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# ESTIMATE OF THE HAUSDORFF MEASURE OF THE SINGULAR SET OF A SOLUTION FOR A SEMI-LINEAR ELLIPTIC EQUATION ASSOCIATED WITH SUPERCONDUCTIVITY

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Abstract. We study the boundedness of the Hausdorff measure of the singular set of any solution for a semi-linear elliptic equation in general dimensional Euclidean space $\mathbb{R}^n$. In our previous paper, we have clarified the structures of the nodal set and singular set of a solution for the semi-linear elliptic equation. In particular, we showed that the singular set is $(n-2)$-rectifiable. In this paper, we shall show that under some additive smoothness assumptions, the $(n-2)$-dimensional Hausdorff measure of singular set of any solution is locally finite.

## 1. Introduction

We consider a semi-linear elliptic equation

$$-\nabla_{\boldsymbol{A}}^2\psi = f(|\psi|^2)\psi \quad \text{in} \quad \Omega \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $f$ is a real-valued, bounded function on $\overline{\mathbb{R}}_+ = [0,\infty)$. Here $\boldsymbol{A}$ is a real vector-valued function (called magnetic potential), $\psi$ is a complex-valued function. $\nabla_{\boldsymbol{A}}$ and $\nabla_{\boldsymbol{A}}^2$ are defined by $\nabla_{\boldsymbol{A}} = \nabla - i\boldsymbol{A}$, $\nabla$ is the gradient operator and

$$\nabla_{\boldsymbol{A}}^2\psi = \Delta\psi - i[2\boldsymbol{A}\cdot\nabla\psi + (\operatorname{div}\boldsymbol{A})\psi] - |\boldsymbol{A}|^2\psi\,.$$

This type of operator is considered in Aramaki [1, 2, 3, 5] and Pan and Kwek [24].

Associated with the magnetic potential $\boldsymbol{A} = (A_1, A_2, \dots, A_n)$, define an anti-symmetric $n\times n$ matrix $B = (B_{ij})$ called the magnetic vector field by

$$B_{ij} = \partial_{x_i}A_j - \partial_{x_j}A_i \quad \text{for} \quad i,j = 1,2,\dots,n\,.$$

Let us recall that superconductivity in two or three dimensional space can be described by a pair $(\psi, \boldsymbol{A})$, where $\psi$ is a complex-valued function called the order parameter and $\boldsymbol{A}$ is a real vector-valued function called the magnetic potential,

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which is a minimizer of the so-called Ginzburg-Landau functional. The Euler equation becomes

$$\begin{cases} -\nabla_{\boldsymbol{A}}^2\psi = \kappa^2(1-|\psi|^2)\psi & \text{in} \quad \Omega\,, \\ \boldsymbol{\nu}\cdot\nabla_{\boldsymbol{A}}\psi = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ with $n = 2$ or $n = 3$ is a bounded domain and $\boldsymbol{\nu}$ is the outer unit normal vector at $\partial\Omega$. It is well known that any solution of (1.2) satisfies $|\psi| \le 1$ in $\Omega$. If we choose a bounded function $f$ on $[0, \infty)$ so that $f(t) = \kappa^2(1-t)$ for $|t| \le 1$, the first equation of (1.2) is of the form (1.1).

In the superconductivity theory or the Landau-de Gennes model of liquid cristal, it is important to know the third critical field $H_{c_3}$ or $Q_{c_3}$. It is associated with the lowest eigenvalue of the magnetic Schrödinger operator of type $-\nabla_{q\boldsymbol{A}}^2$, i.e.,

$$\begin{cases} -\nabla_{q\boldsymbol{A}}^2\psi = \mu(q\boldsymbol{A})\psi & \text{in} \quad \Omega\,, \\ \boldsymbol{\nu}\cdot\nabla_{q\boldsymbol{A}}\psi = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.3}$$

If we put $f(t) = \mu(q\boldsymbol{A})$ which is a constant, the first equation of (1.3) is also of form (1.1). For the superconductivity theory, see Lu and Pan [17], [18] and Pan [22]. For the theory of liquid cristal with $\boldsymbol{A} = \boldsymbol{n}$ which is a unit vector field, see Pan [21]. Helffer and Mohamed [15] and Helffer and Morame [16] have extensively considered the eigenvalue problem for the magnetic Schrödinger operator of type $-\nabla_{\boldsymbol{A}}^2$ for $n \ge 2$.

In the equation (1.2), the nodal set $\{x \in \Omega\,;\, \psi(x) = 0\}$ means the normal state there. Pan [23] has studied the structure of the nodal set and the singular set $\{x \in \Omega\,;\, \psi(x) = 0,\, \nabla\psi(x) = 0\}$ of any non-trivial solution of (1.1) in the three dimensional domain.

In the previous paper Aramaki [4], we showed that the nodal set and the singular set of any non-trivial solution of (1.1) in the general $n$ dimensional domain are $(n-1)$ and $(n-2)$-rectifiable, respectively.

For the second order linear elliptic equations with the real coefficients, there are many articles on the nodal set or the singular set. For example, see Garofalo and Lin [9], Han [11], [12] and Han et al. [13]. In particular, Hardt et al. [14] proved that for any non-trivial solution of a linear elliptic equation with real smooth coefficients, the $(n-2)$-dimensional Hausdorff measure of the singular set is locally finite.

However it seems that there are not many articles on the structure of the singular set of complex-valued solutions of equations of type (1.1) (cf. Elliot et al. [7]).

In this paper, we shall estimate the $(n-2)$-dimensional Hausdorff measure of singular set of any non-trivial complex-valued solution $\psi$ of (1.1).

We assume that

(H) $\boldsymbol{A} \in L^\infty(\Omega; \mathbb{R}^n), \operatorname{div}\boldsymbol{A} \in L^q_{\mathrm{loc}}(\Omega)$ with $q > n/2$ if $n \ge 4$ and $q \ge 2$ if $n = 3$, and $B \in L^\infty(\Omega; \mathbb{R}^{n^2})$.

Our main result on the singular set is the following.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ be a bounded domain, assume that the hypothesis* (H) *holds, and let $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C})$ be any non-trivial complex-valued weak solution of* (1.1) *with $\nabla_{\boldsymbol{A}}\psi \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C}^n)$ and $f_\infty := \|f(|\psi|^2)\|_{L^\infty(\Omega)} < \infty$.*

*Then there exists an integer $M > 0$ depending on $\psi$, $f_\infty$ and $\|B\|_{L^\infty(\Omega)}$ such that if, in addition, $\boldsymbol{A} \in C^M(\Omega; \mathbb{R}^n)$, $\operatorname{div} \boldsymbol{A} \in C^M(\Omega)$ and $f \in C^M([0,\infty))$, then for any $\Omega' \Subset \Omega$, there exists a constant $C > 0$ depending on $M, \psi$, $f_\infty$, $\|\boldsymbol{A}\|_{L^\infty(\Omega)}$, $\Omega'$, $C^M(\Omega'')$ norms of $\boldsymbol{A}$, $\operatorname{div} \boldsymbol{A}$, $f(|\psi|^2)$ for some $\Omega' \Subset \Omega'' \Subset \Omega$ such that*

$$\mathcal{H}^{n-2}\big(\{x \in \Omega'; \psi(x) = 0, \nabla\psi(x) = 0\}\big) \le C \tag{1.4}$$

*where $\mathcal{H}^{n-2}$ is the $(n-2)$-dimensional Hausdorff measure.*

## 2. Preliminaries

In this section, we shall list up some propositions which are needed later and held under the hypothesis (H). All the propositions and theorem are found in [4] (c.f. [23]).

At first, we have the regularity of the solution.

**Proposition 2.1.** *Assume that the hypothesis* (H) *holds and let $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C})$ be any weak solution of* (1.1). *Then $\psi \in W^{2,q}_{\mathrm{loc}}(\Omega; \mathbb{C}) \cap C^\alpha_{\mathrm{loc}}(\Omega; \mathbb{C})$ for some $\alpha \in (0,1)$, and for any $B_{2R}(x_0) \Subset \Omega$ and $1 < p \le q$, there exists a constant $C > 0$ depending on $p, q$ and $\|\boldsymbol{A}\|_{L^\infty(\Omega)}$ such that*

$$\begin{aligned} &R^2\|D^2\psi\|_{L^p(B_R(x_0))} + R\|\nabla\psi\|_{L^p(B_R(x_0))} \\ &\qquad\qquad \le C\big\{\|\psi\|_{L^p(B_R(x_0))} + R^2\|f(|\psi|^2)\psi - i(\operatorname{div} \boldsymbol{A})\psi\|_{L^p(B_R(x_0))}\big\}\,. \end{aligned}$$

Next, we state the doubling property of solutions. Let $\psi \not\equiv 0$ be any weak solution of (1.1). For any $B_r(x_0) \Subset \Omega$, we define some quantities.

$$\begin{aligned} I(x_0,r) &= \int_{B_r(x_0)} \{|\nabla_{\boldsymbol{A}}\psi|^2 - f(|\psi|^2)|\psi|^2\}\,dx\,, \\ H(x_0,r) &= \int_{\partial B_r(x_0)} |\psi|^2\,dS_r\,, \quad D(x_0,r) = \int_{B_r(x_0)} |\nabla\psi|^2\,dx\,, \\ M(x_0,r) &= \frac{rI(x_0,r)}{H(x_0,r)}\,, \quad N(x_0,r) = \frac{rD(x_0,r)}{H(x_0,r)} \quad \text{if} \quad H(x_0,r) \ne 0 \end{aligned} \tag{2.1}$$

where $dS_r$ denotes the surface area of $\partial B_r(x_0)$. Then we have

**Proposition 2.2.** *Assume that the conditions of Theorem 1.1 hold for any non-trivial weak solution $\psi \in W^{1,2}(\Omega; \mathbb{C})$. Then there exist $r_0, c_0, N > 0$ where $r_0$ depends only on $f_\infty$, and $c_0$ and $N$ depend only on $\Omega, \psi, f_\infty$ and $\|B\|_{L^\infty(\Omega)}$ such that for any $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$, we have the following.*

(i) $M(x_0,r) \le c_0$,

(ii)

$$\int_{B_r(x_0)} |\psi|^2\,dx \le r\int_{\partial B_r(x_0)} |\psi|^2\,dS\,,$$

*and the doubling property:*

(iii)

$$\int_{B_{2r}(x_0)} |\psi|^2\,dx \le 4^N\int_{B_r(x_0)} |\psi|^2\,dx\,.$$

If $\psi \not\equiv 0$ is a weak solution of (1.1) and satisfies that $\int_{\partial B_r(x_0)} |\psi|^2 dS_r = 0$ for some $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$, it follow from (ii) that $\int_{B_r(x_0)} |\psi|^2 dx = 0$. Therefore, from the unique continuation theorem (cf. Aronszajn [6]) or the doubling property (iii) (cf. [9]), $\psi \equiv 0$ in $\Omega$. Thus we have

$$H(x_0, r) = \int_{\partial B_r(x_0)} |\psi|^2 \, dS_r \neq 0$$

for any $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$, and so we see that $M(x_0, r)$ and $N(x_0, r)$ are well defined. From Proposition 2.2 (i) we see that $M(x_0, r) \le c_0$ for any $0 < r \le r_0/2$ with $B_{2r}(x_0) \Subset \Omega$.

We get an important fact.

**Proposition 2.3** ([4] or [23]). *Assume that the conditions of Theorem 1.1 for any non-trivial weak solution $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{C})$ of (1.1). Then we have*

$$\lim_{r \to 0} M(x_0, r) = \lim_{r \to 0} N(x_0, r) \quad \textit{for any } x_0 \in \Omega \tag{2.2}$$

*and the limit is a non-negative integer.*

From this proposition, we can define the vanishing order of $\psi$ at $x_0 \in \Omega$ by

$$\mathcal{O}_\psi(x_0) = \lim_{r \to 0} M(x_0, r) = \lim_{r \to 0} N(x_0, r)\,.$$

Of course, if $\psi$ is smooth enough, we see that

$$\begin{cases} D^\alpha \psi(x_0) = 0 & \text{for any } \alpha \text{ with } \ |\alpha| < \mathcal{O}_\psi(x_0)\,, \\ D^\beta \psi(x_0) \neq 0 & \text{for some } \beta \ \text{ with } |\beta| = \mathcal{O}_\psi(x_0) \end{cases}$$

where $D^\alpha = \partial^\alpha = \partial^{|\alpha|}/\partial x^\alpha$.

We note that the vanishing order of $\psi$ is uniformly bounded in $\Omega$, i.e.,

$$\mathcal{O}_\psi(x) \le c_0 \quad \text{for} \quad x \in \Omega \tag{2.3}$$

where $c_0$ is the constant as in Proposition 2.2 (i) and depends only on $\psi$, $\Omega$, $f_\infty$ and $\|B\|_{L^\infty(\Omega)}$.

Next, we state the decomposition of the solution of (1.1).

**Proposition 2.4** (cf. [4], [23] and [11]). *Assume that the conditions of Theorem 1.1 hold for any non-trivial weak solution $\psi$ of (1.1). Then for any $0 < R \le r_0/2$ with $B_{2R}(x_0) \Subset \Omega$, there exists an integer $m \ge 0$ such that we can write*

$$\psi(x + x_0) = P_m(x) + \phi(x)\,, \quad x \in B_R(0) \tag{2.4}$$

*where $P_m$ is a non-zero, complex-valued homogeneous, harmonic polynomial of degree $m$, and $\phi$ satisfies*

$$|\phi(x)| \le C|x|^{m+\alpha} \quad \textit{in} \quad B_R(0) \tag{2.5}$$

*for some $\alpha \in (0, 1)$, and a constant $C > 0$ which depends only on $m$, $\|\boldsymbol{A}\|_{L^\infty(\Omega)}$, $\|\mathrm{div}\, \boldsymbol{A}\|_{L^2(B_{2R}(x_0))}$ and $f_\infty$.*

We call $P_m = \psi_{x_0}$ the leading polynomial of $\psi$ at $x_0$. We see that $m$ is the vanishing order of $\psi$ at $x_0$, so $m = \mathcal{O}_\psi(x_0) \le c_0$. Now we define the singular set of $\psi$ by

$$\mathcal{S}(\psi) = \{x \in \Omega; \mathcal{O}_\psi(x) \ge 2\}.$$

In the previous paper [4], we showed

**Theorem 2.5.** *Assume the the conditions of Theorem 1.1 hold for any non-trivial weak solution* $\psi \in W^{1,2}(\Omega;\mathbb{C})$. *Then for any* $\Omega' \Subset \Omega$, $\mathcal{S}(\psi) \cap \Omega'$ *is countably* $(n-2)$*-rectifiable, more precisely, if we define*

$$\begin{aligned}\mathcal{S}_*(\psi) = \{x \in \mathcal{S}(\psi);\ &\textit{the leading polynomial of } \psi \textit{ at } x \textit{ is a polynomial}\\ &\textit{of two variables after some rotation of coordinates}\},\end{aligned}$$

*then* $\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ *is countably* $(n-3)$*-rectifiable. Thus*

$$\mathcal{H}^{n-2}(\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)) = 0. \tag{2.6}$$

## 3. Estimate of the singular set

In this section, we shall estimate the Hausdorff measure of the singular set of any non-trivial weak solution of (1.1). In addition to the hypothesis (H), we assume that for an integer $M \ge 1$,

(K)$_M$ $\boldsymbol{A} \in C^M(\Omega;\mathbb{R}^n), \operatorname{div}\boldsymbol{A} \in C^M(\Omega)$ and $f \in C^M([0,\infty))$.

In the following, for any given $\Omega' \Subset \Omega$, we always choose

$$\Omega'' = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \min(r_0, \operatorname{dist}(\Omega', \partial\Omega))/3\},$$

where $r_0$ is as in Proposition 2.2, and define

$$\Lambda(\Omega'') = |\boldsymbol{A}|_{C^M(\Omega'')} + |\boldsymbol{A}|^2_{C^M(\Omega'')} + |\operatorname{div}\boldsymbol{A}|_{C^M(\Omega'')} + |f(|\psi|^2)|_{C^M(\Omega'')}, \tag{3.1}$$

if $\psi \in C^M_{\mathrm{loc}}(\Omega;\mathbb{C})$. We also use the notations

$$|D^j\psi(x)| = \sum_{|\beta|=j} |D^\beta_x \psi(x)|,$$
$$|D^j\psi(x) - D^j\psi(y)| = |D^j(\psi(x) - \psi(y))|.$$

At first, we obtain the regularity of any solution of (1.1) under the hypotheses (H) and (K)$_M$.

**Proposition 3.1** (Regularity)**.** *Addition to the hypothesis* (H)*, assume that* (K)$_M$ *holds for some integer* $M \ge 1$. *Let* $\psi \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{C})$ *be any weak solution of* (1.1). *Then* $\psi \in C^{M+1,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{C})$ *for some* $\alpha \in (0,1)$. *Moreover, we have the Schauder type estimate: for any* $\Omega' \Subset \Omega$, *there exists* $R_0 > 0$ *depending on* $n, M, \alpha, \Omega$ *and* $\Lambda(\Omega'')$ *such that for all* $0 < R \le R_0$ *and* $x_0 \in \Omega'$,

$$\begin{aligned}&\sum_{j=1}^{M+1} R^j \sup_{x \in B_R(x_0)} |D^j\psi(x)|\\ &\qquad + R^{M+1+\alpha} \sup_{\substack{x,y \in B_R(x_0)\\ x \ne y}} \frac{|D^{M+1}\psi(x) - D^{M+1}\psi(y)|}{|x-y|^\alpha} \le C \sup_{x \in B_{2R}(x_0)} |\psi(x)|\end{aligned}$$

*where the constant $C$ depends on $n$, $M$, $\alpha$, $\Omega$, $\Omega'$ and $\Lambda(\Omega'')$.*

**Proof.** By Proposition 2.1, we see that $\psi \in C^{\alpha}_{\mathrm{loc}}(\Omega;\mathbb{C}) \cap W^{2,q}_{\mathrm{loc}}(\Omega;\mathbb{C})$ for some $\alpha \in (0,1)$ and $\psi$ satisfies the equation (1.1). We note that $\boldsymbol{A} \in C^{M}(\Omega;\mathbb{R}^n) \hookrightarrow C^{M-1,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Similarly, $\operatorname{div}\boldsymbol{A}$, $|\boldsymbol{A}|^2$ belong to $C^{M-1,\alpha}_{\mathrm{loc}}(\Omega)$. Since $f \in C^M(\overline{\mathbb{R}}_+) \hookrightarrow C^{M-1,1}_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$, we have $f(|\psi|^2)\psi \in C^{\alpha}_{\mathrm{loc}}(\Omega;\mathbb{C})$. Therefore, it follows from Gilbarg and Trudinger [10, Theorem 9.19] that $\psi \in C^{2,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{C})$. By the boot-strap method, we see that $\psi \in C^{M+1,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{C})$.

Next, we shall get the estimate. In order to do so, we write (1.1) into the form:

$$-\Delta\psi + 2i\boldsymbol{A}\cdot\nabla\psi + \big(i(\operatorname{div}\boldsymbol{A}) + |\boldsymbol{A}|^2\big)\psi = f(|\psi|^2)\psi \quad \text{in} \quad \Omega\,. \tag{3.2}$$

We simply write $3R_1 = \min\big(r_0, \operatorname{dist}(\Omega',\partial\Omega)\big)$ and we choose $\Omega''$ as above. Then for any $0 < R < R_1$ and $x_0 \in \Omega'$, $B_{2R}(x_0) \subset \Omega''$. We shall apply the Schauder estimate [10, p.142] in $B_{2R}(x_0)$. We use the following notations as in [10]. For $g \in C^{k,\alpha}\big(B_r(x_0)\big)$,

$$|g|^{(\sigma)}_{k,\alpha,B_r(x_0)} = |g|^{(\sigma)}_{k,B_r(x_0)} + [g]^{(\sigma)}_{k,\alpha,B_r(x_0)}\,,$$

where

$$\begin{aligned}
|g|^{(\sigma)}_{k,B_r(x_0)} &= \sum_{j=0}^{k} \sup_{x\in B_r(x_0)} d_x^{j+\sigma}|D^j g(x)|\,,\\
[g]^{(\sigma)}_{k,\alpha,B_r(x_0)} &= \sup_{\substack{x,y\in B_r(x_0)\\ x\neq y}} d_{x,y}^{k+\alpha+\sigma}\frac{|D^k g(x) - D^k g(y)|}{|x-y|^\alpha}\,,\\
|g|^{*}_{k,\alpha,B_r(x_0)} &= |g|^{(0)}_{k,\alpha,B_r(x_0)}\,,\\
d_x = \operatorname{dist}&\big(x,\partial B_r(x_0)\big)\,, \quad d_{x,y} = \min(d_x,d_y)\,.
\end{aligned}$$

It follows from the hypothesis $(\mathrm{K})_M$ that

$$\begin{aligned}
|\boldsymbol{A}|^{(1)}_{M-1,\alpha,B_{2R}(x_0)} &+ |\operatorname{div}\boldsymbol{A}|^{(2)}_{M-1,\alpha,B_{2R}(x_0)} + ||\boldsymbol{A}|^2|^{(2)}_{M-1,\alpha,B_{2R}(x_0)}\\
&\le C\Lambda(\Omega'') < \infty
\end{aligned}$$

for some constant $C$ depending only on $\Omega$. Therefore we can apply the result of [10] to (3.2) to get

$$|\psi|^{*}_{M+1,\alpha,B_{2R}(x_0)} \le C\big(|\psi|_{0,B_{2R}(x_0)} + |f(|\psi|^2)\psi|^{(2)}_{M-1,\alpha,B_{2R}(x_0)}\big) \tag{3.3}$$

where $C$ depends on $n, M, \alpha, \Omega$ and $\Lambda(\Omega'')$.

We estimate the last term in the right hand side of (3.3). Since $\psi \in C^{M+1,\alpha}(B_{2R}(x_0)) \subset C^{M+1,\alpha}(\overline{\Omega''})$ $(M \geq 1)$, we have

$$\begin{aligned}\sum_{j=0}^{M-1} \sup_{x\in B_{2R}(x_0)} d_x^{j+2}|D^j[f(|\psi(x)|^2)\psi(x)]| &\leq C \sum_{j=0}^{M-1} \sup_{x\in B_{2R}(x_0)} d_x^{j+2}|D^j\psi(x)| \\ &\leq C_1R^2 \sum_{j=0}^{M-1} \sup_{x\in B_{2R}(x_0)} d_x^{j}|D^j\psi(x)|\end{aligned}$$

where the constant $C_1$ depends on $\Lambda(\Omega'')$. Similarly we can estimate

$$\begin{aligned}&\sup_{\substack{x,y\in B_{2R}(x_0)\\ x\neq y}} d_{x,y}^{M+1+\alpha} \frac{|D^{M-1}[f(|\psi(x)|^2)\psi(x)] - D^{M-1}[f(|\psi(y)|^2)\psi(y)]|}{|x-y|^\alpha} \\ &\quad= \sup_{\substack{x,y\in B_{2R}(x_0)\\ x\neq y}} d_{x,y}^{M+1+\alpha} \frac{|D^{M-1}[f(|\psi(x)|^2)\psi(x)] - D^{M-1}[f(|\psi(y)|^2)\psi(y)]|}{|x-y|} \\ &\qquad \times |x-y|^{1-\alpha} \\ &\quad\leq C_2 \sum_{j=0}^{M} \sup_{x,y\in B_{2R}(x_0)} \sup_{z\in B_{2R}(x_0)} d_{x,y}^{M+1+\alpha}|D^j\psi(z)||x-y|^{1-\alpha} \\ &\quad\leq C_3R^2 \sum_{j=0}^{M} \sup_{z\in B_{2R}(x_0)} d_z^{j}|D^j\psi(z)|\end{aligned}$$

where the constant $C_3$ also depends on $\Lambda(\Omega'')$. Thus we see that

$$|f(|\psi|^2)\psi|^{(2)}_{M-1,\alpha,B_{2R}(x_0)} \leq C_4R^2|\psi|^*_{M+1,\alpha,B_{2R}(x_0)}$$

where $C_4$ depends on $\Lambda(\Omega'')$. If we choose $R_0 > 0$ so that $CC_4R_0^2 < 1/2$ where $C$ is as in (3.3) and $R_0 < R_1$, then it follows from (3.3) that for all $0 < R \leq R_0$

$$|\psi|^*_{M+1,\alpha,B_{2R}(x_0)} \leq C|\psi|_{0,B_{2R}(x_0)}$$

where the constant $C$ depends on $n$, $M$, $\alpha$, $\Omega$ and $\Lambda(\Omega'')$. Since $d_x = \operatorname{dist}(x, \partial B_{2R}(x_0)) \geq R$ for $x \in B_R(x_0)$, we obtain the conclusion. □

We choose an integer $M \geq 1$ in Theorem 1.1 so that

$$M \geq 2c_0^2 \tag{3.4}$$

where $c_0$ is the constant as in Proposition 2.2 (i). We note that it follows from (2.3) that the vanishing order of $\psi$ is uniformly bounded in $\Omega$ : $\mathcal{O}_\psi(x) \leq c_0$ for all $x \in \Omega$.

Let $\psi$ be any non-trivial weak solution $\psi$ of (1.1) and $\Omega' \Subset \Omega$. Then for all $x_0 \in \Omega'$ and $0 < R < R_0$ where $R_0$ is as in Proposition 3.1, $\psi$ has a decomposition in $B_{2R}(x_0) \Subset \Omega''$:

$$\psi(x + x_0) = P_m(x) + \phi(x), \quad x \in B_R(0) \tag{3.5}$$

where $P_m$ is a non-zero complex-valued homogeneous, harmonic polynomial of degree $m$ and $\phi$ satisfies (2.5).

We estimate the remainder term $\phi$.

**Lemma 3.2.** *Assume that the hypotheses* (H) *and* $(\mathrm{K})_M$ *hold. Then in the decomposition of* $\psi$ *in* (3.5), $\phi$ *satisfies*

$$|D^j\phi(x)| \le \begin{cases} CR^{m-j+\alpha}, & j=0,1,\dots,m, \\ C, & j=m+1,\dots,M+1 \end{cases}$$

*in* $B_R(0)$ *where the constant* $C$ *depends on* $n, M, \psi, \Omega$ *and* $\Lambda(\Omega'')$.

**Proof.** Since $P_m$ is harmonic in $\mathbb{R}^n$, we have

$$\Delta\phi(x) = -\Delta\psi(x+x_0) \quad \text{in} \quad B_R(0).$$

Therefore we can apply the Schauder estimate as in the proof of Proposition 3.1, so we can get

$$\begin{aligned} &\sum_{j=0}^{M+1} R^j \sup_{x\in B_R(0)} |D^j\phi(x)| \\ &\qquad \le C\Big\{ \sup_{x\in B_R(0)} |\phi(x)| + \sum_{j=0}^{M-1} R^{j+2} \sup_{x\in B_R(x_0)} |D^j\Delta\psi(x)| \\ &\qquad\qquad + R^{M+1+\alpha} \sup_{\substack{x,y\in B_R(x_0)\\ x\neq y}} \frac{|D^j\Delta\psi(x) - D^j\Delta\psi(y)|}{|x-y|^\alpha}\Big\} \end{aligned} \tag{3.6}$$

where $C$ depends on $n$, $M$ and $\Omega$. We write the equation (1.1) into the form

$$\Delta\psi = 2i\boldsymbol{A}\cdot\nabla\psi + i(\operatorname{div}\boldsymbol{A})\psi + |\boldsymbol{A}|^2\psi - f(|\psi|^2)\psi.$$

Then applying Proposition 3.1, we shall estimate the last two terms in (3.6). In the following we denote constants depending only on $\Omega$ and $\Lambda(\Omega'')$ by $C$ which may vary from line to line. For $0\le j\le M-1$, we have

$$\begin{aligned} &R^{j+2} \sup_{x\in B_R(x_0)} |D^j\Delta\psi(x)| \\ &= R^{j+2} \sup_{x\in B_R(x_0)} \big|D^j[2i\boldsymbol{A}\cdot\nabla\psi + i(\operatorname{div}\boldsymbol{A})\psi + |\boldsymbol{A}|^2\psi - f(|\psi|^2)\psi]\big| \\ &\le CR^{j+2}\sum_{k=0}^{j+1} \sup_{x\in B_R(x_0)} |D^k\psi(x)| \\ &\le CR\sum_{k=0}^{j+1} R^k \sup_{x\in B_R(x_0)} |D^k\psi(x)| \\ &\le CR \sup_{x\in B_{2R}(x_0)} |\psi(x)| \le CR^{m+1}. \end{aligned}$$

We can similarly estimate the last term in (3.6). Thus we get

$$\sum_{j=0}^{M+1} R^j \sup_{x\in B_R(0)} |D^j\phi(x)| \le C\{ \sup_{x\in B_R(0)} |\phi(x)| + R^{m+1}\} \le CR^{m+\alpha}.$$

Therefore, for $j = 0, 1, \dots, m$, we have

$$|D^j \phi(x)| \le C R^{m-j+\alpha} \quad \text{in} \quad B_R(0).$$

For $j = m+1, \dots, M+1$, since $D^j P_m \equiv 0$, we see that $D^j \phi(x) = D^j \psi(x + x_0)$ in $B_R(0)$. Thus we have

$$|D^j \phi(x)| \le \sup_{x \in B_R(x_0)} |D^j \psi(x)| \le \sup_{x \in \Omega''} |D^j \psi(x)| \le C_j \quad \text{in} \quad B_R(0)\,.$$

This completes the proof. □

Now we show a property of a complex-valued harmonic polynomial.

**Lemma 3.3** (cf. [13]). *Let $P$ be a complex-valued non-zero homogeneous, harmonic polynomial of degree $m \ge 2$, and of two variables in $\mathbb{R}^n$. Then there exist $\delta_*, r_* > 0$ depending on $P$ such that if $\varphi \in C^{2m^2}(B_1(0); \mathbb{C})$ satisfies $|\varphi - P|_{C^{2m^2}(B_1(0);\mathbb{C})} < \delta_*$, then*

$$\mathcal{H}^{n-2}(|\nabla \varphi|^{-1}\{0\} \cap B_r(0)) \le c(n)(m-1)^2 r^{n-2}$$

*for all $0 < r \le r_*$.*

**Proof.**

Step 1. It suffices to prove the case where $P$ and $\varphi$ are real-valued.

In fact, assume that Lemma 3.3 holds for the case where $P$ and $\varphi$ are real-valued. Let $P$ and $\varphi$ be complex-valued functions satisfying the hypotheses in the lemma. Since either of $\Re P$ or $\Im P$ is non-zero, let $\Re P \not\equiv 0$. We choose $\delta_*$ and $r_*$ corresponding to $\Re P$. If $|\varphi - P|_{C^{2m^2}(B_1(0);\mathbb{C})} < \delta_*$, then $|\Re\varphi - \Re P|_{C^{2m^2}(B_1(0);\mathbb{C})} < \delta_*$. Since $|\nabla \varphi|^{-1}\{0\} \subset |\nabla \Re\varphi|^{-1}\{0\}$, we get the conclusion.

Step 2. We shall show the lemma for the real case. Though the proof is identical as [13, Lemma 3.2], we introduce an outline of the proof. We choose a coordinates $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n) \in \mathbb{R}^n$, and the polar coordinates $\widetilde{x}_1 = r\cos\theta, \widetilde{x}_2 = r\sin\theta$ in $\mathbb{R}^2$. By the hypothesis on $P$, we may assume that

$$P(\widetilde{x}) = r^m \cos m\theta\,.$$

Then we have

$$D_{\widetilde{x}_1} P(\widetilde{x}) = m r^{m-1} \cos(m-1)\theta\,,$$

$$D_{\widetilde{x}_2} P(\widetilde{x}) = m r^{m-1} \sin(m-1)\theta\,.$$

By the formulae:

$$\cos(m-1)\theta = 2^{m-2} \prod_{r=1}^{m-1} \sin\left(\theta + \frac{(2r-1)\pi}{2(m-1)}\right),$$

$$\sin(m-1)\theta = 2^{m-2} \prod_{r=1}^{m-1} \sin\left(\theta + \frac{(r-1)\pi}{m-1}\right),$$

there exists $2(m-1)$ non-zero vectors $\nu_k^i \in \mathbb{R}^2$ $(i=1,2,k=1,2,\dots,m-1)$ such that

$$D_{\widetilde{x}_i}P(\widetilde{x}) = \prod_{k=1}^{m-1}\big(\nu_k^i\cdot(\widetilde{x}_1,\widetilde{x}_2)\big) \quad \text{for} \quad i=1,2\,.$$

We note that

$$\det(\nu_k^i,\nu_l^j)\neq 0 \quad \text{for} \quad (i,k)\neq(j,l) \tag{3.7}$$

and $D_{\widetilde{x}_i}P(\widetilde{x})=0$ for $i=3,4,\dots,n$. Here for $\nu_k^i=(\nu_{k,1}^i,\nu_{k,2}^i)\in\mathbb{R}^2$, $\det(\nu_k^i,\nu_l^j)=\det\begin{pmatrix}\nu_{k,1}^i & \nu_{l,1}^j\\ \nu_{k,2}^i & \nu_{l,2}^j\end{pmatrix}$. We take a change of coordinates $\widetilde{x}=Ox$ with an orthogonal matrix $O=(o_{ij})$ to be chosen. Let $\eta_i=(o_{1i},o_{2i})\in\mathbb{R}^2$, $i=1,2,\dots,n$. Then we get

$$\begin{aligned} D_{x_i}P(x) &= o_{1i}\prod_{k=1}^{m-1}\big((\eta_1\cdot\nu_k^1)x_1+\cdots+(\eta_n\cdot\nu_k^1)x_n\big)\\ &\quad + o_{2i}\prod_{k=1}^{m-1}\big((\eta_1\cdot\nu_k^2)x_1+\cdots+(\eta_n\cdot\nu_k^2)x_n\big) \end{aligned} \tag{3.8}$$

for $i=1,2,\dots,n$. We note that if $D_{x_i}P(x)$ does not vanish, it is a homogeneous polynomial of degree $m-1$, and that (3.8) contains only first two rows of $O$. For any $1\le i<j\le n$ and $p\in\mathbb{R}^n$, we define two dimensional planes

$$\mathbb{P}_{ij}(p)=\big\{(p_1,\dots,p_{i-1},x_i,p_{i+1},\dots,p_{j-1},x_j,p_{j+1},\dots,p_n)\big\}$$

and $\mathbb{P}_{ij}=\mathbb{P}_{ij}(0)$. Then it follows from (3.8) that

$$\begin{pmatrix} D_{x_i}P|_{\mathbb{P}_{ij}}\\ D_{x_j}P|_{\mathbb{P}_{ij}}\end{pmatrix} = \begin{pmatrix} o_{1i} & o_{2i}\\ o_{1j} & o_{2j}\end{pmatrix}\begin{pmatrix}\prod_{k=1}^{m-1}\big((\eta_i\cdot\nu_k^1)x_i+(\eta_j\cdot\nu_k^1)x_j\big)\\ \prod_{k=1}^{m-1}\big((\eta_i\cdot\nu_k^2)x_i+(\eta_j\cdot\nu_k^2)x_j\big)\end{pmatrix}. \tag{3.9}$$

If we require that

$$\det(\eta_i,\eta_j)=\det\begin{pmatrix} o_{1i} & o_{1j}\\ o_{2i} & o_{2j}\end{pmatrix}\neq 0\,, \tag{3.10}$$

then that $D_{x_i}P=D_{x_j}P=0$ on $\mathbb{P}_{ij}$ is equivalent to

$$\prod_{k=1}^{m-1}\big((\eta_i\cdot\nu_k^1)x_i+(\eta_j\cdot\nu_k^1)x_j\big)=\prod_{k=1}^{m-1}\big((\eta_i\cdot\nu_k^2)x_i+(\eta_j\cdot\nu_k^2)x_j\big)=0\,.$$

By (3.7) and (3.10), for any $1\le k,l\le m-1$,

$$\det\begin{pmatrix}\eta_i\cdot\nu_k^1 & \eta_j\cdot\nu_k^1\\ \eta_i\cdot\nu_l^2 & \eta_j\cdot\nu_l^2\end{pmatrix}=\det(\eta_i,\eta_j)\det(\nu_k^1,\nu_l^2)\neq 0\,.$$

Thus if we require that in the orthogonal matrix $O$,

$$\det\begin{pmatrix} o_{1i} & o_{2i}\\ o_{1j} & o_{2j}\end{pmatrix}\neq 0 \quad \text{for all} \quad 1\le i<j\le n\,,$$

we obtain that for fixed $1\le i<j\le n$, $f_{ij}=(D_{x_i}P,D_{x_j}P)|_{\mathbb{P}_{ij}}:\mathbb{R}^2\to\mathbb{R}^2$ has only one zero at $x_i=x_j=0$. If we replace $(x_i,x_j)$ in (3.9) with $(z_i,z_j)\in\mathbb{C}^2$, we see

that $f_{ij}\colon \mathbb{C}^2 \to \mathbb{C}^2$ has also only zero at $z_i = z_j = 0$. We apply [13, Theorem 4.1] with $n = 2$ due to Hilbert's Nullstellensatz. For fixed $1 \le i < j \le n$, there exist $\delta_{ij} > 0$ and $r_{ij} > 0$ depending on $f_{ij}$ such that for any $v \in C^M(B^2_{1/2}(0);\mathbb{R}^2)$ with $|v - f_{ij}|_{C^M(B_{1/2}(0);\mathbb{R}^2)} < \delta_{ij}$, we have

$$\operatorname{card}\big(v^{-1}\{0\} \cap B^2_{r_{ij}}(0)\big) \le (m-1)^2 \tag{3.11}$$

where $B^2_r(0)$ denotes the ball centered at the origin with radius $r$ in $\mathbb{R}^2$ and $M = 2(m-1)^2$ which is independent of $i, j$. Put $\delta_* = \frac{1}{2}\min_{1\le i<j\le n}\delta_{ij}$ and $r_* = \min_{1\le i<j\le n} r_{ij}$. Moreover, we assume that $\varphi \in C^{2m^2}(B_1(0);\mathbb{R})$ satisfies $|\varphi - P|_{C^{2m^2}(B_1(0);\mathbb{R})} < \delta_*$. If we take $r_*$ smaller if necessary, for any $0 < r < r_*$ and for any $p \in B_r(0)$, $v_{ij,p} := (D_{x_i}\varphi, D_{x_j}\varphi)|_{\mathbb{P}_{ij}(p)}$ satisfies $|v_{ij,p} - f_{ij}|_{C^M(B^2_{1/2}(0);\mathbb{R})} < 2\delta \le \delta_{ij}$. Hence from (3.11)

$$\operatorname{card}(v^{-1}_{ij,p}\{0\} \cap B^2_r(0)) \le (m-1)^2\,.$$

Since $|\nabla\varphi|^{-1}\{0\} \cap \mathbb{P}_{ij}(p) \subset v^{-1}_{ij,p}\{0\}$, if we set the projection

$$\pi_{ij}(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1},\dots,x_n)\,,$$

then for any $q \in B^{n-2}_r(0) \subset \mathbb{R}^{n-2}$ and any $1 \le i < j \le n$,

$$\operatorname{card}\big(|\nabla\varphi|^{-1}\{0\} \cap \pi^{-1}_{ij}(q) \cap B_r(0)\big) \le (m-1)^2\,.$$

By the general area-coarea formula (cf. Federer [8, 3.3.22] or Morgan [20, 3.13]), we have

$$\begin{aligned}
&\mathcal{H}^{n-2}\big(|\nabla\varphi|^{-1}\{0\} \cap B_r(0)\big)\\
&\quad\le \sum_{1\le i<j\le n}\int_{B^{n-2}_r(0)} \operatorname{card}\big(|\nabla\varphi|^{-1}\{0\} \cap \pi^{-1}_{ij}(q) \cap B_r(0)\big)\, d\mathcal{H}^{n-2}q\\
&\quad\le c(n)(m-1)^2 r^{n-2}\,. \qquad (3.12)
\end{aligned}$$

Since $M + 1 = 2(m-1)^2 + 1 \le 2m^2$, this completes the proof. □

Now we can get the following

**Proposition 3.4.** *Assume that the conditions of Theorem 1.1 hold. Then for any $\Omega' \Subset \Omega$ and any $\varepsilon > 0$, there exist $C(\varepsilon) = C\varepsilon^{n-2}$ and $\gamma(\varepsilon) = \gamma\varepsilon^{n-2}$ where $C$ and $\gamma$ depend on $\psi$, $\Omega'$ and $\Lambda(\Omega'')$, and a collection of finitely many balls $\{B_{r_i}(x_i)\}_i$ with $r_i \le \varepsilon$, $x_i \in \mathcal{S}(\psi) \cap \Omega'$ such that*

$$\begin{aligned}
&\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \setminus \bigcup_i B_{r_i}(x_i)\Big) \le C(\varepsilon)\,,\\
&\sum_i r_i^{n-2} \le \gamma(\varepsilon)\,.
\end{aligned} \tag{3.13}$$

**Proof.** By (2.6), we have $\mathcal{H}^{n-2}\big(\mathcal{S}(\psi)\cap\Omega'\setminus\mathcal{S}_*(\psi)\big) = 0$. Therefore $(n-2)$-spherical measure of $\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ is equal to zero (cf. Mattila [19, p. 75]). Thus for any $\varepsilon > 0$, there exist at most countably many balls $\{B_{r_i}(x_i)\}$ with $r_i \le \varepsilon$ and $x_i \in \mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ such that

$$\sum_i r_i^{n-2} \le \gamma(\varepsilon, \psi, \Omega')$$

where $\gamma(\varepsilon, \psi, \Omega') \to 0$ as $\varepsilon \to 0$.

Then we shall show the following claim.

Claim: For any $y \in S_*(\psi) \cap \Omega'$, there exist $R = R(y, \psi, \Omega')$ and $c = c(y, \psi, \Omega')$ with $R \le R_0$ where $R_0$ is as in Proposition 3.1 such that for any $0 < r < R$,

$$\mathcal{H}^{n-2}(\mathcal{S}(\psi) \cap \Omega' \cap B_r(y)) \le cr^{n-2}\,.$$

Here $R$ and $c$ depend only on $y$, $\psi$, $\Omega'$ but also on $\Lambda(\Omega'')$.

We prove the claim. Let $y \in \mathcal{S}_*(\psi) \cap \Omega'$. By the construction of $\Omega''$, $B_{2R_0}(y) \Subset \Omega''$. If $R < R_0$, it follows from Proposition 2.4 that we can write

$$\psi(x + y) = P_m(x) + \phi(x) \quad \text{in} \quad B_R(0)$$

where $P_m$ is a non-zero homogeneous, harmonic polynomial of degree $m \ge 2$ of two variables after some rotation of coordinates, and from Lemma 3.2, we have

$$|D^j \phi(x)| \le \begin{cases} CR^{m-j+\alpha} & j = 0, 1, \dots, m\,, \\ C & j = m+1, \dots M+1 \end{cases}$$

in $B_R(0)$. If we choose $R = R(y, \psi, \Omega')$ small enough with $R \le R_0$, we have

$$\Big|\frac{1}{R^m}\phi\Big|^*_{C^{M+1}(B_R(0));\mathbb{C}} < \delta_* \tag{3.14}$$

where $\delta_*$ is as in Lemma 3.3. In fact,

$$\begin{aligned} \Big|\frac{1}{R^m}\phi\Big|^*_{C^{M+1}(B_R(0);\mathbb{C})} &= \sum_{j=0}^{M+1} \frac{R^j}{R^m} \sup_{x \in B_R(0)} |D^j \phi(x)| \\ &= \sum_{j=0}^{m} + \sum_{j=m+1}^{M+1} \\ &\le C\Big[\sum_{j=0}^{m} \frac{R^j}{R^m} R^{m-j+\alpha} + \sum_{j=m+1}^{M+1} \frac{R^j}{R^m}\Big] \\ &\le CR^\alpha \end{aligned}$$

where $C$ depends on $y$, $\psi$, $\Omega$ and $\Lambda(\Omega'')$. Thus if we choose $R > 0$ small enough, we get (3.14). That is to say, we have

$$\Big|\frac{1}{R^m}(\psi(\cdot + y) - P_m)\Big|^*_{C^{M+1}(B_R(0);\mathbb{C})} < \delta_*\,.$$

By scaling: $x \mapsto Rx$ and using the homogeneity of $P_m$, we have

$$\Big|\frac{1}{R^m}\psi(y + Rx) - P_m(x)\Big|_{C^{M+1}(B_1(0);\mathbb{C})} < \delta_*\,.$$

Since from (3.4) and (2.3), $M \geq 2c_0^2 \geq 2m^2$, we can apply Lemma 3.3 to $\frac{1}{R^m}\psi(y + Rx)$ and get

$$
\begin{aligned}
c(n)(m-1)^2 r^{n-2} &\geq \mathcal{H}^{n-2}\big(\{x; \nabla_x \psi(y+Rx) = 0\} \cap B_r(0)\big) \\
&= \mathcal{H}^{n-2}\Big(\Big\{\frac{z-y}{R}; \nabla_z \psi(z) = 0\Big\} \cap \Big\{\frac{z-y}{R}; \Big|\frac{z-y}{R}\Big| < r\Big\}\Big) \\
&= \frac{1}{R^{n-2}}\mathcal{H}^{n-2}\big(|\nabla\psi|^{-1}\{0\} \cap B_r(y)\big)
\end{aligned}
$$

for all $0 < r < r_*$. Here we may assume that $R < 1$. Therefore, we get

$$
\mathcal{H}^{n-2}\big(|\nabla\psi|^{-1}\{0\} \cap B_r(y)\big) \leq c(n)(m-1)^2 r^{n-2} \tag{3.15}
$$

for all $0 < r < Rr_*$. Since we can replace $R$ with a smaller one, we can take $Rr_* \leq \varepsilon$. Thus the claim holds.

Therefore, since

$$
\mathcal{S}_*(\psi) \subset \bigcup_{y \in \mathcal{S}_*(\psi)} B_{r(y)}(y)\,,
$$

we have

$$
\begin{aligned}
\mathcal{S}(\psi) \cap \Omega' &= (\mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)) \cup \mathcal{S}_*(\psi) \\
&\subset \bigcup_i B_{r_i}(x_i) \cup \bigcup_{y \in \mathcal{S}_*(\psi)} B_{r(y)}(y)\,.
\end{aligned}
$$

Since $\mathcal{S}(\psi) \cap \Omega'$ is relatively compact, there exist finitely many $x_i \in \mathcal{S}(\psi) \cap \Omega' \setminus \mathcal{S}_*(\psi)$ $(i = 1, 2, \ldots, k = k(\varepsilon, \psi))$ and $y_j \in \mathcal{S}_*(\psi)$ $(j = 1, 2, \ldots, l = l(\varepsilon, \psi))$ such that

$$
\mathcal{S}(\psi) \cap \Omega' \subset \bigcup_{i=1}^{k} B_{r_i}(x_i) \cup \bigcup_{j=1}^{l} B_{s_j}(y_j)
$$

and

$$
\sum_{i=1}^{k} r_i^{n-2} \leq \sum_{i=1}^{k} \varepsilon^{n-2} := \gamma(\varepsilon, \psi) = k\varepsilon^{n-2}\,.
$$

Thus it follows from the claim that

$$
\begin{aligned}
&\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \setminus \bigcup_{i=1}^{k} B_{r_i}(x_i)\Big) \\
&\quad \leq \sum_{j=1}^{l} \mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap B_{s_j}(y_j)\big) \\
&\quad \leq C\sum_{j=1}^{l} s_j^{n-2} \leq C\sum_{j=1}^{l} \varepsilon^{n-2} = Cl\varepsilon^{n-2}\,.
\end{aligned}
$$

This completes the proof. $\square$

Finally, we have

**Theorem 3.5.** *Assume that the conditions in Theorem 1.1 hold. For any $\Omega' \Subset \Omega$, there exists a constant $C > 0$ depending on $\psi$, $\Omega'$ and $\Lambda(\Omega'')$ such that*

$$\mathcal{H}^{n-2}(\mathcal{S}(\psi) \cap \Omega') \leq C\,.$$

**Proof.** Let $0 < R < R_0$ where $R_0$ is as in Proposition 3.1. Since $\{B_R(x)\}_{x\in\Omega'}$ is an open covering of a compact set $\overline{\Omega'}$, there exists finitely many points $x_1, \dots, x_{k_0} \in \Omega'$ such that $\overline{\Omega'} \subset \bigcup_{i=1}^{k_0} B_R(x_i)$. We put a collection of the balls $\phi_0 = \{B_R(x_i)\}_{i=1}^{k_0}$. Fix any $\varepsilon > 0$. Then we have the following

Claim: There exist collections of balls $\phi_1, \phi_2, \dots$ such that for any $l \geq 1$,

(i) $\mathrm{rad}\,(B) \leq (2\varepsilon)^l R_0$ for all $B \in \phi_l$ where $\mathrm{rad}\,(B)$ denotes the radius of the ball $B$.

(ii) The center of $B$ is contained in $\Omega'$ for all $B \in \phi_l$.

(iii) $\sum_{B\in\phi_l} (\mathrm{rad}\,(B))^{n-2} \leq \gamma(\varepsilon)^l$.

(iv) $\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap \big(\bigcup_{B\in\phi_{l-1}} B \sim \bigcup_{B\in\phi_l} B\big)\big) \leq C(\varepsilon)\gamma(\varepsilon)^{l-1}$ where $\gamma(\varepsilon)$ and $C(\varepsilon)$ are as in Proposition 3.4.

First, we show that the claim implies Theorem 3.5. In order to do so, we show that

$$\begin{aligned}\mathcal{S}(\psi) \cap \Omega' \subset & \bigcup_{l=1}^{\infty} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{B\in\phi_{l-1}} B \sim \bigcup_{B\in\phi_l} B\Big)\Big) \\ & \cup \bigcap_{l=0}^{\infty} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=l}^{\infty} \bigcup_{B\in\phi_j} B\Big)\Big). \end{aligned} \tag{3.16}$$

In fact, let $p \in \mathcal{S}(\psi) \cap \Omega'$ and assume that

$$p \notin \bigcup_{B\in\phi_{l-1}} B \sim \bigcup_{B\in\phi_l} B \tag{3.17}$$

for all $l \geq 1$. Since $p \in \bigcup_{B\in\phi_0} B$, clearly $p \in \mathcal{S}(\psi) \cap \Omega' \cap \big(\bigcup_{j=0}^{\infty} \bigcup_{B\in\phi_j} B\big)$. It suffices to show that for any $k \geq 0$,

$$p \in \bigcap_{l=0}^{k} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=l}^{\infty} \bigcup_{B\in\phi_j} B\Big)\Big). \tag{3.18}$$

We show $(3.18)_k$ by induction on $k$. When $k = 1$, by (3.17), $p \notin \big(\bigcup_{B\in\phi_0} B\big) \setminus \big(\bigcup_{B\in\phi_1} B\big)$ and $p \notin \big(\bigcup_{B\in\phi_1} B\big) \setminus \big(\bigcup_{B\in\phi_0} B\big)$. Since $p \in \bigcup_{B\in\phi_0} B$, we see that

$$p \in \mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{B\in\phi_1} B\Big) \subset \mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=1}^{\infty} \bigcup_{B\in\phi_j} B\Big).$$

Thus we have

$$p \in \bigcap_{l=0}^{1} \Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big(\bigcup_{j=l}^{\infty} \bigcup_{B\in\phi_j} B\Big)\Big).$$

Therefore $(3.18)_1$ holds. Assume that $(3.18)_j$ holds for $j \leq k$ $(k \geq 1)$. Then $p \in \mathcal{S}(\psi) \cap \Omega'$ and $p \in \bigcup_{j=k}^{\infty} \bigcup_{B\in\phi_j} B$. That is to say, for some $j \geq k$, $p \in \bigcup_{B\in\phi_j} B$.

If $j \geq k+1$, then $p \in \bigcup_{j=k+1}^{\infty} \bigcup_{B \in \phi_j} B$. Hence $(3.18)_{k+1}$ holds. If $j = k$, then $p \in \bigcup_{B \in \phi_k} B$. From (3.17), $p \in \bigcup_{B \in \phi_{k+1}} B \subset \bigcup_{j=k+1}^{\infty} \bigcup_{B \in \phi_j} B$. Thus $(3.18)_{k+1}$ holds. Therefore, $(3.18)_k$ holds for any $k \geq 0$.

From (3.16), we see that

$$\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega'\big) \leq \sum_{l=1}^{\infty} \mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \cap \Big( \bigcup_{B \in \phi_{l-1}} B \sim \bigcup_{B \in \phi_l} B\Big)\Big) + \inf_{l \geq 0} \sum_{j=l}^{\infty} \sum_{B \in \phi_j} \mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap B\big)\,.$$

Here it follows from (3.15) that

$$\begin{aligned}\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega' \cap B\big) &\leq \mathcal{H}^{n-2}\big(|\nabla\psi|^{-1}\{0\} \cap B\big) \\ &\leq c(n)(m-1)^2\big(\operatorname{rad}(B)\big)^{n-2} \leq c(n)M\big(\operatorname{rad}(B)\big)^{n-2}\,.\end{aligned}$$

Therefore, from the claim, we have

$$\begin{aligned}&\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega'\big) \\ &\quad \leq C(\varepsilon) \sum_{l=1}^{\infty} \gamma(\varepsilon)^{l-1} + c(n)M \inf_{l \geq 0} \sum_{j=l}^{\infty} \sum_{B \in \phi_j} \big(\operatorname{rad}(B)\big)^{n-2} \\ &\quad \leq C(\varepsilon) \sum_{l=1}^{\infty} \gamma(\varepsilon)^{l-1} + c(n)M \inf_{l \geq 0} \sum_{j=l}^{\infty} \gamma(\varepsilon)^{j}\,.\end{aligned}$$

If we choose $\varepsilon > 0$ small enough so that $\gamma(\varepsilon) \leq 1/2$, we have

$$\mathcal{H}^{n-2}\big(\mathcal{S}(\psi) \cap \Omega'\big) \leq 2C(\varepsilon)\,.$$

Thus the theorem holds.

Finally we prove the claim. Note that $\phi_0 = \{B_R(x_i)\}_{i=1}^{k_0}$ is independent of $\varepsilon$. We shall construct $\{\phi_l\}_{l \geq 1}$ by induction on $l$. Assume that we have constructed $\phi_0, \phi_1, \ldots, \phi_{l-1}$ $(l \geq 1)$. Let $B = B_r(y) \in \phi_{l-1}$, $(y \in \Omega')$. If we choose $\varepsilon > 0$ small enough, if necessary, by induction hypothesis, $r \leq (2\varepsilon)^{l-1}R_0 \leq R_0$. Then $B_{2r}(y) \Subset \Omega$ and $\psi$ satisfies the equation

$$-\nabla_{\boldsymbol{A}}^2 \psi = f(|\psi|^2)\psi \quad \text{in} \quad B_{2r}(y)\,.$$

By Proposition 3.4 replaced $\varepsilon$ with $\varepsilon r$, there exist a finitely many balls $\{B_{r_i}(x_i)\}_i$ with $r_i \leq \varepsilon r$ and $x_i \in \mathcal{S}(\psi) \cap \Omega' \cap B_r(y)$ such that

$$\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi) \cap \Omega' \cap B_r(y) \setminus \Big(\bigcup_i B_{r_i}(x_i)\Big)\Big) \leq C(\varepsilon) r^{n-2} \tag{3.19}$$

and

$$\sum_i r_i^{n-2} \leq \gamma(r\varepsilon)^{n-2} = \gamma(\varepsilon) r^{n-2}\,.$$

Let $\phi_l^B = \{B_{r_i}(x_i)\}_i$ and $\phi_l = \{\phi_l^B\}_{B \in \phi_{l-1}}$. Then

$$r_i \leq \varepsilon r \leq \varepsilon(2\varepsilon)^{l-1}R_0 \leq (2\varepsilon)^l R_0\,.$$

Thus (i) in the claim holds. (ii) is clear. Since

$$\sum_{B\in\phi_l}\big(\operatorname{rad}(B)\big)^{n-2}=\sum_{B\in\phi_{l-1}}\sum_i r_i^{n-2}\le\gamma(\varepsilon)\gamma(\varepsilon)^{l-1}=\gamma(\varepsilon)^l\,,$$

(iii) holds. It follows from (3.19) and the induction hypothesis that

$$\begin{aligned}&\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi)\cap\Omega'\cap\Big(\bigcup_{B\in\phi_{l-1}}B\sim\bigcup_{B\in\phi_l}B\Big)\Big)\\&\quad\le\sum_{B_r(y)\in\phi_{l-1}}\mathcal{H}^{n-2}\Big(\mathcal{S}(\psi)\cap\Omega'\cap B_r(y)\setminus\Big(\bigcup_i B_{r_i}(x_i)\Big)\Big)\\&\quad\le C(\varepsilon)\sum_{B_r(y)\in\phi_{l-1}}r^{n-2}\le C(\varepsilon)\gamma(\varepsilon)^{l-1}\,.\end{aligned}$$

This proves (iv). Thus the claim holds. This completes the proof of Theorem 3.5. □

It is clear that this Theorem 3.5 implies Theorem 1.1.

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