3AC\,, \\[2mm]
E_2(I) = \dfrac{b+\eta}{2} - 5C^2\,.
\end{array}
\right.
\tag{6.12}
$$

Therefore, we rewrite (6.8) taking into account (6.11) and we obtain the following

Theorem 6.2. *A curvature homogeneous Lorentzian three-manifold* (M,g) *with Ricci operator of degenerate Segre type* $\{21\}$*, is conformally flat if and only if there exists* (*at least, locally*) *a null frame* $\{E_1,E_2,E_3\}$*, such that*

$$[E_1,E_2]=0\,,\quad [E_1,E_3]=-A\,E_1-GE_2+CE_3\,,\qquad\qquad (6.13)$$
$$[E_2,E_3]=-3C\,E_1-I\,E_2\,,$$

where A*,* C*,* G*,* I *are smooth functions satisfying* (6.12)*. If at least one among* C*,* I *is not constant, then* (M,g) *is not curvature homogeneous up to order one.*

By Theorem 6.2, the class of curvature homogeneous conformally flat pseudo-symmetric Lorentzian three-spaces is quite large. To provide some semi-symmetric examples, we remark that if we assume $A=0$, then system (6.12) easily implies $C=0$ and $a=b+\eta=0$ (that is, Q is two-step nilpotent) and reduces to

$$E_1(G)=2\eta\,,\qquad E_2(G)=0\,,\qquad E_1(I)=E_2(I)=0\,.\qquad\qquad (6.14)$$

For any nonconstant function I satisfying (6.14), we have a curvature homogeneous conformally flat pseudo-symmetric Lorentzian three-space. Hence, we proved the following

Corollary 6.3. *For any smooth functions* G *and* I *satisfying* (6.14)*, the Lorentzian three-manifold* (M,g)*, admitting* (*locally*) *a null basis* $\{E_1,E_2,E_3\}$*, such that*

$$[E_1,E_2]=0\,,\qquad [E_1,E_3]=-GE_2\,,\qquad [E_2,E_3]=-I\,E_2\,,\qquad\qquad (6.15)$$

is curvature homogeneous, conformally flat and semi-symmetric. Unless I *is constant,* (M,g) *is not curvature homogeneous up to order one.*

Remark 6.4. In the Riemannian case, there are not conformally flat curvature homogeneous spaces which are not locally homogeneous [12]. By Theorems 4.2, 5.1, 6.1 and 6.2, the same conclusion holds for pseudo-symmetric Lorentzian three-spaces with diagonalizable Ricci operator. In fact, in all cases listed in Theorems above, the Ricci operator is two-step nilpotent.

7. Examples which are not curvature homogeneous

Pseudo-symmetry (in particular, semi-symmetry) does not imply curvature homogeneity. In fact, it is a pointwise condition which forces the Ricci operator at each point to be of degenerate Segre type either $\{11,1\}$ or $\{21\}$. For this reason, it is interesting to find some explicit examples of conformally flat pseudo-symmetric Lorentzian spaces which are not curvature homogeneous, also in order to exhibit examples of pseudo-symmetric spaces of nonconstant type.

7.1. **Three-dimensional Robertson-Walker spacetimes.** A three-dimensional *Robertson-Walker spacetime* is a warped product $M=I\times_f S$, whose base is an open interval I of $\mathbb{R}$, with negative definite metric $(I,-dt^2)$, the fiber is a (connected) Riemannian surface $(S,\bar g)$ of constant Gaussian curvature $\bar\kappa$ and the warping function is any positive function $f>0$ on I. So, M is the product manifold $I\times S$, endowed with the Lorentzian metric

$$g_f=-dt^2+f^2(t)\bar g\,.\qquad\qquad (7.1)$$

As proved in [2], Robertson-Walker spacetimes (of any dimension) are conformally flat. We now denote by $\nabla, \bar{\nabla}$ the Levi Civita connection of $g_f, \bar{g}$ respectively, and by $R, \bar{R}$ their Riemannian curvature tensors.

Consider a local basis $\{e_0 = \partial_t, e_1, e_2\}$, where $\partial_t = \frac{\partial}{\partial t}$ is the natural basis of I and $\{e_1, e_2\}$ a local orthonormal basis of Ricci eigenvectors on S. Starting from g of the form (7.1), a standard calculation gives the following description of ∇ (see also [19]):

$$\nabla_{\partial_t} e_i = \frac{f'}{f}\, e_i\,, \quad \nabla_{e_i} e_j = f f' \delta_{ij}\, \partial_t + \bar{\nabla}_{e_i} e_j\,, \qquad \text{for all} \quad i, j \geq 1\,.$$

Then, it is easy to deduce the curvature components of $M = I \times_f S$. We have

$$R_{i0j0} = -f f'' \delta_{ij}, \qquad R_{ijkl} = f^2(\bar{\kappa} + (f')^2)(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}), \qquad \text{for all } i, j, k, l \geq 1.$$

Next, we can calculate the components of the Ricci tensor ϱ of $(I \times_f S, g_f)$ with respect to $\{e_i\}$ and we obtain

$$\varrho_{00} = -\frac{2f''}{f}\,, \qquad \varrho_{i0} = 0\,, \qquad \varrho_{ij} = (\bar{\kappa} + f f'' + (f')^2)\delta_{ij}\,, \tag{7.2}$$

for all $i, j \geq 1$. Applying (7.1) and (7.2), we find that with respect to the local pseudo-orthonormal frame field $\{e_0, \frac{1}{f}e_1, \frac{1}{f}e_2\}$, on $M = I \times_f S$, the Ricci operator is given by

$$\begin{pmatrix} -\frac{2f''}{f} & 0 & 0 \\ 0 & \frac{1}{f^2}(\bar{\kappa} + f f'' + (f')^2) & 0 \\ 0 & 0 & \frac{1}{f^2}(\bar{\kappa} + f f'' + (f')^2) \end{pmatrix} \tag{7.3}$$

Therefore, by Theorem 2.2, $M = I \times_f S$ is always pseudo-symmetric, since its Ricci operator is of degenerate segre type $\{11, 1\}$. Moreover, by by Theorem 2.1, M is semi-symmetric if and only if $f'' = 0$ (excluding the trivial case when f is constant and so, M is a direct product).

Finally, by (7.3) it also follows that M is not curvature homogeneous, except in the very special case when there exist two real constants α, β, such that

$$\begin{cases} f'' = \alpha f\,, \\ \frac{1}{f^2}(\bar{\kappa} + f f'' + (f')^2) = \beta\,. \end{cases} \tag{7.4}$$

Therefore, we have the following

Theorem 7.1. *All three-dimensional Robertson-Walker spacetimes $M = I \times_f S$ are conformally flat and pseudo-symmetric. Whenever $f'' \neq 0$, M is not semi-symmetric. Unless f is a solution of* (7.4)*, M is not curvature homogeneous.*

7.2. **Examples with two-step nilpotent Ricci operator.** On an open subset of $\mathbb{R}^3[w, x, y]$, we consider conformally flat Lorentzian metrics obtained as conformal deformations of the flat Lorentzian metric $dw^2 + dx\, dy$, that is,

$$g = A^{-2}(dw^2 + dx\, dy)\,, \tag{7.5}$$

where A is an arbitrary nonvanishing smooth function of (x, y). It is easily seen that vector fields

$$e_1 = A\frac{\partial}{\partial w}\,, \qquad e_2 = A\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)\,, \qquad e_3 = A\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big) \tag{7.6}$$

form a pseudo-orthonormal frame field with respect to g, with e_3 timelike. Starting from (7.6), a straightforward calculation gives

$$\begin{aligned} &[e_1, e_2] = -(A'_x + A'_y)\, e_1\,, \qquad [e_1, e_3] = -(A'_x - A'_y)\, e_1\,, \\ &[e_2, e_3] = -(A'_x - A'_y)\, e_2 + (A'_x + A'_y)\, e_3\,. \end{aligned} \tag{7.7}$$

Using (7.7) and the Koszul formula (4.4), we can now describe the Levi Civita connection associated to g. Explicitly, we obtain

$$\begin{aligned} &\nabla_{e_1} e_1 = (A'_x + A'_y)\, e_2 - (A'_x - A'_y)\, e_3\,, && \nabla_{e_2} e_1 = 0\,, \\ &\nabla_{e_1} e_2 = -(A'_x + A'_y)\, e_1\,, && \nabla_{e_2} e_2 = -(A'_x - A'_y)\, e_3\,, \\ &\nabla_{e_1} e_3 = -(A'_x - A'_y)\, e_1\,, && \nabla_{e_2} e_3 = -(A'_x - A'_y)\, e_2\,, \\ &\nabla_{e_3} e_1 = 0\,, \\ &\nabla_{e_3} e_2 = -(A'_x + A'_y)\, e_3\,, \\ &\nabla_{e_3} e_3 = -(A'_x + A'_y)\, e_2\,. \end{aligned} \tag{7.8}$$

Next, we use (7.8) to calculate the curvature components with respect to the orthonormal frame field $\{e_i\}$ and we get

$$\begin{aligned} &R_{1212} = A\left(A''_{xx} + 2A''_{xy} + A''_{yy}\right) - 4A'_x A'_y\,, && R_{1213} = A\left(A''_{xx} - A''_{yy}\right)\,, \\ &R_{1313} = A\left(A''_{xx} - 2A''_{xy} + A''_{yy}\right) + 4A'_x A'_y\,, && R_{1323} = 0\,, \\ &R_{2323} = 4\left(A'_x A'_y - AA''_{xy}\right)\,, && R_{1223} = 0\,, \end{aligned} \tag{7.9}$$

We now use (7.9) to describe the Ricci operator of the metric (7.5) with respect to $\{e_i\}$. A direct calculation gives

$$Q = \begin{pmatrix} 2p & 0 & 0 \\ 0 & p+q+r & s \\ 0 & -s & p-q+r \end{pmatrix}\,, \tag{7.10}$$

where we put

$$\begin{aligned} &p = 2\left(AA''_{xy} - 2A'_x A'_y\right)\,, && q = A\left(A''_{xx} + A''_{yy}\right)\,, \\ &r = 4\left(AA''_{xy} - A'_x A'_y\right)\,, && s = A\left(A''_{xx} - A''_{yy}\right)\,. \end{aligned} \tag{7.11}$$

Calculating the Ricci eigenvalues by (7.10), we easily find that if $r = p \neq 0$ and $q^2 = s^2 \neq 0$, then Q is of degenerate Segre type $\{21\}$, with triple Ricci eigenvalue $\lambda = 2p$. By (7.11), these conditions are equivalent to

$$\begin{cases} A''_{xy} = 0 \neq A'_x A'_y\,, \\ \text{either } \; A''_{xx} = 0 \neq A''_{yy} \; \text{ or conversely.} \end{cases} \tag{7.12}$$

Without loss of generality, we assume $A''_{xx} = 0$. So, system (7.12) is satisfied for any function $A = A(x, y)$ satisfying $A''_{xx} = A''_{xy} = 0$, that is, for which $A'_x = \alpha \neq 0$ is a real constant. Integrating, we get

$$A(x, y) = \alpha x + \beta(y)\,,$$

for an arbitrary function β with $\beta' \neq 0$, in order to ensure $A'_x A'_y \neq 0$. Moreover, $s \neq 0$ if and only if $\beta'' \neq 0$. Summarizing, we proved the following

Theorem 7.2. *Every Lorentzian metric g of the form* (7.5)*, determined by a smooth function $A(x, y) = \alpha x + \beta(y)$, where $\alpha \neq 0$ is a real constant and β an arbitrary function of one variable, is a conformally flat pseudo-symmetric Lorentzian metric, with associated Ricci operator Q of degenerate Segre type* $\{21\}$*, defined on the open subset of $\mathbb{R}^3[w, x, y]$ where $\beta'\beta''(y) \neq 0$. The triple Ricci eigenvalue is nonconstant and given by*

$$\lambda = -8\alpha\beta'(y) \neq 0$$

and so, g is neither curvature homogeneous nor semi-symmetric.

7.3. **n-dimensional conformally flat semi-symmetric pseudo-Riemannian spaces.** In the Riemannian case, the only existing examples of conformally flat semi-symmetric spaces which are not locally symmetric are given by real cones [1]. Adapting their construction to pseudo-Riemannian settings, we can exhibit a class of n-dimensional conformally flat semi-symmetric pseudo-Riemannian spaces, which does not fit in the classification given in [17].

Let $(\bar{M}, \bar{g})$ be a $(n-1)$-dimensional pseudo-Riemannian manifold of constant sectional curvature $\bar{\kappa} \neq -1$, and $\mu(t)$ the unique solution of the differential equation $\frac{d\mu}{dt} = -\mu^2$ with initial condition $\mu(0) = \mu_0 > 0$, that is, $\mu(t) = (t + (1/\mu_0))^{-1}$. Put $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > -1/\mu_0\}$ and consider $M = \mathbb{R}_+ \times \bar{M}$ equipped with the metric

$$g = -dt^2 + \mu(t)^{-2}\bar{g}\,.$$

Note that if $\bar{M}$ is of signature $(p, n-1-p)$, then M is of signature $(p, n-p)$. Similarly to the Riemannian case, we call $(M = \mathbb{R}_+ \times \bar{M}, g)$ a *real cone over* $(\bar{M}, \bar{g})$. The name is due to the fact that hypersurfaces defined by $\mu =$constant are homothetic to one another. Since (M, g) is a pseudo-Riemannian warped product with one-dimensional base and fiber of constant curvature, it is conformally flat [2].

At any point p of (M, g), we fix a pseudo-orthonormal basis of tangent vectors $\{e_0, e_1 = \mu e'_1, \dots, e_{n-1} = \mu e'_{n-1}\}$, where $e_0 = \partial_t$ and $\{e'_1, \dots, e'_{n-1}\}$ are pseudo-orthonormal vectors tangent to $(\bar{M}, \bar{g})$, with $\varepsilon_i = \bar{g}(e'_i, e'_i) = \pm 1$ for all i. A standard calculation gives that the curvature components with respect to $\{e_0, \dots, e_{n-1}\}$ are given by

$$\begin{cases} R_{ijkh} = 0 \quad \text{if} \quad 0 \in \{i, j, k, h\}\,, \\ R_{ijkh} = \mu^2(\bar{\kappa} + 1)(\bar{g}_{ik}\bar{g}_{jh} - \bar{g}_{jk}\bar{g}_{ih}) \quad \text{if} \quad i, j, k, h \geq 1\,, \end{cases} \tag{7.13}$$

where $\bar{g}_{ii} = \mu^2\varepsilon_i = \pm\mu^2$ and $\bar{g}_{ij} = \varepsilon_i\mu^2\delta_{ij}$, for all i, j. By (7.13) it easily follows that the Ricci components with respect to $\{e_0, e_1, \dots, e_{n-1}\}$ are given by

$$\begin{cases} \varrho_{0i} = 0 \quad \text{for all} \quad i\,, \\ \varrho_{ik} = (n-2)(\bar{\kappa}+1)\bar{g}_{ik} \quad \text{if} \quad i, j \geq 1 \end{cases} \tag{7.14}$$

Then, a straightforward calculation proves that

$$\varrho(R(e_i, e_j)e_k, e_h) + \varrho(R(e_i, e_j)e_h, e_k) = 0\,, \quad \text{for all} \quad i, j, k, h\,. \tag{7.15}$$

Since (M, g) is conformally flat, pseudo-symmetry condition (7.15) is sufficient to conclude that (M, g) is semi-symmetric. So, we have the following

Theorem 7.3. *All real cones $(M = \mathbb{R}_+ \times \bar{M}, g = -dt^2 + \mu(t)^{-2}\bar{g})$ over a pseudo-Riemannian space form $(\bar{M}, \bar{g})$ of constant curvature $\bar{\kappa} \neq -1$, are semi-symmetric conformally flat pseudo-Riemannian manifolds.*

Remark 7.4. Examples given in Theorem 7.3 do not fit in the classification result given in [17] but are compatible with it. In fact, as (M, g) does not contain the "vertex" of the cone, such manifold is never complete.

By (7.14) it follows that the scalar curvature of (M, g) is given by $\tau = (n-1)(n-2)(\bar{\kappa}+1)\mu^2$ and so, it can not be constant. Hence, (M, g) is never curvature homogeneous.

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Università degli Studi di Lecce, Dipartimento di Matematica
Via Provinciale Lecce-Arnesano, 73100 Lecce, Italy
E-mail: giovanni.calvaruso@unile.it