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ON a-KASCH SPACES

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Abstract. If X is a Tychonoff space, $C(X)$ its ring of real-valued continuous functions. In this paper, we study non-essential ideals in $C(X)$. Let a be a infinite cardinal, then X is called a-Kasch (resp. $\bar{a}$-Kasch) space if given any ideal (resp. z-ideal) I with $\operatorname{gen}(I) < a$ then I is a non-essential ideal. We show that X is an $\aleph_0$-Kasch space if and only if X is an almost P-space and X is an $\aleph_1$-Kasch space if and only if X is a pseudocompact and almost P-space. Let $C_F(X)$ denote the socle of $C(X)$. For a topological space X with only a finite number of isolated points, we show that X is an a-Kasch space if and only if $\frac{C(X)}{C_F(X)}$ is an a-Kasch ring.

1. Introduction

Throughout this paper, $C(X)$ will denote the ring of real valued continuous functions defined on a completely regular space X and $C^*(X)$ will be the subring of bounded functions. Let us first recall some general notation from [8]. For $f \in C(X)$, let $Z(f) = \{x \in X : f(x) = 0\}$ be the **zero set** of f and $\operatorname{Coz}(f) = X \setminus Z(f)$ be its **cozero set**. Whenever S is a subset in $C(X)$, then $Z[S] = \{Z(f) : f \in S\}$ and $\bigcap Z[S] = \bigcap_{f\in S} Z(f)$. An ideal I of $C(X)$ is called **fixed ideal** if $\bigcap Z[I]$ is nonempty, otherwise I is called a **free ideal**. A subset Y of X is said to be **C-embedded** (resp. **C^*-embedded)** in X if the map that sends each element of $C(X)$ (resp. $C^*(X)$) to its restriction to Y is onto $C(Y)$ ($C^*(Y)$). An ideal I of $C(X)$ is called a **z-ideal** (resp. **z°-ideal)** if $Z(f) = Z(g)$ (resp. $\operatorname{int} Z(f) = \operatorname{int} Z(g)$) and $g \in I$ imply that $f \in I$. X is called **extremally** (resp. **basically) disconnected** if each open (resp. cozero) set has an open closure and it is called **P-space** if each finitely generated ideal of $C(X)$ is a direct summand, and so it is called an **almost P-space** if every non-empty G_δ has a non-empty interior, see [12]. For undefined terms and notations, see [8], and so [7].

Throughout this paper, R is a commutative ring with unit. For each subset S of R, the annihilator of S denoted by $\operatorname{Ann}(S)$ is defined by $\operatorname{Ann}(S) = \{r \in R : rs = 0 \text{ for all } s \in S\}$.

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An ideal I of R is said to be a**-generated**, where a is a cardinal number, if it admit a generating set A such that $|A| \leq a$. The least element in the set of cardinal numbers of all generating set of I is denoted by $\mathrm{gen}\,(I)$.

A ring R is **Kasch** if every maximal ideal of R has a non zero annihilator; equivalently, every simple R-module embeds in R. In order to get more information, see [11].

An ideal I of R is called **essential** in R if $I \cap J \neq (0)$ holds for every non-zero ideal J of R. Hence every reduce commutative ring is Kasch ring provided that every non-zero ideal is a non-essential ideal.

In [1], F. Azarpanah study essential ideal in $C(X)$, he proved that for completely regular space X, the following statements are equivalent:

(1) X is finite.

(2) Every ideal in $C(X)$ is not essential ideal.

It is clear that $C(X)$ is Kasch ring if and only if every ideal in $C(X)$ is not essential ideal.

Now it is natural to ask, under which condition on X, every ideal I in $C(X)$ with $\mathrm{gen}\,(I) < a$ is not essential ideal. By this question we introduce a-Kasch rings and study their properties.

2. Preliminaries

In this section we review some propositions which will be used in the later sections. Also we prove that If every fixed ideal is a countable generated ideal then X is a countable discrete space.

It is not hard to prove that, an ideal B in a reduce commutative ring is an essential ideal if and only if $\mathrm{Ann}\,(B) = 0$. We cite the following results which will be frequently referred to in the sequel. For the proof of the following results see [1] and [12].

Proposition 2.1. *If E is a nonzero ideal in $C(X)$, then the following statements are equivalent:*

(1) *E is an essential ideal in $C(X)$.*

(2) $\mathrm{Ann}\,(E) = 0$.

(3) $\mathrm{Int} \bigcap Z[E] = \emptyset$.

Proposition 2.2. *X is an almost P-space if and only if every non-empty zero set has non-empty interior.*

Also, for the proof of the following results see [8].

Proposition 2.3. *For every f, $g \in C(X)$,*

(1) *If $Z(f)$ is a neighborhood of $Z(g)$, then f is a multiple of g, that is, $f = hg$ for some $h \in C(X)$.*

(2) *If $|f| \leq |g|^r$ for some real $r > 1$, then f is a multiple of g.*

Proposition 2.4. *Let X be a finite space.*

(1) *f is a multiple of g if and only if $Z(g) \subseteq Z(f)$*

(2) *Every ideal is a z-ideal.*

(3) *Every ideal is principal, and in fact, is generated by an idempotent.*

Proposition 2.5. *A point p of X is isolated if and only if the ideal M_p (resp. M_p^*) is principal.*

In the theory of rings, many structure results were obtained with the help of minimal ideals, and the socle of a ring seems to be most efficient. The intersection of all essential ideals in any commutative ring R, or the sum of all minimal ideals of R is the **socle** of R, see [9]. Let $C_F(X)$ denote the socle of $C(X)$. The socle $C_F(X)$ of $C(X)$ was first characterized via the following proposition in [10].

Proposition 2.6. *The socle $C_F(X)$ of $C(X)$ is a z-ideal, consisting of all functions that vanish everywhere except on a finite number of points of X.*

Theorem 2.7. *The following statements are equivalent:*

(1) *$C(X)$ is a Kasch ring.*

(2) *Every fixed ideal in $C(X)$ is finitely generated ideal.*

(3) *X is finite discrete space.*

(4) *Every z-ideal in $C(X)$ is non-essential ideal.*

Proof. (1) $\Rightarrow$ (2) By the hypothesis every maximal ideal of M in $C(X)$ is non-essential ideal and since $|\bigcap Z[M]| \leq 1$, we can then conclude from of Proposition 2.1 that $|\operatorname{int} \bigcap Z[M]| = 1$. It follows that there exists an isolated point $p \in X$ such that $M = M_p$. So that X is a pseudocompact discrete space, see [8] Theorem 5.8(b), we infer that X is a finite discrete space, and by Proposition 2.4, this completes the proof.

(2) $\Rightarrow$ (3) For every $p \in X$, M_p is a finitely generated ideal and since M_p is a z-ideal, we conclude that M_p is principal ideal, see [13]. By Proposition 2.5, p is an isolated point, i.e., X is a discrete space. Let $x_0 \in X$ and for every $x_0 \neq p \in X$ define $f_p \in C(X)$ by $f_p(p) = 1$ and $f_p(X \setminus \{p\}) = 0$. Now put $I = \sum_{x_0 \neq p \in X} f_p C(X)$ then clearly I is a fixed ideal and in view of our hypothesis it is finitely generated ideal, and since X is discrete, we conclude that there exists idempotent $e \in C(X)$ such that $I = eC(X)$, i.e., $Z(e) = \bigcap Z[I] = \{x_0\}$. We also note that $e \in I$, so that there exists $g_1, \ldots, g_n \in C(X)$ and $f_{p_1}, \ldots, f_{p_n} \in I$ such that $e = g_1 f_{p_1} + \cdots + g_n f_{p_n}$. This implies that $X \setminus \{p_1, \ldots, p_n\} = \bigcap_{i=1}^n Z(f_{p_i}) \subset Z(e) = \{x_0\}$. This means that $X = \{p_1, \ldots, p_n, x_0\}$ is a finite discrete space.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) If I is an ideal of $C(X)$, then there exists a maximal ideal of M in $C(X)$ such that $I \subseteq M$, and in view of our hypothesis M is a non-essential ideal, therefore $\emptyset \neq \operatorname{int} \bigcap Z[M] \subseteq \operatorname{int} \bigcap Z[I]$, i.e., I is a non-essential ideal of $C(X)$ (by Proposition 2.1). $\square$

In view of statement (2) of Theorem 2.7, it is natural to ask that what is the space X if every fixed ideal in $C(X)$ is a countable generated ideal. We prove that X is a countable discrete space in this case.

Lemma 2.8. *For every cardinal number $a \geq \aleph_0$, if X is a discrete space and every fixed ideal in $C(X)$ is* $\operatorname{gen}(I) \leq a$, *then* $|X| \leq a$.

Proof. Let us, in contrary, assume that $|X| > a$. Let $x_0 \in X$. For every $x_0 \neq p \in X$ define $f_p \in C(X)$ by $f_p(p) = 1$ and $f_p(X \setminus \{p\}) = 0$. Now we put $I = \sum_{x_0 \neq p \in X} f_p C(X)$ then clearly I is a fixed ideal and in view of our hypothesis there exists generating set of S for I such that $|S| \leq a$. We also note that $\bigcup_{f \in S} \operatorname{Coz}(f) = X \setminus \{x_0\}$ and since $|X \setminus \{x_0\}| = |X| > a \geq \aleph_0$, we conclude that there exist $f \in S$ such that $|\operatorname{Coz}(f)| = |X|$, so that there exist $g_1, \ldots, g_n \in C(X)$ and $f_{p_1}, \ldots, f_{p_n} \in I$ such that $f = g_1 f_{p_1} + \cdots + g_n f_{p_n}$. This implies that $X \setminus \{p_1, \ldots, p_n\} = \bigcap_{i=1}^{n} Z(f_{p_i}) \subseteq Z(f)$. This means that $\operatorname{Coz}(f) \subseteq \{p_1, \ldots, p_n\}$ is a finite set, which is a contradiction. □

Proposition 2.9. *If every fixed ideal is a countable generated ideal then X is a countable discrete space.*

Proof. By Lemma 2.8, it suffices to show that X is a discrete space. For every $p \in X$, M_p is a fixed maximal ideal. By our hypothesis there exist $f_1, f_2, \ldots \in M$ such that $M_p = (f_1, f_2, \ldots)$. Define

$$g = \sum_{i=1}^{\infty} \frac{f_i^{2/3}}{2^i(1 + f_i^{2/3})}.$$

Because of uniform convergence, $g \in C(X)$. Clearly $g \in M_p$ and since $|f_i| \leq (2^i(1 + f_i^{2/3})g)^{2/3}$, we can then conclude from of Proposition 2.3 that f_i is a multiple g, for all i. So that $M_p = gC(X)$ and in view of Proposition 2.5, p is an isolated point, i.e., X is a discrete space. □

3. When is X an a-Kasch space

In this section the concept of an a-Kasch space is introduced and some fundamental properties are considered.

Definition 3.1. Let a be a infinite cardinal, then a ring R is said to be an **a-Kasch ring** if given any proper ideal I with $\operatorname{gen}(I) < a$, then I is a non-essential ideal. X is called **a-Kasch space** if $C(X)$ is a-Kasch ring and it is called **$\bar{a}$-Kasch space** if given any proper z-ideal I with $\operatorname{gen}(I) < a$, then I is a non-essential ideal.

Theorem 3.2. *X is an almost P-space if and only if X is an $\aleph_0$-Kasch space.*

Proof. Necessity. Let $I = (f_1, \ldots, f_n)$ be a proper ideal of $C(X)$. Hence $\operatorname{int} \cap Z[I] = \operatorname{int} Z(f_1^2 + \cdots + f_n^2) \neq \emptyset$ and now by Proposition 2.1, we infer that I is a non-essential ideal.

Sufficiently. In view of Proposition 2.2, it suffices to show that for every $f \in C(X)$, if $Z(f) \neq \emptyset$ then $\operatorname{int} Z(f) \neq \emptyset$, and since $I = fC(X)$ is non-essential ideal, by Proposition 2.1, we infer that $\operatorname{int} Z(f) \neq \emptyset$. □

Theorem 3.3. *X is an $\aleph_1$-Kasch space if and only if X is a pseudocompact and almost P-space.*

Proof. Necessity. By Theorem 3.2, X is an almost P-space, then by Theorem 5.8(b) and 5.14 in [8], it suffices to show that if M is maximal ideal of $C(X)$, then $Z[M]$ has the countable intersection property. Let $f_1, f_2, \dots$ are belong to M and put $I = (f_1, f_2, \dots)$ then $\text{gen}(I) < \aleph_1$ and I is a proper ideal of $C(X)$. Now by our hypothesis we infer that $\emptyset \neq \text{int} \bigcap_{i=1}^{\infty} Z(f_i) \subseteq \bigcap_{i=1}^{\infty} Z(f_i)$.

Sufficiently. By our hypothesis and 6.6(b) in [8], $C(X) = C^*(X) \cong C(\beta X)$ and βX is an almost P-space. So if $I = (f_1, f_2, \ldots)$ is an ideal of $C(\beta X)$, then there exists $g \in C(\beta X)$ such that $\bigcap Z[I] = \bigcap_{i=1}^{\infty} Z(f_i) = Z(g) \supseteq \text{int } Z(g) \neq \emptyset$, see 1.14(a) and Theorem 4.11 in [8], i.e., I is a non-essential ideal of $C(\beta X)$. Thus βX is an $\aleph_1$-Kasch space and since $C(X) \cong C(\beta X)$, we conclude that X is an $\aleph_1$-Kasch space. □

Corollary 3.4. *βX is an almost P-space if and only if X is a pseudocompact almost P-space.*

The fact that $C(X) \cong C(\upsilon X)$ (see 8.8(a) in [8]) implies that X is a-Kasch space if and only if υX is a-Kasch space. Clearly $\mathbb{N}$ is $\aleph_0$-Kasch, but $\beta\mathbb{N}$ is not $\aleph_0$-Kasch, see [12]. However, we have the following.

Corollary 3.5. *For cardinal number $a \geq \aleph_1$, X is an a-Kasch space if and only if βX is an a-Kasch space.*

Proposition 3.6. *The following statements are equivalent:*

(1) *X is a-Kasch (resp. $\overline{a}$-Kasch) space.*

(2) *Every ideal (resp. z-ideal) of I in $C(X)$ with* $\text{gen}\,(I) < a$ *is in a non-essential principal ideal.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal (a z-ideal) of $C(X)$ with $\text{gen}\,(I) < a$. Hence I is a non-essential ideal of $C(X)$ and now by Proposition 2.1, $\text{int} \bigcap Z[I] \neq \emptyset$. Then by 3.2(b) in [8], there exists $g \in C(X)$ such that $\emptyset \neq \text{int } Z(g) \subseteq Z(g) \subseteq \text{int} \bigcap Z[I] \subseteq Z(f)$, for all $f \in I$. Hence by Propositions 2.1 and 2.3, $gC(X)$ is a non-essential ideal which containing of I.

(2) $\Rightarrow$ (1) Let I be an ideal (resp. a z-ideal) of $C(X)$ with $\text{gen}\,(I) < a$. Thus there exists $g \in C(X)$ such that $I \subseteq gC(X)$ and $gC(X)$ is a non-essential ideal, i.e., $\text{int } Z(g) \neq \emptyset$, and since $\text{int } Z(g) \subseteq \cap Z[I]$, we can then conclude from of Proposition 2.1 that I is a non-essential ideal of $C(X)$. □

Lemma 3.7. *For every non-empty G_δ set of G in X, there exists a countable generated z°-ideal of I in $C(X)$ such that $\bigcap Z[I] \subseteq G$.*

Proof. Let $x \in G = \bigcap_{i=1}^{\infty} G_i$, where each G_i is an open subset of X. In view of 3.2(b) in [8], we define by induction a sequence of $f_n \in C(X)$ as follows,
there is a $f_1 \in C(X)$ such that $x \in \text{int } Z(f_1) \subseteq Z(f_1) \subseteq G_1$. Now suppose that there exists $f_{n-1} \in C(X)$ such that $x \in \text{int } Z(f_{n-1}) \subseteq Z(f_{n-1}) \subseteq \bigcap_{i=1}^{n-1} G_i \cap \text{int } Z(f_{n-2})$. Since $x \in \bigcap_{i=1}^{n} G_i \cap \text{int } Z(f_{n-1})$, we conclude that there is a $f_n \in C(X)$ such that $x \in \text{int } Z(f_n) \subseteq Z(f_n) \subseteq \bigcap_{i=1}^{n} G_i \cap \text{int } Z(f_{n-1})$. So that for every $n \in \mathbb{N}$, $\text{int } Z(f_n) \subseteq Z(f_n) \subseteq Z(f_{n-1})$. Hence $I = (f_1, f_2, \ldots)$ is a z°-ideal such that $\bigcap Z[I] \subseteq G$. □

Clearly, every completely regular space is $\overline{\aleph_0}$-Kasch, because every finitely generated semiprime ideal is generated by an idempotent. Hence every finite generated z-ideal is a non-essential ideal, see [8, Theorem 2.8], and also we have the following.

Theorem 3.8. *The following statements are equivalent:*

(1) *X is an $\overline{\aleph_1}$-Kasch space.*

(2) *X is an almost P-space and every countable generated z-ideal is fixed.*

(3) *Every countable generated z°-ideal is a non-essential ideal.*

Proof. (1) $\Rightarrow$ (2) It suffices to show that every non-empty G_δ set has non-empty interior. Let G be a non-empty G_δ set in X, in view of Lemma 3.7, there exists a countable generated z°-ideal of I in $C(X)$ such that $\bigcap Z[I] \subseteq G$, and since every z°-ideal is z-ideal, we can then conclude from our hypothesis that $\emptyset \neq \operatorname{int} \bigcap Z[I] \subseteq \operatorname{int} G$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) In view of Lemma 3.7 and by our hypothesis is an almost P-space. So that every z-ideal is a z°-ideal, see [4], and the proof is complete. □

4. Properties of a-Kasch spaces

Throughout this section all spaces are assumed to be infinity completely regular. A σ-compact space (i.e., it is the union of at most countably many compact) and locally compact space is $\aleph_1$-Kasch, see 8.2 and Theorem 14.17 in [8] and so [12].

In view of Theorem 3.3, the one-point compactification of an uncountable discrete space is an $\aleph_1$-Kasch, see [12]. Now it becomes of interest to ask: Under what condition a $\aleph_1$-Kasch space is a one-point compactification of a discrete space. We give a solution to this question, see Proposition 4.3. Before we proceed, we recall some definitions. The set of all cardinal numbers of the from $|\mathcal{B}|$, where $\mathcal{B}$ is a basis for X, has the smallest element, this cardinal number is called the **weight** of X and is denoted by $\mathcal{W}(X)$. The **character of a point** x in X is defined as the smallest cardinal number of the from $|\mathcal{B}(x)|$, where $\mathcal{B}(x)$ is a basis for X at the point x, this cardinal number is denoted by $\chi(x, X)$, see [7]. It is not hard to prove that $\chi(x, X) = a$ if and only if $\operatorname{gen}(O_x) = a$. So that we have the following.

Proposition 4.1. *For every a-Kasch space X and $x \in X$, $\chi(x, X) < a$ if and only if x is an isolated point in X.*

Corollary 4.2. *Suppose that X is an $\aleph_0$-Kasch space, X is a first countable if and only if it is a discrete space.*

By previous results, if X is an a-Kasch space and $a \geq \aleph_1$, then X is not a second countable space and also it is not a first countable space.

Suppose that b is a infinite cardinal number and it is a immediate successor of a. If X is a completely regular space and $|X| = a$, then X is not b-Kasch space. Otherwise we get a contradiction. Let $x_0 \in X$, by completely regularity of X, for every $y \in x \setminus \{x_0\}$ there exists $f_y \in M_{x_0}$ such that $f_y(y) \neq 0$. If $I = \sum_{y \in X \setminus \{x_0\}} f_y C(X)$, then $\operatorname{gen}(I) \leq a < b$, thus I is a non-essential ideal in

$C(X)$ and by Proposition 2.1, $\emptyset \neq \operatorname{int} \bigcap Z[I] = \operatorname{int}\{x_0\}$, i.e., x_0 is an isolated point. It follows that X is a discrete space and since $b \geq \aleph_1$, we can then conclude from of Theorem 3.3 that X is a finite space, which is impossible.

Proposition 4.3. *For an $\aleph_1$-Kasch space X, the following statements are equivalent:*

(1) *X is a one-point compactification of a discrete space.*

(2) *There exists the unique $x \in X$ such that $\chi(x, X) \geq \aleph_1$ and every neighborhood x has a compact complement.*

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) For every $y \in X \setminus \{x\} = Y$, by Proposition 4.1, y is an isolated point in X. So that Y is a discrete space. It is clear that X is a one-point compactification of Y. $\square$

By Proposition 3.2 a dense subset or an open subset of an $\aleph_0$-Kasch space is an $\aleph_0$-Kasch space (see [12]) and also we have the following.

Proposition 4.4. *Let X be a topological space.*

(1) *Every dense subset C-embedded of X is a-Kasch space if and only if X is a-Kasch space.*

(2) *The free union $\dot{\bigcup}_{s\in S} X_s$ (see [7]) is $\aleph_0$-Kasch if and only if X_s is $\aleph_0$-Kasch space for all $s \in S$.*

(3) *If f is a open continuous function from a-Kasch space X onto Y, then Y is a-Kasch space.*

(4) *A C-embedded open subset in an $\aleph_1$-Kasch space is an $\aleph_1$-Kasch space.*

Proof. (1) Let A be a dense subset C-embedded of X. Then every $f \in C(A)$ has the unique extension $\bar{f} \in C(X)$. So that the mapping $f \to \bar{f}$ is an isomorphism of $C(X)$ onto $C(A)$ and the proof is complete.

(2) Trivial.

(3) Let I be an ideal in $C(Y)$ with $\operatorname{gen}(I) < a$. There exists a generating set S for I such that $|S| < a$. Now we put $T = \{g\circ f : g \in S\}$ and $J = TC(X)$, then J is a non-essential ideal in $C(X)$, i.e., $\emptyset \neq \operatorname{int}_X \bigcap Z[J] = \operatorname{int}_X \bigcap Z[T] = A$ (by Proposition 2.1). Since f is a open continuous function, we conclude that $f(A)$ is a open subset in Y and it is clear that $\emptyset \neq f(A) \subseteq \operatorname{int}_Y \bigcap Z[I]$, so that by Proposition 2.1, I is a non-essential ideal in $C(Y)$.

(4) It is clear. $\square$

The **density** of a space X is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X, this cardinal number is denoted by $d(X)$.

Proposition 4.5. *If X is an a-Kasch space then*

(1) *$\mathcal{W}(X) \geq a$.*

(2) *$d(X) \geq a$.*

(3) *If A is nowhere dense closed in X, then $|X \setminus A| \geq a$.*

(4) *If* $C_K(X) = \{f \in C(X) : \overline{X \setminus Z(f)}$ *is a compact space*$\} = (0)$ *and* A *is compact in* X*, then* $|X \setminus A| \geq a$.

(5) *Every prime ideal of* P *in* $C(X)$ *is an isolated maximal ideal or* $\operatorname{gen}(I) \geq a$ *and* $\operatorname{Ann}(I) = (0)$.

(6) *For every* $x \in \beta X$*,* O^x *is an isolated maximal ideal or* $\operatorname{gen}(O^x) \geq a$ *and* $\operatorname{Ann}(O^x) = (0)$.

(7) *Every ideal of* I *with* $\operatorname{gen}(I) < a$*, which* $O^x \subset I$ *for some* $x \in \beta X$ *is an isolated maximal ideal and* $O^x = I$.

Proof. (1) Let $\mathcal{W}(X) < a$. Let $p \in X$, there exists $S \subseteq C(X)$ with $|S| < a$ such that $\{p\} = \bigcap Z[S]$, it follows that $\emptyset \neq \operatorname{int}\{p\} = \operatorname{int}\bigcap Z[SC(X)]$. Thus p is an isolated point, i.e., X is a discrete space. So that $\mathcal{W}(X) = |X| < a$. If $a = \aleph_0$ then X is finite space and if $a \geq \aleph_1$, in view of Theorem 3.3, X is pseudocompact, it follows that X is finite space, which is a contradiction.

(2) Let X is a discrete space and $d(X) < a$. It is clear that $\operatorname{gen}(C_F(X)) = |X| = d(X) < a$, thus $C_F(X)$ is a non-essential ideal in $C(X)$, it follows that $C_F(X)$ is a fixed ideal. But by Corollary 3.6 in [10], $C_F(X)$ is a free ideal, which is a contradiction.

If X is not a discrete space, there exists a dense proper subset of A in X. Let $x_0 \in X \setminus A$, for every $p \in A$ there exists $f_p \in C(X)$ such that $f_p(p) = 1$ and $f_p(x_0) = 0$. So that $I = \sum_{p\in A} f_pC(X) \subseteq M_{x_0}$. We claim that $F = \operatorname{int}\bigcap Z[I] = \emptyset$, for otherwise there exists $p \in F \cap A$. This means that $f_p(p) = 0$ and $f_p(p) = 1$, which is the desired contradiction. So that I is a essential ideal with $\operatorname{gen}(I) \leq d(X)$, it follows that $a < \operatorname{gen}(I) \leq d(X)$, and the proof is complete.

(3) For every $x \in X \setminus A$ there exists $f_x \in C(X)$ such that $f_x(A) = 0$ and $f_x(x) = 1$, so that $I = \sum_{x\in X\setminus A} f_xC(X)$ is an ideal with $\operatorname{gen}(I) \leq |X \setminus A|$. Since $\operatorname{int}\bigcap Z[I] = \operatorname{int} A = \emptyset$, we can then conclude from of Proposition 2.1 that I is an essential ideal. It follows that $|X \setminus A| \geq \operatorname{gen}(I) \geq a$.

(4) By previous part it suffices to show that $\operatorname{int} A = \emptyset$. Let us assume that $x \in \operatorname{int} A$ and seek a contradiction. By completely regularity X there exists $g \in C(X)$ such that $g(x) = 1$ and $g(X \setminus \operatorname{int} A) = 0$, thus $X \setminus Z(g) \subseteq \operatorname{int} A \subseteq A$, and since A is a compact space we conclude that $\overline{X \setminus Z(g)}$ is compact, i.e., $0 \neq g \in C_K(X)$, which is the desired contradiction.

(5), (6) and (7) It is clear. □

In view of proposition next if X is an a-Kasch space, where $a \geq \aleph_1$, then X is not a P-space and also it is not a basically disconnected space.

Proposition 4.6. *Let* X *is an* $\aleph_0$*-Kasch space, then the following statements are equivalent:*

(1) X *is a* P*-space.*

(2) *For every non-unit element of* f *in* $C(X)$*,* $\operatorname{Ann}(f)$ *is a principal ideal.*

(3) X *is a basically disconnected space.*

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Let $f \in C(X)$, it suffices to show that $\overline{\mathrm{Coz}\,(f)}$ is an open set. If $Z(f) = \emptyset$ or $Z(f) = X$ then $\overline{\mathrm{Coz}\,(f)}$ is an open set. Now let us assume that $X \neq Z(f) \neq \emptyset$, by our hypothesis $\mathrm{Ann}\,(f)$ is a nonzero principal z-ideal, so that $\mathrm{Ann}\,(f) = eC(X)$ for some an idempotent of e in $C(X)$, see [13]. Clearly, $\overline{\mathrm{Coz}\,(f)} \subseteq Z(e)$ therefore if $x \in Z(e) \setminus \overline{\mathrm{Coz}\,(f)}$ we get a contradiction. By completely regularity X there exists $g \in C(X)$ such that $g(x) = 1$ and $g(\overline{\mathrm{Coz}\,(f)}) = 0$, which means that $g \in \mathrm{Ann}\,(f) = eC(X)$, and other hand $Z(e) \subseteq Z(g)$ then $g(x) = 0$, which is impossible.

(3) $\Rightarrow$ (1) See Proposition 2.8 in [3]. $\square$

Proposition 4.7. *If $\frac{C(X)}{C_F(X)}$ is an $\aleph_0$-Kasch ring then X is an $\aleph_0$-Kasch space.*

Proof. Let $I = (f_1, \dots, f_n)$ is a ideal in $C(X)$. If $f_1, \dots, f_n \in C_F(X)$, in view of Proposition 2.6, $A = \bigcup_{i=1}^n \mathrm{Coz}\,(f_i)$ is finite clopen set in X. Now we define $g \in C(X)$ by $g(x) = 0$ for all $x \in A$ and $g(x) = 1$ if $x \notin A$, see 1A in [8]. Hence $g \in \mathrm{Ann}\,(I)$ and $g \neq 0$. Now let us assume that $f_i \notin C_F(X)$ for some f_i and put $\bar{I} = \sum_{i=1}^n \bar{f}_i \frac{C(X)}{C_F(X)}$, where $\bar{f}_i = f_i + C_F(X)$, it is clear that $\bar{I} \neq (0)$, and $\mathrm{gen}\,(\bar{I}) < \aleph_0$.

If $\bar{I} \neq \frac{C(X)}{C_F(X)}$, there exits $g \in C(X) \setminus C_F(X)$ such that $\bar{g}\bar{I} = (0)$. BY Proposition 2.6 we get that $B = \mathrm{Coz}\,(g) \cap (\bigcup_{i=1}^n \mathrm{Coz}\,(f_i))$ is finite clopen set in X. Thus if we define $g_0(x) = 0$ for all $x \in B$ and $g_0(x) = g(x)$ if $x \notin B$, then $0 \neq g_0 \in C(X)$ and $g \in \mathrm{Ann}\,(I)$.

If $\bar{I} = \frac{C(X)}{C_F(X)}$, then there exists $h \in I$ such that $\bar{h} = \bar{1}$. Thus in view of Proposition 2.6, $\mathrm{Coz}\,(h-1)$ is a finite clopen set in X. Since $\bigcap Z[I] \subseteq Z(h)$ and $Z(h) \subseteq \mathrm{Coz}\,(h-1)$, we conclude that $\bigcap Z[I]$ is a finite clopen set. If $\bigcap Z[I] \neq \emptyset$ then by Proposition 2.1, I is a non-essential ideal in $C(X)$. If $\bigcap Z[I] = \emptyset$ we get a contradiction. Let $Z(h) = \{x_1, \dots, x_n\}$, thus for every $x_i \in Z(h), \exists h_i \in I$ such that $h_i(x_i) \neq 0$. So that $Z(h_1^2 + \cdots + h_n^2 + h^2) = \emptyset$ and $h_1^2 + \cdots + h_n^2 + h^2 \in I$, which is the desired contradiction. $\square$

Lemma 4.8. *If $\mathrm{gen}\,(C_F(X)) < a$ and X is an a-Kasch space then $\frac{C(X)}{C_F(X)}$ is an a-Kasch ring.*

Proof. let $\bar{I} = \frac{I}{C_F(X)}$ such that $\mathrm{gen}\,(\bar{I}) < a$, where I is an ideal in $C(X)$ which is contains $C_F(X)$. By our hypothesis $\mathrm{gen}\,(I) \leq \mathrm{gen}\,(\bar{I}) + \mathrm{gen}\,(C_F(X)) < a$, thus I is a non-essential ideal in $C(X)$, i.e., $A = \mathrm{int} \bigcap Z[I] \neq \emptyset$ (by Proposition 2.1). By completely regularity X, if $x \in A$, there exists $f \in C(X)$ such that $f(x) = 1$ and $f(X \setminus A) = (0)$, it follows that $f \in I$ and $J = \bar{f} \frac{C(X)}{C_F(X)} \neq (0)$. Now let $\frac{I}{C_F(X)}$ be an essential ideal in $\frac{C(X)}{C_F(X)}$ and seek a contradiction. Thus $J \cap \frac{I}{C_F(X)} \neq (0)$ and there exists $g \in I$ such that $\bar{g}\bar{f} \neq 0$. But $X = A \cup (X \setminus A) \subseteq Z(g) \cup Z(f)$, i.e., $\bar{g}\bar{f} = 0$, which is impossible. $\square$

Proposition 4.9. *For a topological space X with only a finite number of isolated points, the following statements are equivalent:*

(1) *X is an a-Kasch space.*

(2) $\frac{C(X)}{C_F(X)}$ *is an a-Kasch ring.*

Proof. $1 \Rightarrow 2$) By Lemma 3.1 and our hypothesis, it is clear.

$2 \Rightarrow 1$) Let I is an ideal in $C(X)$ with $\operatorname{gen}(I) < a$. By our hypothesis $\operatorname{gen}(\frac{C_F(X)+I}{C_F(X)}) < a$.

If $\frac{C_F(X)+I}{C_F(X)} \neq \frac{C(X)}{C_F(X)}$, there exists $0 \neq \bar{g} \in \frac{C(X)}{C_F(X)}$ such that $g(C_F(X) + I) \subseteq C_F(X)$. By Proposition 2.6, we infer that $\operatorname{Coz}(gf) \subseteq H$, for all $f \in I$, where H is set of all isolated points in X. It follows that $A = \bigcup_{f\in I} \operatorname{Coz}(gf) \subseteq H$ is a finite clopen set in X (by hypothesis). By $1A$ in [8], we can define $g_0 \in C(X)$ by $g_0(x) = 0$ if $x \in A$ and $g_0(x) = g(x)$ if $x \notin A$. Since $\bar{g} \neq 0$, $\operatorname{Coz}(g)$ is a infinity set, we conclude that $0 \neq g_0 \in \operatorname{Ann}(I)$, i.e., I is a non-essential ideal in $C(X)$.

If $\frac{C_F(X)+I}{C_F(X)} = \frac{C(X)}{C_F(X)}$, there exists $g \in C_F(X)$ and $f \in I$ such that $g + f = 1$ then $1 - f = g \in C_F(X)$. By Proposition 2.6, $\bigcap Z[I] \subseteq Z(f) \subseteq \operatorname{Coz}(1 - f) \subseteq H$. It follows that $Z(f)$ and $\bigcap Z[I]$ are finite clopen set in X. If $\bigcap Z[I] \neq \emptyset$ then by Proposition 2.1, I is a non-essential ideal in $C(X)$, and if $\bigcap Z[I] = \emptyset$, as in the proof of the previous proposition we get a contradiction.

A subset S of R is said to be orthogonal provided $xy = 0$ for all $x, y \in S, x \neq y$. If X is a connected space and S is a orthogonal subset in $C(X)$ with more than two element, then $I = \sum_{f\in S} fC(X) \neq C(X)$, otherwise there exists $f_1, \ldots, f_n \in S$ and $g_1, \ldots, g_n \in C(X)$ such that $g_1f_1 + \cdots + g_nf_n = 1$ and $g_1f_1 \neq 0$. Clearly $g_1f_1 = (g_1f_1)^2$ and by 1A in [8], $g_1f_1 = 1$, i.e., f_1 is a unit element in $C(X)$, it follows that $|S| \leq 2$, which is the desired contradiction. □

Proposition 4.10. *Let X is an a-Kasch space and S is a maximal orthogonal subset in $C(X)$.*

(1) *S is a finite set if and only if $C(X) = \sum_{f\in S} fC(X)$.*

(2) *If S is a infinity subset in $C(X)$ then $|S| \geq a$ and $I = \sum_{f\in S} fC(X)$ is an essential ideal in $C(X)$.*

Proof. (1) **Necessity.** If $I = \sum_{f\in S} fC(X) \neq C(X)$, since $\operatorname{gen}(I) < \aleph_0$, we conclude that there exists $0 \neq g \in \operatorname{Ann}(I) = \operatorname{Ann}(S)$. Thus $A = S \cup \{g\}$ is a orthogonal subset in $C(X)$. So that A properly contains S , which violates the maximality of S.

Sufficiency. Let S is a infinity subset in $C(X)$ we get a contradiction. Clearly, there exists $f_1, \ldots, f_n \in S$ and $g_1, \ldots, g_n \in C(X)$ such that $g_1f_1 + \cdots + g_nf_n = 1$. Since S is a infinity set, we conclude that there exists $0 \neq h \in S \setminus \{f_1, \ldots, f_n\}$. But $h = g_1f_1h + \cdots + g_nf_nh = 0$ which is impossible.

(2) In view of the proof of previous case, $\operatorname{Ann}(S) = (0)$ and $I = \sum_{f\in S} fC(X)$ is a proper ideal in $C(X)$. So that I is an essential ideal in $C(X)$ and $|S| = \operatorname{gen}(I) \geq a$. □

If X is a pseudocpmpact then $\beta\colon C(X) \to C(\beta X)$ is a isomorphism onto, 6.6(b) in see [8]. It is not hard to prove that for every $A \subseteq X, \bigcap Z[O_A] = cl_X A$ and also see [5, Lemma 6.1], so that we have the following.

Proposition 4.11. *Suppose that X is an a-Kasch space, if $\beta X(\upsilon X)$ has a nowhere dense zeroset then $a = \aleph_0$.*

Proof. Let $a \geq \aleph_1$. By Theorem 3.3 and 8A in [8], $\upsilon X = \beta X$. Let A be a nowhere dense zero set in βX. Now put $O_\beta^A = \{f^\beta : A \subseteq \mathrm{int}_{\beta X} Z(f^\beta) \text{ and } f \in C(X)\}$, thus $\mathrm{int} \bigcap Z[O_\beta^A] = \mathrm{int}\ _{\beta X} cl_{\beta X} A = \emptyset$, by Proposition 2.1 we infer that O_β^A is an essential ideal in $C(\beta X)$. In view of Lemma 2.1 in [13], $O^A = \{f \in C(X) : A \subseteq \mathrm{int}\ _{\beta X} cl_{\beta X} Z(f)\}$ is a countable generated ideal in $C(X)$. Hence $\mathrm{gen}\,(O_\beta^A) = \mathrm{gen}\,(\beta(O^A)) = \mathrm{gen}\,(O^A) < \aleph_1$ and in view of Corollary 3.3, O_β^A is a non-essential ideal in $C(\beta X)$, which is a contradiction. $\square$

Proposition 4.12. *If X is a pseudocompact space then every countable z-ideal in $C(X)$ is of the form O^A, where $A \in Z[\beta X]$, and hence it is a z°-ideal.*

Proof. Let $I = (f_1, f_2, \ldots)$ be a z-ideal in $C(X)$. Clearly $\beta(I) = (f_1^\beta, f_2^\beta, \ldots)$ is a z-ideal in $C(\beta X)$. By Corollary in [13] there exists $A \in Z[\beta X]$ such that $\beta(I) = O_\beta^A = \{f^\beta : A \subseteq \mathrm{int}\ _{\beta X} Z(f^\beta) \text{ and } f \in C(X)\}$. It follows that $I = O^A = \{f \in C(X) : A \subset \mathrm{int}\ _{\beta X} cl_{\beta X} Z(f)\}$ is a z°-ideal, see [4]. $\square$

Lemma 4.13. *If I is a countable generated z-ideal and $Z[I]$ is closed under countable intersection then I is generated by an idempotent.*

Proof. Let $I = (f_1, f_2, \ldots)$ and put $g = \sum_{i=1}^\infty \frac{f_i^{\frac{2}{3}}}{2^i(1+f_i^{\frac{2}{3}})}$. Then by uniform convergence, $g \in C(X)$ and $Z(g) = \bigcap_{i=1}^\infty Z(f_i)$. But $Z[I]$ is closed under countable intersection, hence $Z(g) \in Z[I]$ implies that $g \in I$, for I is a z-ideal. Since $|f_n| \leq (2^n(1 + f_n^{\frac{2}{3}})g)^{\frac{2}{3}}$, we can then conclude from Proposition 2.3 that f_n is a multiple of g, for all $n \in N$. We infer that $I = gC(X)$, hence I is generated by an idempotent, see [13]. $\square$

If for every $j \in \Lambda, I_j$ is a z-ideal and $Z[I_j]$ is closed under countable intersection then $\bigcap_{j\in\Lambda} I_j$ is a z-ideal and $Z[\bigcap_{j\in\Lambda} I_j]$ is closed under countable intersection. By Lemma 3.2 and Theorem 5.14, 5.8 in [8] if X is a pseudocompact space then every countably generated intersection of maximal ideals is generated by an idempotent, see [12].

Proposition 4.14. *If X is a pseudocompact space, then every countably generated ideal which contains finite intersection maximal ideals is generated by an idempotent.*

References

[1] Azarpanah, F., *Essential ideals in $C(X)$*, Period. Math. Hungar. **3** (12) (1995), 105–112.

[2] Azarpanah, F., *Intersection of essential ideals in $C(X)$*, Proc. Amer. Math. Soc. **125** (1997), 2149–2154.

[3] Azarpanah, F., *On almost P-space*, Far East J. Math. Sci. Special volume (2000), 121–132.

[4] Azarpanah, F., Karamzadeh, O. A. S., Aliabad, A. R., *On z°-ideals in $C(X)$*, Fund. Math. **160** (1999), 15–25.

[5] Dietrich, W. E., Jr., *On the ideal structure of $C(X)$*, Trans. Amer. Math. Soc. **152** (1970), 61–77.

[6] Donne, A. Le, *On a question concerning countably generated z-ideal of $C(X)$*, Proc. Amer. Math. Soc. **80** (1980), 505–510.

[7] Engelking, R., *General topology*, mathematical monographs, vol. 60 ed., PWN Polish Scientific publishers, 1977.

[8] Gillman, L., Jerison, M., *Rings of continuous functions*, Springer-Verlag, 1979.

[9] Goodearl, K. R., *Von Neumann regular rings*, Pitman, San Francisco, 1979.

[10] Karamzadeh, O. A. S., Rostami, M., *On the intrinsic topology and some related ideals of $C(X)$*, Proc. Amer. Math. Soc. **93** (1985), 179–184.

[11] Lam, Tsit-Yuen, *Lectures on Modules and Rings*, Springer, 1999.

[12] Levy, R., *Almost P-spaces*, Can. J. Math. **29** (1977), 284–288.

[13] Marco, G. De, *On the countably generated z-ideal of $C(X)$*, Proc. Amer. Math. Soc. **31** (1972), 574–576.

[14] Nunzetta, P., Plank, D., *Closed ideal in $C(X)$*, Proc. Amer. Math. Soc. **35** (2) (1972), 601–606.

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