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MODULE (φ,ψ)-AMENABILITY OF BANACH ALGEBRAS

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Abstract. Let S be an inverse semigroup with the set of idempotents E and $S/\approx$ be an appropriate group homomorphic image of S. In this paper we find a one-to-one correspondence between two cohomology groups of the group algebra $\ell^1(S)$ and the semigroup algebra $\ell^1(S/\approx)$ with coefficients in the same space. As a consequence, we prove that S is amenable if and only if $S/\approx$ is amenable. This could be considered as the same result of Duncan and Namioka [5] with another method which asserts that the inverse semigroup S is amenable if and only if the group homomorphic image $S/\sim$ is amenable, where $\sim$ is a congruence relation on S.

1. Introduction

For a discrete semigroup S, $\ell^\infty(S)$ is the Banach algebra of bounded complex-valued functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, let $l_a f$ and $r_a f$ denote the left and the right translations of f by a, that is $(l_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$, for each $s \in S$. Then a linear functional $m \in (\ell^\infty(S))^*$ is called a *mean* if $\|m\| = \langle m, 1\rangle = 1$; m is called a *left* (*right*) *invariant mean* if $m(l_a f) = m(f)$ ($m(r_a f) = m(f)$, respectively) for all $s \in S$ and $f \in \ell^\infty(S)$. A discrete semigroup S is called *amenable* if there exists a mean m on $\ell^\infty(S)$ which is both left and right invariant (see [5]).

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach A-module is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach A-module X, where $H^1(\mathcal{A}, X^*)$ is the *first Hochschild cohomology group* of A with coefficients in X^*. This concept was introduced by Barry Johnson in [7]. He showed that discrete group G is amenable if and only if the Banach algebra $\ell^1(G)$ is amenable. This fails to be true for discrete semigroups. M. Amini in [1] introduced the concept of module amenability for a class of Banach algebras, and showed that for an inverse semigroup S, the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$, where E is the set of idempotents of S, if and only if S is amenable (see also [3]). If $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, then each $\mathfrak{A}$-module endomorphism φ (not necessarily $\mathbb{C}$-linear) on $\mathcal{A}$ induces a continuous endomorphism $\widehat{\varphi}$ on $\mathcal{A}/J$, where J is a closed ideal of $\mathcal{A}$, in particular generated by $\alpha \cdot (ab) - (ab) \cdot \alpha$ for

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all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. In section two, for each pair $\mathfrak{A}$-module endomorphism φ and ψ on $\mathcal{A}$, we define module (φ, ψ)-amenability of Banach algebras, and among other results investigate the relation between module (φ, ψ)-amenability of $\mathcal{A}$ and $(\widehat{\varphi}, \widehat{\psi})$-amenability of $\mathcal{A}/J$. In section three we show that if S is an inverse semigroup with a directed upward set of idempotents E, then there exists a one-to-one correspondence between the quotient spaces $H^{\ell^1(E)}_{(\varphi,\psi)}(\ell^1(S), X^*)$, the first relative (to $\ell^1(E)$) (φ, ψ)-cohomology group of $\ell^1(S)$ with coefficients in X^* and $H_{(\widehat{\varphi},\widehat{\psi})}(\ell^1(S/\approx), X^*)$, the first $(\widehat{\varphi}, \widehat{\psi})$-cohomology group of $\ell^1(S/\approx)$, where $S/\approx$ is the maximal group homomorphic image of S, which $s \approx t$ whenever $\delta_s - \delta_t$ belongs to the closed linear span of the set

$$\{\delta_{\mathrm{set}} - \delta_{st} : s, t \in S, e \in E\}.$$

Finally we show that S is amenable if and only if $S/\approx$ is amenable. The fact S is amenable if and only if $S/\sim$ is amenable, where $\sim$ is a congruence relation on S is first proved by Duncan and Namioka [5] (see also [10, Propositin A.0.5]). Also it was proved in [12] that with a two-sided stable equivalence relation $\simeq$ on S, if the semigroup $S/\simeq$ has a left invariant mean, then so does S.

2. Module (φ, ψ)-amenability of Banach algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. We denote by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ the metric space of all bounded homomorphisms from $\mathcal{A}$ into $\mathcal{B}$, with the metric derived from Ł$(\mathcal{A}, \mathcal{B})$; the bounded linear operators from $\mathcal{A}$ into $\mathcal{B}$, and denote $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$ by $\mathrm{Hom}(\mathcal{A})$.

Let X be a $\mathcal{A}$-module and let σ and τ be in $\mathrm{Hom}(\mathcal{A})$. A bounded linear mapping $D\colon \mathcal{A} \to X$ is called a (σ, τ)-*derivation* if

$$D(ab) = D(a) \cdot \sigma(b) + \tau(a) \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

A bounded linear mapping $D\colon \mathcal{A} \to X$ is called a (σ, τ)-*inner derivation* if there exists $x \in X$ such that

$$D(a) = x \cdot \sigma(a) - \tau(a) \cdot x \qquad (a \in \mathcal{A}).$$

We use notations $Z_{(\sigma,\tau)}(\mathcal{A}, X)$ for the space of all continuous (σ, τ)-derivations $D\colon \mathcal{A} \to X$, and $B_{(\sigma,\tau)}(\mathcal{A}, X)$ for those which are (σ, τ)-inner derivation. Also we use notation $H_{(\sigma,\tau)}(\mathcal{A}, X)$ for the quotient space $Z_{(\sigma,\tau)}(\mathcal{A}, X)/B_{(\sigma,\tau)}(\mathcal{A}, X)$ which call the first (σ, τ)-cohomology group of $\mathcal{A}$ with coefficients in X. Derivations of these forms are studied in [8].

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in \mathcal{A},\ \alpha \in \mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \qquad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$
$$(\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \qquad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X)$$

and the same for the right or two-sided actions. Then we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover

$$\alpha\cdot x = x\cdot\alpha \qquad (\alpha\in\mathfrak{A}, x\in X)$$

then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If X is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined by

$$\langle\alpha\cdot f, x\rangle = \langle f, x\cdot\alpha\rangle\,, \quad \langle a\cdot f, x\rangle = \langle f, x\cdot a\rangle \qquad (a\in\mathcal{A}, \alpha\in\mathfrak{A},\ x\in X, f\in X^*)$$

and the same for the right actions.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathfrak{A}$-module, as we have not assumed the compatibility condition

$$a\cdot(\alpha\cdot b) = (a\cdot\alpha)\cdot b \qquad (\alpha\in\mathfrak{A}, a, b\in\mathcal{A})\,.$$

If $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

If $\mathcal{A}$ is a Banach $\mathfrak{A}$-module with compatible actions, then so is the dual space $\mathcal{A}^*$. If moreover $\mathcal{A}$ is a commutative $\mathfrak{A}$-module, then $\mathcal{A}^*$ is commutative $\mathcal{A}$-$\mathfrak{A}$-module.

Now let $\mathcal{A}$ and $\mathcal{B}$ be $\mathfrak{A}$-modules. Then a $\mathfrak{A}$-module morphism from $\mathcal{A}$ to $\mathcal{B}$ is a norm-continuous map $T\colon\mathcal{A}\longrightarrow\mathcal{B}$ with $T(a\pm b) = T(a)\pm T(b)$ which is multiplicative, that is

$$T(\alpha{\cdot}a) = \alpha{\cdot}T(a)\,, \quad T(a{\cdot}\alpha) = T(a){\cdot}\alpha\,, \quad T(ab) = T(a)T(b)\,, \qquad (a, b\in\mathcal{A}, \alpha\in\mathfrak{A})\,.$$

We denote by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$, the space of all such morphisms and denote $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},\mathcal{A})$ by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Suppose that φ and ψ in $\mathrm{Hom}_{\mathfrak{A}}(A)$. A bounded map $D\colon\mathcal{A}\longrightarrow X$ is called a *module* (φ,ψ)*-derivation* if

$$D(a\pm b) = D(a)\pm D(b)\,, \quad D(ab) = D(a)\cdot\varphi(b) + \psi(a)\cdot D(b) \qquad (a, b\in\mathcal{A})\,,$$

and

$$D(\alpha\cdot a) = \alpha\cdot D(a)\,, \quad D(a\cdot\alpha) = D(a)\cdot\alpha \qquad (a\in\mathcal{A}, \alpha\in\mathfrak{A})\,.$$

Note that $D\colon\mathcal{A}\longrightarrow X$ is bounded if there exist $M>0$ such that $\|D(a)\|\le M\|a\|$ for all $a\in\mathcal{A}$. Although D is not necessarily linear, but still its boundedness implies its norm continuity (since D preserves subtraction). There is a similar justification for $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$. When X is commutative $\mathcal{A}$-$\mathfrak{A}$-module, each $x\in X$ defines a module (φ,ψ)-derivation $D^x_{(\varphi,\psi)}(a) = x\cdot\varphi(a) - \psi(a)\cdot x$ on $\mathcal{A}$. These are called *module* (φ,ψ)*-inner derivations.*

Definition 2.1. Let φ and ψ be in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. The Banach algebra $\mathcal{A}$ is called module (φ,ψ)-amenable (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module (φ,ψ)-derivation $D:\mathcal{A}\longrightarrow X^*$ is (φ,ψ)-inner.

We use the notations $Z^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$ for the space of all module (φ,ψ)-derivations $D\colon\mathcal{A}\longrightarrow X^*$, $B^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$ for those which are inner (φ,ψ)-derivations, and $H^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$ for the quotient space which we call the first relative (to $\mathfrak{A}$) (φ,ψ)-cohomology group of $\mathcal{A}$ with coefficients in X^*. Hence $\mathcal{A}$ is module (φ,ψ)-amenable if and only if $H^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*) = \{0\}$ for all commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X. We note that if φ and ψ are identity maps, then module (φ,ψ)-amenability is the same as module amenability (see [1]).

From [1, Proposition 2.1] we see that (φ,ψ)-amenability of $\mathcal{A}$ implies its module (φ,ψ)-amenability if $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. Therefore (φ,ψ)-amenability is stronger than module (φ,ψ)-amenability.

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra and $\psi,\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If $\mathcal{A}$ is a module (φ,ψ)-amenable, then $\mathcal{A}$ is module $(\lambda\circ\varphi, \mu\circ\psi)$-amenable, for any λ and μ in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.*

Proof. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D\colon \mathcal{A} \to X^*$ be a module $(\lambda\circ\varphi, \mu\circ\psi)$-derivation. We consider another $\mathcal{A}$-module structure on X via

$$a \bullet x = \lambda(a)\cdot x\,, \quad x \bullet a = x\cdot\mu(a) \qquad (a\in\mathcal{A}, x\in X)\,.$$

It is easy to check that X with this product is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. We have

$$D(ab) = D(a)\cdot(\lambda\circ\varphi)(b) + (\mu\circ\psi)(a)\cdot D(b) = D(a)\bullet\varphi(b) + \psi(a)\bullet D(b)\,, \qquad (a,b\in\mathcal{A})\,.$$

Thus D is a module (φ,ψ)-derivation, and so, there exists $f\in X^*$ such that $D(a) = f\bullet\varphi(a) - \psi(a)\bullet f$. Therefore $D(a) = f\cdot(\lambda\circ\varphi)(a) + (\mu\circ\psi)(a)\cdot f$. $\square$

Corollary 2.3. *If $\mathcal{A}$ is module amenable (as an $\mathfrak{A}$-module), then $\mathcal{A}$ is module (φ,ψ)-amenable, for each φ and ψ in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.*

In the following proposition we show that the converse of Corollary 2.3 in a special case.

Proposition 2.4. *Let $\mathcal{A}$ be an Banach $\mathfrak{A}$-module and $\varphi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If φ is an epimorphism and $\mathcal{A}$ is module (φ,φ)-amenable, then $\mathcal{A}$ is module amenable.*

Proof. Assume that X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D\colon\mathcal{A}\to X^*$ is a module derivation. Obviously $d = D\circ\varphi$ is a module (φ,φ)-derivation and so, by module (φ,φ)-amenability of $\mathcal{A}$ there exists $f\in X^*$ such that $d(a) = f\cdot\varphi(a) - \varphi(a)\cdot f$ for all $a\in\mathcal{A}$. Let $b\in\mathcal{A}$, then there exist $a\in\mathcal{A}$ such that $\varphi(a) = b$. Hence $D(b) = D(\varphi(a)) = d(a) = f\cdot\varphi(a) - \varphi(a)\cdot f = f\cdot b - b\cdot f$. Therefore D is a module inner derivation. $\square$

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules and $\varphi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$, $\psi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{B})$. If there is λ in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$ such that $\lambda\circ\varphi = \psi\circ\lambda$ and range of λ is a dense subset of $\mathcal{B}$, then module (φ,φ)-amenability of $\mathcal{A}$ implies module (ψ,ψ)-amenability of $\mathcal{B}$.*

Proof. Let X be a commutative Banach $\mathcal{B}$-$\mathfrak{A}$-module and $D\colon\mathcal{B}\to X^*$ be a module (ψ,ψ)-derivation. X can be considered as Banach $\mathcal{A}$-module by the following actions

$$a * x = \lambda(a)\cdot x\,, \quad x * a = x\cdot\lambda(a) \qquad (a\in\mathcal{A}, x\in X)\,.$$

Then X with this product is a Banach $\mathcal{A}$-$\mathfrak{A}$-module, so $\overline{D} = D \circ \lambda \colon \mathcal{A} \to X^*$ is (φ,φ)-derivation because

$$\begin{aligned}\overline{D}(ab) = D\big(\lambda(a)\lambda(b)\big) &= D(a) \cdot \psi\big(\lambda(b)\big) + \psi\big(\lambda(a)\big) \cdot D\big(\lambda(b)\big)\\ &= D\big(\lambda(a)\big) \cdot \lambda\big(\varphi(b)\big) + \lambda\big(\varphi(a)\big) \cdot D\big(\lambda(b)\big)\\ &= \overline{D}(a) * \varphi(b) + \varphi(a) * \overline{D}(b)\,,\end{aligned}$$

for all a, b in $\mathcal{A}$. Due to module (φ,φ)-amenability of $\mathcal{A}$, there exist $f \in X^*$ such that $\overline{D}(a) = f * \varphi(a) - \varphi(a) * f$. Thus $D(\lambda(a)) = f \cdot \psi(\lambda(a)) + \psi(\lambda(a)) \cdot f$. By density of range of λ and continuity of D, D is inner. □

By using the Proposition 2.5, if φ and ψ are identity map and $\lambda(\mathcal{A})$ is dense in $\mathcal{B}$, then module amenability of $\mathcal{A}$ implies module amenability $\mathcal{B}$. Therefore Proposition 2.5 could be considered as a generalization of [1, Proposition 2.5].

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with compatible actions and J be the closed ideal of $\mathcal{A}$ generated by elements $(\alpha \cdot a)b - a(b \cdot \alpha)$ for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, then the quotient Banach algebra $\mathcal{A}/J$ is Banach $\mathfrak{A}$-module with compatible actions. Suppose that $\varphi, \psi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that $\varphi(J) \subseteq J, \psi(J) \subseteq J$. Then one can define maps $\widehat{\varphi}$, $\widehat{\psi} \colon \mathcal{A}/J \to \mathcal{A}/J$ by $\widehat{\varphi}(a+J) = \varphi(a) + J$ and $\widehat{\psi}(a+J) = \psi(a) + J$.

We say that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ if there is a bounded net $\{\zeta_j\}$ in $\mathfrak{A}$ such that $\|\zeta_j \cdot a - a\| \to 0$ and $\|a \cdot \zeta_j - a\| \to 0$, for each $a \in \mathcal{A}$.

Proposition 2.6. *Let φ, ψ be in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$, then $(\widehat{\varphi},\widehat{\psi})$-amenability of $\mathcal{A}/J$ implies module (φ,ψ)-amenability $\mathcal{A}$.*

Proof. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \to X^*$ be a module (φ,ψ)-derivation. We can show that $J \cdot X = X \cdot J = 0$, so X is a Banach $\mathcal{A}/J$-module with module actions

$$(a+J) \cdot x := a \cdot x\,, \quad x \cdot (a+J) := x \cdot a \qquad (x \in X, a \in \mathcal{A})\,.$$

Consider $\widehat{D} \colon \mathcal{A}/J \to X$, defined by $\widehat{D}(a+J) = D(a)$ for all $a \in \mathcal{A}$. $\widehat{D}$ is well defined because

$$\begin{aligned}D(\alpha \cdot ab - ab \cdot \alpha) &= \alpha \cdot D(ab) - D(ab) \cdot \alpha\\ &= \alpha \cdot \big(D(a) \cdot b + a \cdot D(b)\big) - \big(D(a) \cdot b + a \cdot D(b)\big) \cdot \alpha\\ &= \alpha \cdot \big(D(a) \cdot b\big) - \big(D(a) \cdot b\big) \cdot \alpha\\ &\quad + \alpha \cdot \big(a \cdot D(b)\big) - \big(a \cdot D(b)\big) \cdot \alpha = 0\,.\end{aligned}$$

Now for each a, b in $\mathcal{A}$ we have

$$\begin{aligned}\widehat{D}(ab+J) = D(ab) &= D(a) \cdot \varphi(b) + \psi(a) \cdot D(b)\\ &= \widehat{D}(a+J) \cdot \big(\varphi(b) + J\big) + \big(\psi(a) + J\big) \cdot \widehat{D}(b+J)\\ &= \widehat{D}(a+J) \cdot \widehat{\varphi}(b+J) + \widehat{\psi}(a+J) \cdot \widehat{D}(b+J)\,.\end{aligned}$$

Since $\mathfrak{A}$ has the bounded approximate identity (ζ_i) for $\mathcal{A}$, then for each $\rho \in \mathbb{C}$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}\widehat{\varphi}(\rho a + J) = \varphi(\rho a) + J &= \lim_i \varphi(\rho a \cdot \zeta_i) + J = \lim_i \varphi(a \cdot \rho\zeta_i) + J \\ &= \lim_i \rho\varphi(a \cdot \zeta_i) + J = \rho\varphi(a) + J = \rho\widehat{\varphi}(a + J)\,.\end{aligned}$$

Now, it follows from the proof of [1, Proposition 2.1] that $\widehat{D}$ is $\mathbb{C}$-linear, and so it is $(\widehat{\varphi}, \widehat{\psi})$-inner. Hence there exist $f \in X^*$ such that

$$D(a) = \widehat{D}(a + J) = f \cdot \widehat{\varphi}(a + J) - \widehat{\psi}(a + J) \cdot f = f \cdot \varphi(a) - \psi(a) \cdot f\,.$$

Therefore D is a module (φ, ψ)-inner. □

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$, where f is a continuous linear functional on $\mathfrak{A}$.

Proposition 2.7. *Let φ, ψ be as above and $\mathcal{A}$ be module (φ, ψ)-amenable as an $\mathfrak{A}$-module with trivial left action. If $\mathcal{A}/J$ has an identity, then $\mathcal{A}/J$ is $(\widehat{\varphi}, \widehat{\psi})$-amenable.*

Proof. Without loss of generality we assume that X is an unital $\mathcal{A}/J$-module and $\widehat{D}\colon \mathcal{A}/J \to X^*$ be an $(\widehat{\varphi}, \widehat{\psi})$-derivation. Then X is a $\mathcal{A}$-module via

$$a \cdot x := (a + J) \cdot x, \quad x \cdot a := x \cdot (a + J) \qquad (x \in X, a \in \mathcal{A})\,,$$

and also X is $\mathfrak{A}$-module with trivial actions, that is $\alpha \cdot x = x \cdot \alpha = f(\alpha)x$ for all $x \in \mathcal{X}$ and $\alpha \in \mathfrak{A}$. Since $f(\alpha)a - a \cdot \alpha \in J$ [2, Lemma 3.1], we have $f(\alpha)a + J = a \cdot \alpha + J$ for all $\alpha \in \mathfrak{A}$. Hence the actions of $\mathfrak{A}$ and $\mathcal{A}$ on X are compatible. Therefore X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. Consider $D = \widehat{D} \circ \pi\colon \mathcal{A} \to X^*$ where $\pi\colon \mathcal{A} \to \mathcal{A}/J$ is the natural $\mathfrak{A}$-module. Obviously $D(a \pm b) = D(a) \pm D(b)$ for all $a, b \in \mathcal{A}$, and $\mathcal{A}/J$ has an identity, so by [2, Theorem 3.2], $D(a \cdot \alpha) = D(a) \cdot \alpha, D(\alpha \cdot a) = D(a) = \alpha \cdot D(a)$ for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Also we have

$$\begin{aligned}D(ab) = \widehat{D}\big(\pi(ab)\big) &= \widehat{D}(a + J) \cdot \widehat{\varphi}(b + J) + \widehat{\psi}(a + J) \cdot \widehat{D}(b + J) \\ &= D(a) \cdot (\varphi(b) + J) + (\psi(a) + J) \cdot D(b) \\ &= D(a) \cdot \varphi(b) + \psi(a) \cdot D(b)\,,\end{aligned}$$

for all a, $b \in \mathcal{A}$, that is $D \in Z^{\mathfrak{A}}_{(\varphi,\psi)}(\mathcal{A}, X^*)$. Hence there exist $f \in X^*$ such that $D(a) = f \cdot \varphi(a) - \psi(a) \cdot f$ for all $a \in \mathcal{A}$. Therefore $\widehat{D}$ is $(\widehat{\varphi}, \widehat{\psi})$-inner. □

3. Module (φ, ψ)-amenability of semigroup algebras

Recall that a discrete semigroup S is called an *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E. We start this section with a definition of inverse semigroups.

Throughout this section, S is an inverse semigroup with set of idempotents E. There is a natural order on E, defined by

$$e \leq d \Longleftrightarrow ed = e \qquad (e, d \in E)\,.$$

It is easy to see that E is indeed a commutative subsemigroup of S (see [6, Theorem V.1.2]). In particular $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible canonical actions [1]. However, for technical reasons, here we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e\cdot\delta_s=\delta_s\,,\quad \delta_s\cdot\delta_e=\delta_{se}=\delta_s*\delta_e \qquad (s\in S, e\in E)\,.$$

In this case, J is the closed linear span of $\{\delta_{\mathrm{set}}-\delta_{st}\ s,t\in S, e\in E\}$. We consider the following equivalence relation on S

$$s\approx t\Longleftrightarrow \delta_s-\delta_t\in J \qquad (s,t\in S)\,.$$

Recall that E is called *directed upward* if for every e, $f\in E$ there exist $g\in E$ such that $eg=e$ and $fg=f$. This is precisely the assertion that S satisfies the condition D_1 of Duncan and Namioka [5]. It is shown in [2] that if E is directed upward, then the quotient $S/\approx$ is a discrete group. Unital inverse semigroups have a directed upward set of idempotents. Also if E is totally ordered, it is clearly directed upward. The examples of the latter include the bicyclic semigroup and the semigroup of natural numbers with *max* operation. On the other hand, the set of idempotents of the free inverse semigroup on two generators is not directed upward. Indeed, if the generators are a and b, there is no idempotent which is bigger than both aa^* and bb^*.

Consider the quotient map $\pi\colon S\to S/\approx$, $s\mapsto[s]$. As in [11, Theorem 3.3], we may observe that $\ell^1(S)/J\cong\ell^1(S/\approx)$. Now, if $\varphi\in\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, by using the discussion before Proposition 2.6 we define $\widehat{\varphi}$ in $\mathrm{Hom}(\ell^1(S/\approx))$ by $\widehat{\varphi}(\delta_s+J)=\varphi(\delta_s)+J$ and extended by linearity (see also the proof of Proposition 2.6). The following result is the main aim of this section. In fact we show that there exists a one-to-one correspondence between $H^{\ell^1(E)}_{(\varphi,\psi)}(\ell^1(S),X^*)$ and $H_{(\widehat{\varphi},\widehat{\psi})}(\ell^1(S/\approx),X^*)$, where φ and ψ are in $\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$ and X is a commutative Banach $\ell^1(S)$-$\ell^1(E)$-module.

Theorem 3.1. *Let S be an inverse semigroup with a directed upward set of idempotents E. If $\ell^1(S)$ is an $\ell^1(E)$-module with trivial left action and φ, ψ are in* $\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, *then*

$$H^{\ell^1(E)}_{(\varphi,\psi)}\big(\ell^1(S),X^*\big)\cong H_{(\widehat{\varphi},\widehat{\psi})}\big(\ell^1(S/\approx),X^*\big)\,.$$

Proof. Since $S/\approx$ is a discrete group, $\ell^1(S)/J\cong\ell^1(S/\approx)$ has an identity. Also S is an inverse semigroup with a directed upward set of idempotents E, so E satisfies condition D_1 of Duncan and Namioka. Hence $\ell^1(E)$ has a bounded approximate identity [5]. Now, if (ζ_j) is a bounded approximate identity of $\ell^1(E)$, then $\zeta_j*\delta_s=\zeta_j*\delta_{ss^*s}=(\zeta_j\cdot\delta_{ss^*})*\delta_s\to\delta_s$, and similarly for the right side multiplication. Therefore $\ell^1(E)$ has a bounded approximate identity for $\ell^1(S)$. It follows from Proposition 2.6 and Proposition 2.7 that the map

$$Z^{\ell^1(E)}_{(\varphi,\psi)}\big(\ell^1(S),X^*\big)\to Z_{(\widehat{\varphi},\widehat{\psi})}\big(\ell^1(S/\approx),X^*\big)\,,\quad (D\mapsto\widehat{D})\,.$$

induce an isomorphism between the quotient spaces $H^{\ell^1(E)}_{(\varphi,\psi)}(\ell^1(S), X^*)$ and $H_{(\widehat{\varphi},\widehat{\psi})}(\ell^1(S/\approx), X^*)$. $\square$

The following corollary is a result of Theorem 3.1 in which φ and ψ are identity maps.

Corollary 3.2. *With the hypothesis of above theorem, S is amenable if and only if $S/\approx$ is amenable.*

Proof. The discrete group $S/\approx$ is amenable if and only if $\ell^1(S/\approx)$ is amenable by Johnson's theorem [7]. We conclude from Theorem 3.2, $\ell^1(S/\approx)$ is amenable if and only if $\ell^1(S)$ is module amenable (as an $\ell^1(E)$-module). Now the result follows from [1, Theorem 3.1]. $\square$

Let $\sim$ be the congruence relation on S where $s \sim t$ if and only if there exist $e \in E$ such that $se = te$. The quotient semigroup $G(S) \equiv S/\sim$ is then a group. It is indeed the maximal group homomorphic image of S [9]. Also the inverse semigroup S is amenable if and only if the discrete group $G(S)$ is amenable [5, Theorem 1]. Now consider epimorphisms $P\colon S \to S/\approx$; $s \mapsto [s]$ and $Q\colon S \to G(S)$; $s \mapsto [[s]]$, then maximality of $G(S)$ implies that there is a group homomorphism $R\colon S/\approx \to G(S)$ such that $R \circ P = Q$. Clearly R is onto. To see it is one-to-one, let $[[s]] = [[t]]$, then there is $e \in E$ such that $se = te$. Since $\delta_{sa} - \delta_{sea} \in J$ for all s, $a \in S$ and $e \in E$, $\delta_s - \delta_{se} = \delta_{ss^*s} - \delta_{ss^*se} = \delta_{ss^*s} - \delta_{ses^*s} \in J$. Hence $\delta_s - \delta_t = \delta_s - \delta_{se} + \delta_{se} - \delta_t = (\delta_s - \delta_{se}) - (\delta_t - \delta_{te}) \in J$, so $[s] = [t]$. Therefore R is a group isomorphism. Now if $S/\approx$ is amenable, then by Corollary 3.2 and [1, Theorem 3.1], $\ell^1(S)$ is $\ell^1(E)$-module amenable. Therefore $\ell^1(S)$ is module (φ,ψ)-amenable, for each $\varphi, \psi \in \mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$ by Corollary 2.3. We close this section by some examples.

Example 3.3. (i) Let $(\mathbb{N}, \vee)$ be the commutative semigroup of positive integers with maximum operation $m \vee n = \max(m, n)$, then each element of $\mathbb{N}$ is an idempotent, hence $\mathbb{N}/\approx$ is the trivial group with one element. So $\mathbb{N}/\approx$ is amenable. Therefore $(\mathbb{N}, \vee)$ is amenable by the above corollary.

(ii) Let $\mathcal{C}$ be the bicyclic inverse semigroup generated by a and b, that is

$$\mathcal{C} = \{a^m b^n : m, n \geq 0\}, \quad (a^m b^n)^* = a^n b^m .$$

The set of idempotents of $\mathcal{C}$ is $E_{\mathcal{C}} = \{a^n b^n : n = 0, 1, \ldots\}$ which is totally ordered with the following order

$$a^n b^n \leq a^m b^m \to m \leq n .$$

It have been showed in [2] that $\mathcal{C}/\approx$ is isomorphic to integer numbers $\mathbb{Z}$, hence it is amenable. Therefore $\ell^1(\mathcal{C})$ is module (φ,ψ)-amenable for all $\varphi, \psi \in \mathrm{Hom}_{\ell^1(E_{\mathcal{C}})}\left(\ell^1(\mathcal{C})\right)$.

(iii) Let S be an amenable E-unitary inverse semigroup with infinite number of idempotents (see [6] and [10]). Then $\ell^1(S)$ is module amenable [1]. So if φ and ψ are in $\mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$, then $\ell^1(S)$ is module (φ,ψ)-amenable, but not (φ,ψ)-amenable even when φ and ψ are identity maps [5].

(iv) If S is a *Brandt semigroup* of an amenable group over an infinite index set (see [5] and [10]), then $\ell^1(S)$ is module amenable [1]. Therefore we conclude by

Corollary 2.3 that $\ell^1(S)$ is module (φ,ψ)-amenable for all $\varphi,\psi \in \mathrm{Hom}_{\ell^1(E)}(\ell^1(S))$ without having a bounded approximate identity [5].

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