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FISCHER DECOMPOSITIONS IN EUCLIDEAN AND HERMITEAN CLIFFORD ANALYSIS

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Abstract. Euclidean Clifford analysis is a higher dimensional function theory studying so–called monogenic functions, i.e. null solutions of the rotation invariant, vector valued, first order Dirac operator $\underline{\partial}$. In the more recent branch Hermitean Clifford analysis, this rotational invariance has been broken by introducing a complex structure J on Euclidean space and a corresponding second Dirac operator $\underline{\partial}_J$, leading to the system of equations $\underline{\partial} f = 0 = \underline{\partial}_J f$ expressing so-called Hermitean monogenicity. The invariance of this system is reduced to the unitary group $\mathrm{U}(n)$. In this paper we decompose the spaces of homogeneous monogenic polynomials into $\mathrm{U}(n)$-irrucibles involving homogeneous Hermitean monogenic polynomials and we carry out a dimensional analysis of those spaces. Meanwhile an overview is given of so-called Fischer decompositions in Euclidean and Hermitean Clifford analysis.

1. Introduction

In 1917 Ernst Fischer proved (see [19]) that, given a homogeneous polynomial $q(X)$, $X \in \mathbb{R}^m$, every homogeneous polynomial $P_k(X)$ of degree k can be uniquely decomposed as $P_k(X) = Q_k(X) + q(X)R(X)$, where $Q_k(X)$ is a homogeneous polynomial of degree k satisfying $q(D)Q_k = 0$, D being the differential operator corresponding to X through Fourier identification ($X_j \leftrightarrow \partial_{x_j}, j = 1, \dots, m$) and $R(X)$ is a homogeneous polynomial of suitable degree. If in particular $q(X) = \|X\|^2$, then $q(D)$ is the Laplacian Δ_m and Q_k is harmonic, leading to the decomposition

$$\mathcal{P}(\mathbb{R}^m;\mathbb{C}) = \bigoplus_{k=0}^{\infty}\bigoplus_{p=0}^{\infty} r^{2p}\ \mathcal{H}_k(\mathbb{R}^m;\mathbb{C}) \tag{1}$$

of the space $\mathcal{P}(\mathbb{R}^m;\mathbb{C})$ of complex valued polynomials into the spaces $\mathcal{H}_k(\mathbb{R}^m;\mathbb{C})$ of complex valued harmonic homogeneous polynomials of degree k.

Clifford analysis (see e.g. [3, 15, 20, 22]), in its most basic form being a generalization to higher dimension of holomorphic function theory in the complex plane, offers the possibility for a refinement of this decomposition (1). Indeed, denoting by $(e_1, \dots, e_m)$ an orthonormal basis of $\mathbb{R}^m$, the polynomial $q(\underline{X})$ may be chosen to be $q(\underline{X}) = \underline{X}$, where $\underline{X} = \sum_{\alpha=1}^m e_\alpha X_\alpha$ is a real vector in the complex Clifford algebra $\mathbb{C}_m$ constructed over $\mathbb{R}^m$; the differential operator $q(D)$ then is the Dirac operator

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$\underline{\partial} = \sum_{\alpha=1}^{m} e_\alpha \, \partial_{X_\alpha}$ and Q_k is a k-homogeneous polynomial null solution of $\underline{\partial}$, a so–called spherical monogenic. This leads to the well-known Fischer decomposition in Euclidean Clifford analysis of the space $\mathcal{P}(\mathbb{R}^m; \mathbb{S})$ of homogeneous polynomials taking their values in an irreducible representation $\mathbb{S}$ of $\mathbb{C}_m$. Such a representation $\mathbb{S}$ is called a spinor space and usually realized inside the Clifford algebra using a primitive idempotent (see Section 5). This Fischer decomposition reads:

$$\mathcal{P}(\mathbb{R}^m; \mathbb{S}) = \bigoplus_{k=0}^{\infty} \bigoplus_{p=0}^{\infty} \underline{X}^p \, \mathcal{M}_k(\mathbb{R}^m; \mathbb{S}) \,, \tag{2}$$

where $\mathcal{M}_k(\mathbb{R}^m; \mathbb{S})$ denotes the space of spinor valued monogenic homogeneous polynomials of degree k. In particular each harmonic k-homogeneous polynomial H_k, be it real, complex or spinor valued, may be split as

$$H_k = M_k + \underline{X} M_{k-1} \tag{3}$$

M_k and M_{k-1} being monogenic homogeneous polynomials of the indicated degree.

In the books [23, 12] and the series of papers [24, 10, 1, 2, 7, 16, 17, 18] so-called Hermitean Clifford analysis has emerged as a refinement of Euclidean Clifford analysis. Hermitean Clifford analysis is based on the introduction of an additional datum, a so-called complex structure J, intended to bring the notion of monogenicity closer to complex analysis. This complex structure induces an associated Dirac operator $\underline{\partial}_J$, whence Hermitean Clifford analysis then focusses on the simultaneous null solutions of both operators $\underline{\partial}$ and $\underline{\partial}_J$, called Hermitean monogenic functions. The resulting function theory is still in full development, see [8, 25, 9, 6, 5].

It is clear that the traditional approach sketched above cannot be used to obtain a Fischer decomposition of harmonic homogeneous polynomials in terms of Hermitean monogenic homogeneous polynomials. However, a Hermitean monogenic Fischer decomposition was realized in [7] by means of a representation theoretical approach which will be explained further on. This implies however that it is possible to split any monogenic homogeneous polynomial in terms of homogeneous Hermitean monogenic ones, which was established in [13]. The aim of the underlying paper is threefold: (i) to give an alternative proof of the latter splitting, revealing the match between the monogenic and the Hermitean monogenic decompositions of a given harmonic polynomial; (ii) to use the Fischer decomposition formulae for a dimensional analysis of spaces of monogenic and Hermitean monogenic homogeneous polynomials, and meanwhile (iii) to give an overview of all Fischer decompositions in Euclidean and Hermitean Clifford analysis.

2. Clifford algebra: the basics

Consider a real vector space E of dimension m, equipped with a real symmetric and positive definite bilinear form $B(X, Y)$, $X, Y \in E$, with associated quadratic form $Q(X) = B(X, X)$. The orthogonal group $\mathrm{O}(E)$ and the special orthogonal group $\mathrm{SO}(E)$ are defined as the groups of automorphisms, respectively orientation preserving automorphisms g, leaving the bilinear form B invariant:

$$B(gX, gY) = B(X, Y) \,, \ \forall \, X, Y \in E \,.$$

Now, let $(e_1, \ldots, e_m)$ be a basis of E, which we assume to be orthonormal w.r.t. the bilinear form B, i.e. $B(e_j, e_k) = \delta_{jk},\ j, k = 1, \ldots, m$. The introduction of this basis leads to the identification $\mathrm{O}(E) \simeq \mathrm{O}(m)$, through representation by $(m \times m)$-matrices $g = [g_{jk}]$, naturally satisfying the condition $gg^T = g^T g = \mathbf{1}_m$ with $\mathbf{1}_m$ the unit matrix of order m, while in the case of $\mathrm{SO}(E) \simeq \mathrm{SO}(m)$, the additional condition $\det(g) = 1$ holds.

Turning to the complexification $E_{\mathbb{C}}$ of E and the complexification $B_{\mathbb{C}}$ of B, let us now consider the Clifford algebras $\mathcal{C}\ell(E, -Q)$ over E and $\mathcal{C}\ell(E_{\mathbb{C}}, -Q_{\mathbb{C}})$ over $E_{\mathbb{C}}$. When identifying E with $\mathbb{R}^m$ these Clifford algebras are often denoted $\mathbb{R}_m$ and $\mathbb{C}_m$ respectively. The Clifford or geometric product is associative but non–commutative. With respect to the chosen basis, it is governed by the rules

$$e_\alpha^2 = -B(e_\alpha, e_\alpha) = -1\,,\ \alpha = 1, \ldots, m, \quad e_\alpha e_\beta + e_\beta e_\alpha = 0\,,\ \alpha \neq \beta = 1, \ldots, m\,.$$

In standard Euclidean Clifford analysis, each vector $X \in E$ with components $(X_1, \ldots, X_m) \in \mathbb{R}^m$, is identified with the real Clifford vector $\underline{X} = \sum_{\alpha=1}^m X_\alpha e_\alpha$. Its Fischer dual is the first order Clifford vector valued differential operator $\underline{\partial} = \sum_{\alpha=1}^m e_\alpha \partial_{X_\alpha}$, called the Dirac operator, which may also be obtained in a co-ordinate free way as a generalized gradient, see e.g. [1, 2]. It is precisely this Dirac operator which underlies the notion of monogenicity, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. A smooth function f, defined on E or on $E_{\mathbb{C}}$ and taking values in either the real or the complex Clifford algebra, is called left monogenic if and only if it fulfills the Dirac equation $\underline{\partial}[f] = 0$.

The groups $\mathrm{O}(E)$ and $\mathrm{SO}(E)$ are doubly covered by the so-called pin group $\mathrm{Pin}(E)$ and spin group $\mathrm{Spin}(E)$ of the Clifford algebra, respectively, realized inside $\mathbb{C}_m$ as

$$\mathrm{Pin}(E) = \{s \in \mathcal{C}\ell(E, -Q) : \exists k \in \mathbb{N},\ s = \underline{\omega}_1 \ldots \underline{\omega}_k,\ \underline{\omega}_i \in S^{m-1},\ i = 1, \ldots, k\}$$

$$\mathrm{Spin}(E) = \{s \in \mathcal{C}\ell(E, -Q) : \exists k \in \mathbb{N},\ s = \underline{\omega}_1 \ldots \underline{\omega}_{2k},\ \underline{\omega}_i \in S^{m-1},\ i = 1, \ldots, 2k\}$$

where S^{m-1} is the unit sphere in E; through co–ordinatization it holds that $\mathrm{Pin}(E) \simeq \mathrm{Pin}(m)$ and $\mathrm{Spin}(E) \simeq \mathrm{Spin}(m)$. Taking $g \in \mathrm{SO}(E)$, with corresponding pin element $s_g \in \mathrm{Spin}(E)$, the action of g on a vector in E translates to Clifford language as

$$X' = g[X] \ \longleftrightarrow\ \underline{X}' = s_g \underline{X} s_g^{-1}\,.$$

Considering its induced action on a function F, which is given for a Clifford algebra valued function by the so-called H-representation

$$H(s)[F(\underline{X})] = sF(s^{-1}\underline{X}s)s^{-1}$$

and for a spinor valued function F by the so-called L-representation

$$L(s)[F(\underline{X})] = sF(s^{-1}\underline{X}s)$$

one has the commutation relations $[\underline{\partial}, H(s)] = 0$ and $[\underline{\partial}, L(s)] = 0$, whence it follows that the Dirac operator is invariant under this action, and so is the notion of monogenicity. A similar observation applies to $\mathrm{Pin}(E)$.

We now introduce the building blocks of the Hermitean Clifford setting. To this end, we endow the space (E, B) with a so-called complex structure by choosing an

$\mathrm{SO}(E)$ element J for which $J^2 = -\mathbf{1}$, creating in this way the Hermitean space (E, B, J). Clearly $(\det J)^2 = (-1)^m$, forcing the dimension m of E to be even: in the Hermitean context we thus have to put $m = 2n$.

In $(E_\mathbb{C}, B_\mathbb{C})$ the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ create two isotropic subspaces

$$W^\pm = \left\{ Z^\pm \in E_\mathbb{C} \, : \, Z^\pm = \frac{1}{2}(\mathbf{1} \pm iJ)X, \; X \in E \right\}$$

which constitute the direct sum decomposition $E_\mathbb{C} = W^+ \oplus W^-$. Extending the action of $g \in \mathrm{SO}(E)$ to vectors in $E_\mathbb{C}$ by $Z^\pm \in W^\pm \;\mapsto\; g[Z^\pm] = \frac{1}{2}\left(g[X] \pm ig[JX]\right)$, the subspaces $W^\pm$ will remain invariant if and only if g commutes with the complex structure J, or in other words, if g belongs to

$$\mathrm{SO}_J(E) = \left\{ g \in \mathrm{SO}(E) \; : \; gJ = Jg \right\}.$$

Similarly $\mathrm{O}_J(E) \subset \mathrm{O}(E)$ is defined. Note that the orthonormal basis $(e_1, \ldots, e_{2n})$ of E may always be chosen in such a way that the complex structure J is represented by the matrix

$$J = \begin{bmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{bmatrix}.$$

For an arbitrary $\mathrm{O}_J(E)$ element the commutation relation with J then is reflected in the specific form of the corresponding matrix:

$$G = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

with $AA^T + BB^T = \mathbf{1}_n$ and $AB^T - BA^T = 0$, implying that $A \pm iB$ both belong to the unitary group $\mathrm{U}(n)$. In other words:

$$\mathrm{O}_J(2n) = \left\{ G \in \mathrm{O}(2n) \; : \; GJ = JG \right\}$$

is isomorphic with $\mathrm{U}(n)$, and so is $\mathrm{O}_J(E)$.

By means of the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$, the basis $(e_1, \ldots, e_{2n})$ gives rise to an alternative basis for $E_\mathbb{C}$, called the Witt basis:

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\, e_{n+j}), && j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + ie_{n+j}), && j = 1, \ldots, n. \end{aligned}$$

It splits into separate bases $(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ and $(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)$ for W^+ and W^-, respectively. The Witt basis elements satisfy the Grassmann relations

$$\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = 0, \quad \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0, \qquad j, k = 1, \ldots, n$$

including their isotropy: $\mathfrak{f}_j^2 = 0 = \mathfrak{f}_j^{\dagger\,2}$, $j = 1, \ldots, n$, and the duality relations

$$\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}, \qquad j, k = 1, \ldots, n.$$

Each of the sets $(\mathfrak{f}_1, \ldots, \mathfrak{f}_n)$ and $(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)$ thus generates a Grassmann algebra, respectively denoted by $\mathbb{C}\Lambda_n$ and $\mathbb{C}\Lambda_n^\dagger$. The †–notation corresponds to a Hermitean conjugation in $\mathcal{C}\ell(E_\mathbb{C}, -Q_\mathbb{C})$, defined as follows: take $\mu \in \mathcal{C}\ell(E_\mathbb{C}, -Q_\mathbb{C})$ arbitrarily, i.e. $\mu = a + ib$, with $a, b \in \mathcal{C}\ell(E, -Q)$. Then $\mu^\dagger = \overline{a} - i\overline{b}$ where $\overline{a}$ and $\overline{b}$ are the traditional Clifford conjugates of a and b in $\mathcal{C}\ell(E, -Q)$.

The components of the real vector X are now denoted as $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and the corresponding Clifford vector $\underline{X}$ may thus be rewritten in terms of the Witt basis as

$$\underline{X} = \sum_{j=1}^{n} (x_j e_j + y_j e_{n+j}) = \sum_{j=1}^{n} (z_j \mathfrak{f}_j - z_j^c \mathfrak{f}_j^\dagger)$$

where we have introduced the complex variables $z_j = x_j + iy_j$ and their complex conjugates z_j^c, $j = 1, \ldots, n$. For vectors in the isotropic subspaces $W^\pm$ of $E_\mathbb{C}$ a similar identification results into

$$\begin{aligned} Z^+ = \frac{1}{2}(\mathbf{1} + iJ)X &\longleftrightarrow \quad \underline{z} \ = \ \sum_{j=1}^{n} z_j \mathfrak{f}_j \\ Z^- = \frac{1}{2}(\mathbf{1} - iJ)X &\longleftrightarrow -\underline{z}^\dagger = -\sum_{j=1}^{n} z_j^c \mathfrak{f}_j^\dagger \end{aligned}$$

whence the relation $X = Z^+ + Z^-$ may be rewritten in Clifford language as $\underline{X} = \underline{z} - \underline{z}^\dagger$. Similarly we arrive at the definition of the Hermitean Dirac operators

$$\partial_{\underline{z}} = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \partial_{z_j} \quad \text{and} \quad \partial_{\underline{z}^\dagger} = \sum_{j=1}^{n} \mathfrak{f}_j \partial_{z_j^c} = \partial_{\underline{z}}^\dagger$$

which are the Fischer duals of $\underline{z}$ and $\underline{z}^\dagger$, and may be seen as refinements of the Euclidean Dirac operator since $\underline{\partial} = 2(\partial_{\underline{z}}^\dagger - \partial_{\underline{z}})$.

As a side remark, observe that the above operators may also be obtained in another way, making explicit use of the complex structure J. Indeed, let

$$\underline{X}| = J(\underline{X}) = \sum_{j=1}^{n} J(e_j)x_j + J(e_{n+j})y_j = \sum_{j=1}^{n} (e_j y_j - e_{n+j} x_j)$$

then there arises a second, associated (or "twisted") Dirac operator

$$\underline{\partial}_J = J(\underline{\partial}) = \sum_{\alpha=1}^{2n} J(e_\alpha)\partial_\alpha = \sum_{j=1}^{n} (e_j \partial_{y_j} - e_{n+j} \partial_{x_j})$$

corresponding to $\underline{X}|$. We then have that

$$\begin{aligned} 2\partial_{\underline{z}}^\dagger = &\ \ \frac{1}{2}(\mathbf{1} + iJ)[\underline{\partial}] = \ \ \frac{1}{2}\underline{\partial} + \frac{i}{2}\underline{\partial}_J \\ 2\partial_{\underline{z}} = &-\frac{1}{2}(\mathbf{1} - iJ)[\underline{\partial}] = -\frac{1}{2}\underline{\partial} + \frac{i}{2}\underline{\partial}_J \,. \end{aligned}$$

A smooth function F taking its values in the complex Clifford algebra or in spinor space $\mathbb{S}$ is called Hermitean monogenic (or h-monogenic for short) if it is a simultaneous null solution of both Euclidean Dirac operators, i.e. if it fulfills the system

$$\underline{\partial}[F] = 0 = \underline{\partial}_J[F]$$

or, equivalently, if it is a simultaneous null solution of both Hermitean Dirac operators, i.e. if it fulfills the system

$$\partial_{\underline{z}}[F] = 0 = \partial_{\underline{z}}^{\dagger}[F]\,.$$

Also the two Hermitean Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^{\dagger}$ may be generated (as was the case for the Euclidean Dirac operator $\underline{\partial}$) as generalized gradients, see [26] through projection on the appropriate invariant subspaces, which moreover guarantees the invariance of the considered system under the group action of $\mathrm{O}_J(2n) \simeq \mathrm{U}(n)$, see [1, 2].

For further use, observe that the Hermitean vector variables and Dirac operators are isotropic on account of the properties of the Witt basis elements, i.e.

$$(\underline{z})^2 = (\underline{z}^{\dagger})^2 = 0 \quad \text{and} \quad (\partial_{\underline{z}})^2 = (\partial_{\underline{z}}^{\dagger})^2 = 0$$

whence the Laplacian $\Delta = -\underline{\partial}^2$ allows for the decomposition and factorization

$$\Delta = 4(\partial_{\underline{z}}\partial_{\underline{z}}^{\dagger} + \partial_{\underline{z}}^{\dagger}\partial_{\underline{z}}) = 4(\partial_{\underline{z}}^{\dagger} + \partial_{\underline{z}})^2 = -4(\partial_{\underline{z}}^{\dagger} - \partial_{\underline{z}})^2$$

while also

$$-(\underline{z} - \underline{z}^{\dagger})^2 = (\underline{z} + \underline{z}^{\dagger})^2 = \underline{z}\,\underline{z}^{\dagger} + \underline{z}^{\dagger}\underline{z} = |\underline{z}|^2 = |\underline{z}^{\dagger}|^2 = \|\underline{X}\|^2 = r^2\,.$$

3. Harmonic analysis

We start with the space $\mathcal{P}(\mathbb{R}^m;\mathbb{C})$ of complex valued polynomials defined on $\mathbb{R}^m$, considered as a module over the full orthogonal group $\mathrm{O}(m)$. The action of the group $\mathrm{O}(m)$ on polynomials in $\mathcal{P}(\mathbb{R}^m;\mathbb{C})$ is the regular representation:

$$[g\cdot P](\underline{X}) = P(g^{-1}\cdot\underline{X}), \quad g\in\mathrm{O}(m), \quad P\in\mathcal{P}(\mathbb{R}^m;\mathbb{C}), \quad \underline{X}\in\mathbb{R}^m\,.$$

Denoting by $\mathcal{H}_k(\mathbb{R}^m;\mathbb{C})$ the space of complex valued harmonic k–homogeneous polynomials, each of the spaces

$$r^{2p}\ \mathcal{H}_k(\mathbb{R}^m;\mathbb{C})\,, \quad p\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}\,, \quad k\in\mathbb{N}_0$$

is a subspace of $\mathcal{P}(\mathbb{R}^m;\mathbb{C})$ which is invariant under the $\mathrm{O}(m)$ action. In addition, they form the constituents of the Fischer decomposition (1) of $\mathcal{P}(\mathbb{R}^m;\mathbb{C})$:

$$\mathcal{P}(\mathbb{R}^m;\mathbb{C}) = \bigoplus_{k=0}^{\infty}\bigoplus_{p=0}^{\infty} r^{2p}\ \mathcal{H}_k(\mathbb{R}^m;\mathbb{C})\,. \tag{4}$$

In (4), all $\mathrm{O}(m)$-modules $\mathcal{H}_k(\mathbb{R}^m;\mathbb{C})$ are irreducible and mutually inequivalent. In particular the space $\mathcal{P}_k(\mathbb{R}^m;\mathbb{C})$ of k-homogeneous polynomials decomposes as

$$\mathcal{P}_k(\mathbb{R}^m;\mathbb{C}) = \bigoplus_{p=0}^{\lfloor\frac{k}{2}\rfloor} r^{2p}\ \mathcal{H}_{k-2p}(\mathbb{R}^m;\mathbb{C})\,.$$

Next we consider the space $\mathcal{P}(\mathbb{R}^{2n};\mathbb{C})$ of complex valued polynomials defined on Euclidean space of even dimension, however considered as an $\mathrm{O}_J(2n) \cong \mathrm{U}(n)$-module. The action of $\mathrm{O}_J(2n)$ on polynomials in $\mathcal{P}(\mathbb{R}^{2n};\mathbb{C})$ is given by

$$[u\cdot P](\underline{X}) = P(u^{-1}\cdot\underline{X}), \quad u\in\mathrm{O}_J(2n), \quad P\in\mathcal{P}(\mathbb{R}^{2n};\mathbb{C}), \quad \underline{X}\in\mathbb{R}^{2n}\,.$$

Since each complex valued polynomial in $(x_1,\dots,x_n,y_1,\dots,y_n)$ may be written also as a polynomial in the variables $(z_1,\dots,z_n,z_1^c,\dots,z_n^c)$, i.e.

$$P(\underline{X}) = P(x_1,\dots,x_n,y_1,\dots,y_n) = \widetilde{P}(z_1,\dots,z_n,z_1^c,\dots,z_n^c)$$

we have to determine the polynomials $\widetilde{P}$ which are invariant under the action of $\mathrm{U}(n)$. As is well–known the space of $\mathrm{U}(n)$-invariant polynomials in $\mathcal{P}(\mathbb{R}^{2n};\mathrm{End}(\mathbb{C}))$ is the space with basis $\big(1,r^2,r^4,\dots,r^{2p},\dots\big)$ where

$$r^2 = \sum_{j=1}^{n}\ x_j^2 + y_j^2 = \sum_{j=1}^{n}\ z_j\, z_j^c = \sum_{j=1}^{n}\ |z_j|^2$$

The operator corresponding to the generator r^2 is the Laplacian

$$\Delta = \sum_{j=1}^{n}\ \partial^2_{x_jx_j} + \partial^2_{y_jy_j} = 4\ \sum_{j=1}^{n}\ \partial_{z_j}\partial_{z_j^c}$$

whence we are lead to consider the space of harmonic polynomials in the complex variables $(z_1,\dots,z_n,z_1^c,\dots,z_n^c)$. Its subspace $\mathcal{H}_k^{\mathbb{C}}$ of harmonic k–homogeneous polynomials may be decomposed as

$$\mathcal{H}_k^{\mathbb{C}} = \bigoplus_{a=0}^{k}\ \mathcal{H}_{a,k-a}(\mathbb{R}^{2n};\mathbb{C})$$

where $\mathcal{H}_{a,b}(\mathbb{R}^{2n};\mathbb{C})$ is the space of harmonic polynomials of bidegree (a,b), i.e. a–homogeneous in the variables z_j and b–homogeneous in the variables z_j^c, i.e.

$$H_{a,b}(\lambda z_1,\dots,\lambda z_n,\mu z_1^c,\dots,\mu z_n^c) = \lambda^a\,\mu^b\,H_{a,b}(z_1,\dots,z_n,z_1^c,\dots,z_n^c)\,.$$

This leads to the Fischer decomposition

$$\mathcal{P}(\mathbb{R}^{2n};\mathbb{C}) = \bigoplus_{k=0}^{\infty}\bigoplus_{p=0}^{\infty}\bigoplus_{a=0}^{k}\ r^{2p}\ \mathcal{H}_{a,k-a}(\mathbb{R}^{2n};\mathbb{C}) \tag{5}$$

where the constituents $r^{2p}\,\mathcal{H}_{a,k-a}(\mathbb{R}^{2n};\mathbb{C})$, $p\in\mathbb{N}_0$, $k\in\mathbb{N}_0$, $a=0,\dots,k$, are irreducible invariant subspaces under the action of $\mathrm{U}(n)$. In particular the space $\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{C})$ of k-homogeneous polynomials decomposes as

$$\mathcal{P}_k(\mathbb{R}^m;\mathbb{C}) = \bigoplus_{p=0}^{\lfloor\frac{k}{2}\rfloor}\ \bigoplus_{a=0}^{k-2p}\ r^{2p}\ \mathcal{H}_{a,k-2p-a}(\mathbb{R}^m;\mathbb{C})\,.$$

Comparing the Fischer decompositions (4) and (5), it is clear that changing the symmetry group from $\mathrm{O}(2n)$ to its subgroup $\mathrm{O}_J(2n)\simeq\mathrm{U}(n)$ results in considering the polynomials as functions of the complex variables $(z_1,\dots,z_n,z_1^c,\dots,z_n^c)$ and splitting the spaces of harmonic homogeneous polynomials according to bidegrees of homogeneity:

$$\mathcal{H}_k(\mathbb{R}^{2n};\mathbb{C}) = \bigoplus_{a=0}^{k}\ \mathcal{H}_{a,k-a}(\mathbb{R}^{2n};\mathbb{C})\,.$$

4. Euclidean Clifford analysis

As mentioned in the introduction, the Fischer decompositions in terms of spherical harmonics may be refined by considering the spherical monogenics of Clifford analysis. To that end we consider the space $\mathcal{P}(\mathbb{R}^m;\mathbb{S})$ of spinor valued polynomials and the L–action of the group $\mathrm{Pin}(m)$ on it, given by

$$[L(s)\cdot P](\underline{X}) = s\,P(\widehat{s}^{-1}\underline{X}s)\,, \quad P\in\mathcal{P}(\mathbb{R}^m;\mathbb{S})\,, \quad s\in \mathrm{Pin}(m), \quad \underline{X}\in\mathbb{R}^m\,.$$

We also need the action of $\mathrm{Pin}(m)$ on the space $\mathcal{P}(\mathbb{R}^m;\mathrm{End}(\mathbb{S}))$, where $\mathrm{End}(\mathbb{S})$ is isomorphic as a vector space with the complex Clifford algebra $\mathbb{C}_m$ when m is even, or with its even part when m is odd. Let $s\mapsto\widehat{s}$ denote the main involution on $\mathbb{C}_m$ for which $e_j\mapsto -e_j$; it has eigenvalues ±1, the corresponding eigenspaces being the even and odd part of the Clifford algebra. The action of $\mathrm{Pin}(m)$ on $\mathcal{P}(\mathbb{R}^m;\mathrm{End}(\mathbb{S}))$ then is

$$[s\cdot f](\underline{X}) = s\,f(\widehat{s}^{-1}\underline{X}s)]\,\widehat{s}^{-1}, \quad f\in\mathcal{P}(\mathbb{R}^m;\mathrm{End}(\mathbb{S})), \quad s\in\mathrm{Pin}(m), \quad \underline{X}\in\mathbb{R}^m\,.$$

The space of $\mathrm{Pin}(m)$-invariant polynomials inside $\mathcal{P}(\mathbb{R}^m;\mathrm{End}(\mathbb{S}))$ has the basis $(1,\underline{X},\underline{X}^2,\underline{X}^3,\ldots,\underline{X}^p,\ldots)$, generating a unital superalgebra or $\mathbb{Z}_2$-graded algebra

$$\mathrm{span}_{\mathbb{C}}(1,\underline{X}^2,\underline{X}^4,\ldots)\oplus\mathrm{span}_{\mathbb{C}}(\underline{X},\underline{X}^3,\underline{X}^5,\ldots)$$

which reflects the natural grading of the Clifford algebra by its decomposition into the even subalgebra and the odd subspace.

The $\mathrm{Pin}(m)$-invariant differential operator corresponding, under natural duality, with the generator $\underline{X}$ of this graded algebra, is the Dirac operator $\underline{\partial}$. Its polynomial null solutions are called spherical monogenics; we denote by $\mathcal{M}_k(\mathbb{R}^m;\mathbb{S})$ the space of spinor valued k-homogeneous spherical monogenics. Then each of the spaces $\underline{X}^p\,\mathcal{M}_k(\mathbb{R}^m;\mathbb{S})$, $p\in\mathbb{N}_0$, $k\in\mathbb{N}_0$, is an irreducible invariant subspace of $\mathcal{P}(\mathbb{R}^m;\mathbb{S})$ under the action of $\mathrm{Pin}(m)$, leading to the Fischer decomposition (2) of $\mathcal{P}(\mathbb{R}^m;\mathbb{S})$:

$$\mathcal{P}(\mathbb{R}^m;\mathbb{S}) = \bigoplus_{k=0}^{\infty}\bigoplus_{p=0}^{\infty}\underline{X}^p\,\mathcal{M}_k(\mathbb{R}^m;\mathbb{S})\,. \tag{6}$$

In (6), all $\mathrm{Pin}(m)$-modules $\mathcal{M}_k(\mathbb{R}^m;\mathbb{S})$ are irreducible and mutually inequivalent. To see (6) as a refinement of (4) just take into account the Fischer decomposition of spherical harmonics in terms of spherical monogenics (see also (3))

$$\mathcal{H}_k(\mathbb{R}^m;\mathbb{S}) = \mathcal{H}_k(\mathbb{R}^m;\mathbb{C})\otimes\mathbb{S} = \mathcal{M}_k(\mathbb{R}^m;\mathbb{S})\oplus\underline{X}\mathcal{M}_{k-1}(\mathbb{R}^m;\mathbb{S}) \tag{7}$$

meaning that inside $\mathcal{H}_k(\mathbb{R}^m;\mathbb{S})$ an isomorphic copy of both $\mathrm{Pin}(m)$-irreducible modules $\mathcal{M}_k(\mathbb{R}^m;\mathbb{S})$ and $\mathcal{M}_{k-1}(\mathbb{R}^m;\mathbb{S})$ is realized by the trivial embedding and by the embedding factor $\underline{X}$ respectively; explicitly for $H_k\in\mathcal{H}_k(\mathbb{R}^m;\mathbb{S})$ one has

$$H_k = \Big(\mathbf{1}+\frac{\underline{X}\underline{\partial}}{m+2k-2}\Big)H_k - \frac{\underline{X}\underline{\partial}}{m+2k-2}H_k\,.$$

5. Hermitean Clifford analysis

In this section we further explore the space $\mathcal{P}(\mathbb{R}^{2n};\mathbb{S})$ of $\mathbb{S}$ valued polynomials on Euclidean space of even dimension $\mathbb{R}^{2n}$. Here, we further decompose $\mathbb{S}$ as

$$\mathbb{S} = \bigoplus_{v=0}^{n} \mathbb{S}^{(v)}$$

into its so-called homogeneous parts $\mathbb{S}^{(v)}$, $v = 0,\ldots,n$, i.e. eigenspaces with eigenvalue v for the left multiplication operator $\beta^c = \sum_{j=1}^n \mathfrak{f}_j \mathfrak{f}_j^\dagger$ (see [11]).

We want to obtain a decomposition of the space $\mathcal{P}(\mathbb{R}^{2n};\mathbb{S})$ into irreducible subspaces under the action of the group $\mathrm{Pin}_J(2n)$, which is a double cover of $\mathrm{O}_J(2n)$ inside the Clifford algebra; this group can be defined as

$$\mathrm{Pin}_J(2n) = \{s \in \mathrm{Pin}(2n) \,:\, s s_J = s_J s\}$$

where $s_J = s_1 s_2 \ldots s_n$, with $s_j = \frac{\sqrt{2}}{2}(1 - e_j e_{n+j})$, $j = 1,\ldots,n$, is a $\mathrm{Spin}(2n)$ element corresponding to the complex structure $J \in \mathrm{SO}(2n)$ under the double covering of $\mathrm{SO}(2n)$ by $\mathrm{Spin}(2n)$. The action of $\mathrm{Pin}_J(2n)$ on $\mathcal{P}(\mathbb{R}^{2n};\mathbb{S})$ is given by

$$s \cdot \widetilde{f}(\underline{z},\underline{z}^\dagger) = s\widetilde{f}(\widehat{s}^{-1}\underline{z}s, \widehat{s}^{-1}\underline{z}^\dagger s), \quad \widetilde{f} \in \mathcal{P}(\mathbb{R}^{2n};\mathbb{S})\,, \quad s \in \mathrm{Pin}_J(2n)$$

whereas its action on $\mathcal{P}(\mathbb{R}^{2n};\mathrm{End}(\mathbb{S})) = \mathcal{P}(\mathbb{R}^{2n};\mathbb{C}_{2n})$ is given by

$$s \cdot \widetilde{f}(\underline{z},\underline{z}^\dagger) = s\widetilde{f}(\widehat{s}^{-1}\underline{z}s, \widehat{s}^{-1}\underline{z}^\dagger s)\widehat{s}^{-1}$$

Observe the use of the Hermitean vector variables $\underline{z}$ and $\underline{z}^\dagger$, which are $\mathrm{Pin}_J(2n)$-invariant elements in $\mathcal{P}(\mathbb{R}^{2n};\mathrm{End}(\mathbb{S}))$. In fact it may be proven by invariance theory (see e.g. [21]) that the space of all $\mathrm{Pin}_J(2n)$-invariant polynomials is spanned by all possible words in $\underline{z}$ and $\underline{z}^\dagger$:

$$\begin{aligned}\mathrm{span}_{\mathbb{C}}\big(1, \underline{z}, \underline{z}^\dagger, \underline{z}\,\underline{z}^\dagger, \underline{z}^\dagger\underline{z}, \underline{z}\,\underline{z}^\dagger\underline{z}, \underline{z}^\dagger\underline{z}\,\underline{z}^\dagger, \underline{z}\,\underline{z}^\dagger\underline{z}\,\underline{z}^\dagger, \underline{z}^\dagger\underline{z}\,\underline{z}^\dagger\underline{z}, \ldots\big)\\ = \mathrm{span}_{\mathbb{C}}\big(w_l^{(i)}(\underline{z},\underline{z}^\dagger) : l = 0,1,2,\ldots, i = 1,2\big)\end{aligned}$$

with

$$\begin{aligned} w_{2r}^{(1)}(\underline{z},\underline{z}^\dagger) &= (\underline{z}\,\underline{z}^\dagger)^r = |\underline{z}|^{2r-2}\,\underline{z}\,\underline{z}^\dagger & \qquad w_{2r+1}^{(1)}(\underline{z},\underline{z}^\dagger) &= |\underline{z}|^{2r}\,\underline{z}\\ w_{2r}^{(2)}(\underline{z},\underline{z}^\dagger) &= (\underline{z}^\dagger\underline{z})^r = |\underline{z}|^{2r-2}\,\underline{z}^\dagger\underline{z} & \qquad w_{2r+1}^{(2)}(\underline{z},\underline{z}^\dagger) &= |\underline{z}|^{2r}\,\underline{z}^\dagger \end{aligned}$$

and $w_0^{(1)} = w_0^{(2)} = 1$. This space becomes a unital graded superalgebra, inheriting its grading from the $\mathbb{Z}_2$-grading on $\mathbb{C}_m$.

As a first step towards the decomposition aimed at, we will split $\mathcal{P}(\mathbb{R}^{2n};\mathbb{S})$ according to bidegree of homogeneity and to the homogeneous parts of spinor space:

$$\mathcal{P}(\mathbb{R}^{2n};\mathbb{S}) = \bigoplus_{a,b=0}^{\infty}\,\bigoplus_{v=0}^{n}\,\mathcal{P}_{a,b}^{(v)}$$

with $\mathcal{P}_{a,b}^{(v)} = \mathcal{P}_{a,b}(\mathbb{R}^{2n};\mathbb{C}) \otimes \mathbb{S}^{(v)}$. Under the natural duality the generators $\underline{z}$ and $\underline{z}^\dagger$ of the above superalgebra correspond to the Hermitean Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$. So we consider the spaces $\mathcal{M}_{a,b}^{(v)}$ of $\mathbb{S}^{(v)}$ valued Hermitean monogenic homogeneous polynomials of bidegree (a,b) in the variables $(z_1,\ldots,z_n,z_1^c,\ldots,z_n^c)$, the latter denoted as $(\underline{z},\underline{z}^\dagger)$. This leads to the Fischer decomposition of the space of spinor valued polynomials according to the action of $\mathrm{Pin}_J(2n)$ (see [7]):

$$\mathcal{P}(\mathbb{R}^{2n};\mathbb{S}) = \bigoplus_{a,b=0}^{\infty} \bigoplus_{v=0}^{n} \left(\mathcal{M}_{a,b}^{(v)} \oplus \bigoplus_{p=1}^{\infty} \bigoplus_{i=1,2} w_p^{(i)}(\underline{z},\underline{z}^\dagger)\ \mathcal{M}_{a,b}^{(v)} \right).$$

In particular, for the space $\mathcal{H}_{a,b}^{(v)}$ of $\mathbb{S}^{(v)}$ valued harmonic homogeneous polynomials of bidegree (a,b), this Fischer decomposition reduces to

$$\begin{aligned} \mathcal{H}_{a,b}^{(v)} = \mathcal{M}_{a,b}^{(v)} &\oplus \underline{z}\,\mathcal{M}_{a-1,b}^{(v-1)} \oplus \underline{z}^\dagger\,\mathcal{M}_{a,b-1}^{(v+1)} \\ &\oplus \left(\frac{\underline{z}\underline{z}^\dagger}{b-1+v} - \frac{\underline{z}^\dagger\underline{z}}{a-1+n-v} \right) \mathcal{M}_{a-1,b-1}^{(v)} \end{aligned} \tag{8}$$

where we put $\mathcal{M}_{a,b}^{(v)} = \{0\}$ whenever $a<0$, $b<0$, $v<0$ or $v>n$ and moreover, when $b-1+v=0$ the last summand reduces to $\underline{z}\underline{z}^\dagger\mathcal{M}_{a-1,b-1}^{(v)}$, while, when $a-1+n-v=0$ it reduces to $\underline{z}^\dagger\underline{z}\mathcal{M}_{a-1,b-1}^{(v)}$.

Special attention should be paid to the cases where $v=0$ and $v=n$. Indeed, for $v=0$, Hermitean monogenicity means holomorphy (see [2]), so in this case the spaces of spherical Hermitean monogenics are simply the spaces of scalar valued holomorphic homogeneous polynomials in the variables $(z_1,\ldots,z_n)$, which implies that b must be zero. For $v=n$ Hermitean monogenicity means anti-holomorphy, so in that case we end up with anti-holomorphic homogeneous polynomials in the variables $(z_1^c,\ldots,z_n^c)$, implying that a must be zero. This leads to the following special Fischer decompositions: for $v=0$ one has

- $\mathcal{H}_{a,0}^{(0)} = \mathcal{M}_{a,0}^{(0)}$;
- $\mathcal{H}_{a,1}^{(0)} = \underline{z}^\dagger\,\mathcal{M}_{a,0}^{(1)} \oplus \underline{z}\,\underline{z}^\dagger\,\mathcal{M}_{a-1,0}^{(0)}$;
- $\mathcal{H}_{a,b}^{(0)} = \underline{z}^\dagger\,\mathcal{M}_{a,b-1}^{(1)}$, when $b\neq 0$, $b\neq 1$,

while for $v=n$ one has

- $\mathcal{H}_{0,b}^{(n)} = \mathcal{M}_{0,b}^{(n)}$;
- $\mathcal{H}_{1,b}^{(n)} = \underline{z}\,\mathcal{M}_{0,b}^{(n-1)} \oplus \underline{z}^\dagger\,\underline{z}\,\mathcal{M}_{0,b-1}^{(n)}$;
- $\mathcal{H}_{a,b}^{(n)} = \underline{z}\,\mathcal{M}_{a-1,b}^{(n-1)}$, when $a\neq 0$, $a\neq 1$.

Note that the dimensional analysis carried out in Section 7 confirms these results.

In the next section we will show how the decompositions (7) and (8) fit together, more precisely we will determine the $\mathrm{U}(n)$-irreducible parts of (8) constituting each of the terms in (7).

6. Decomposition of $\mathcal{M}_k$ into U(n)-irreducibles

Let us first decompose the space $\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S})$ of spinor valued k-homogeneous polynomials in the variables $(z_1,\dots,z_n,z_1^c,\dots,z_n^c)$ according to bidegree of homogeneity and to the homogeneous parts of spinor space:

$$\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S}) = \bigoplus_{a+b=k}\bigoplus_{v=0}^{n}\mathcal{P}_{a,b}^{(v)}$$

in this way inducing on a spherical monogenic $M_k \in \mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S})$ the splitting

$$M_k = \sum_{a=0}^{k}\sum_{v=0}^{n} P_{a,k-a}^{(v)}\,. \tag{9}$$

It is important to note that the components $P_{a,k-a}^{(v)}$ are no longer monogenic since

$$\begin{aligned}\underline{\partial}_{\underline{z}} \colon \mathcal{P}_{a,k-a}^{(v)} &\longrightarrow \mathcal{P}_{a-1,k-a}^{(v-1)}\,;\\ \underline{\partial}_{\underline{z}}^{\dagger} \colon \mathcal{P}_{a,k-a}^{(v)} &\longrightarrow \mathcal{P}_{a,k-a-1}^{(v+1)}\end{aligned}$$

whence

$$\underline{\partial} \colon \mathcal{P}_{a,k-a}^{(v)} \longrightarrow \mathcal{P}_{a-1,k-a}^{(v-1)} \oplus \mathcal{P}_{a,k-a-1}^{(v+1)}$$

with $\mathcal{P}_{a,b}^{(v)} = \{0\}$ whenever $a<0$ or $b<0$ or $v<0$ or $v>n$. In other words: the action of the Dirac operator $\underline{\partial}$ mixes up the homogeneous parts of spinor space. Introducing the spaces $\mathcal{M}_{a,k-a}^{(v)} = \mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S}) \cap \mathcal{P}_{a,k-a}^{(v)}$ we clearly have that

$$\bigoplus_{a=0}^{k}\bigoplus_{v=0}^{n} \mathcal{M}_{a,k-a}^{(v)} \subset \mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S})$$

Moreover the polynomials in $\mathcal{M}_{a,k-a}^{(v)}$ satisfy $\underline{\partial} M_{a,k-a}^{(v)} = 0 = \underline{\partial}_{\underline{z}}^{\dagger} M_{a,k-a}^{(v)} - \underline{\partial}_{\underline{z}} M_{a,k-a}^{(v)}$, where $\underline{\partial}_{\underline{z}}^{\dagger} M_{a,k-a}^{(v)} \in \mathcal{P}_{a,k-a-1}^{(v+1)}$ and $\underline{\partial}_{\underline{z}} M_{a,k-a}^{(v)} \in \mathcal{P}_{a-1,k-a}^{(v-1)}$. This means that at the same time $\underline{\partial}_{\underline{z}} M_{a,k-a}^{(v)} = 0$ and $\underline{\partial}_{\underline{z}}^{\dagger} M_{a,k-a}^{(v)} = 0$, or: $M_{a,k-a}^{(v)}$ is Hermitean monogenic, which justifies the notation $\mathcal{M}_{a,k-a}^{(v)}$ for the corresponding space. We thus have

Lemma 1. *On each of the spaces $\mathcal{P}_{a,b}^{(v)}$ the notions of monogenicity and Hermitean monogenicity coincide.*

Introducing the space of spherical Hermitean monogenics of degree k:

$$\mathcal{HM}_k = \left\{Q_k \in \mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S}) : \underline{\partial}_{\underline{z}} Q_k = 0 = \underline{\partial}_{\underline{z}}^{\dagger} Q_k\right\}$$

we thus have obtained that

$$\bigoplus_{a=0}^{k}\bigoplus_{v=0}^{n} \mathcal{M}_{a,k-a}^{(v)} \subset \mathcal{HM}_k \subset \mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S})\,. \tag{10}$$

However, there is more. Denoting the restrictions to $\mathcal{M}_k$ of the Hermitean Dirac operators by $\widetilde{\underline{\partial}_{\underline{z}}}$ and $\widetilde{\underline{\partial}_{\underline{z}}^{\dagger}}$ we have the following result.

Proposition 1. *One has*

$$\mathcal{HM}_k = \text{Ker } \widetilde{\underline{\partial}_z} = \text{Ker } \widetilde{\underline{\partial}_z^\dagger} = \bigoplus_{a=0}^{k} \bigoplus_{v=0}^{n} \mathcal{M}_{a,k-a}^{(v)} \, .$$

Proof. In view of (10) we still need to prove that

$$\bigoplus_{a=0}^{k} \bigoplus_{v=0}^{n} \mathcal{M}_{a,k-a}^{(v)} \supset \mathcal{HM}_k \, .$$

So take $\phi_k \in \mathcal{HM}_k$, then

$$\phi_k = \sum_{a=0}^{k} \sum_{v=0}^{n} \phi_{a,k-a}^{(v)}$$

with $\phi_{a,k-a}^{(v)} \in \mathcal{P}_{a,k-a}^{(v)}$. As $\underline{\partial}_z \phi_{a,k-a}^{(v)} \in \mathcal{P}_{a-1,k-a}^{(v-1)}$ it follows from $\underline{\partial}_z \phi_k = 0$ that $\underline{\partial}_z \phi_{a,k-a}^{(v)} = 0$ for $a = 1, \ldots, k$ and $v = 1, \ldots, n$, while for $a = 0$ or $v = 0$ this equation is trivially satisfied. Similarly it follows from $\underline{\partial}_z^\dagger \phi_k = 0$ that $\underline{\partial}_z^\dagger \phi_{a,k-a}^{(v)} = 0$ for $a = 0, \ldots, k-1$ and $v = 0, \ldots, n-1$, which now is trivial for $a = k$ or $v = n$. We may thus conclude that $\phi_{a,k-a}^{(v)} \in \mathcal{M}_{a,k-a}^{(v)}$ for $a = 0, \ldots, k$ and $v = 0, \ldots, n$, which proves the statement. □

Remark 1. Both for the spherical harmonics and for the spherical Hermitean monogenics, the decomposition according to bidegree of homogeneity and to spinor homogeneity leads to harmonic, respectively Hermitean monogenic components. For spherical monogenics however this is not the case, as already mentioned, since the action of the Dirac operator, in fact a combined action of both Hermitean Dirac operators, mixes up the homogeneous spinor subspaces. We can only say that the corresponding components of a spherical monogenic are in Ker $\underline{\partial}_z \underline{\partial}_z^\dagger =$ Ker $\underline{\partial}_z^\dagger \underline{\partial}_z$.

Our aim now is to decompose $\mathcal{M}_k$ into irreducible subspaces which are invariant under $\text{Pin}_J(2n) \cong \text{U}(n)$. To that end we start from the orthogonal decomposition

$$\mathcal{M}_k = \text{Ker } \widetilde{\underline{\partial}_z} \oplus \left(\text{Ker } \widetilde{\underline{\partial}_z^\dagger} \right)^\perp$$

for which we have already shown in Proposition 1 that

$$\text{Ker } \widetilde{\underline{\partial}_z} = \text{Ker } \widetilde{\underline{\partial}_z^\dagger} = \bigoplus_{a=0}^{k} \bigoplus_{v=0}^{n} \mathcal{M}_{a,k-a}^{(v)} \, .$$

We also know that $(\text{Ker } \widetilde{\underline{\partial}_z^\dagger})^\perp \cong \text{Im } \widetilde{\underline{\partial}_z}$ whence it suffices to determine Im $\widetilde{\underline{\partial}_z}$.

Lemma 2. *One has*

$$\text{Im } \widetilde{\underline{\partial}_z} \subset \bigoplus_{a=0}^{k-1} \bigoplus_{v=1}^{n-1} \mathcal{M}_{a,k-a-1}^{(v)} \, . \tag{11}$$

Proof. Using (9) and invoking that $\partial_{\underline{z}} M_k = \partial_{\underline{z}}^\dagger M_k$ since M_k is monogenic we have

$$\sum_{a=0}^{k-1}\sum_{v=1}^{n-1} \partial_{\underline{z}} P_{a+1,k-a-1}^{(v+1)} = \sum_{a=0}^{k-1}\sum_{v=1}^{n-1} \partial_{\underline{z}}^\dagger P_{a,k-a}^{(v-1)}$$

and hence

$$\begin{aligned} \partial_{\underline{z}} P_{a+1,k-a-1}^{(1)} &= 0\,; \\ \partial_{\underline{z}}^\dagger P_{a,k-a}^{(n-1)} &= 0\,; \\ \partial_{\underline{z}} P_{a+1,k-a-1}^{(v+1)} &= \partial_{\underline{z}}^\dagger P_{a,k-a}^{(v-1)} \in \mathcal{M}_{a,k-a-1}^{(v)} \end{aligned}$$

from which the desired result follows. □

To show that equality holds in (11) we prove the following version of the Poincaré Lemma.

Lemma 3. *Given $\phi_{a,k-a-1}^{(v)} \in \mathcal{M}_{a,k-a-1}^{(v)}$, the polynomial*

$$\psi = \Big(\frac{\underline{z}}{a+n-v} + \frac{\underline{z}^\dagger}{k-a-1+v}\Big)\phi_{a,k-a-1}^{(v)}$$

enjoys the following properties:

(i) $\psi \in \mathcal{M}_k$

(ii) $\partial_{\underline{z}}\psi = \partial_{\underline{z}}^\dagger\psi = \phi_{a,k-a-1}^{(v)}$

Proof. To prove (ii) it suffices invoke the well-known anti–commutation relations (see [7]):

$$\begin{aligned} \{\partial_{\underline{z}}, \underline{z}\} &= \mathbb{E}_{\underline{z}} + n - \beta^c\,; \\ \{\partial_{\underline{z}}, \underline{z}^\dagger\} &= 0\,; \\ \{\partial_{\underline{z}}^\dagger, \underline{z}\} &= 0\,; \\ \{\partial_{\underline{z}}^\dagger, \underline{z}^\dagger\} &= \mathbb{E}_{\underline{z}^c} + \beta^c \end{aligned}$$

the Hermitean Euler operators $\mathbb{E}_{\underline{z}}$ and $\mathbb{E}_{\underline{z}^\dagger}$ having the spaces $\mathcal{P}_{a,b}^{(v)}$ as eigenspaces with respective eigenvalues a and b. Next, (i) follows from (ii). □

Proposition 2. *One has*

$$\mathrm{Im}\ \widetilde{\partial_{\underline{z}}} = \mathrm{Im}\ \widetilde{\partial_{\underline{z}}^\dagger} = \bigoplus_{a=0}^{k-1}\bigoplus_{v=1}^{n-1} \mathcal{M}_{a,k-a-1}^{(v)}\,.$$

Proof. In view of (11) we still have to prove that

$$\mathrm{Im}\ \widetilde{\partial_{\underline{z}}} \supset \bigoplus_{a=0}^{k-1}\bigoplus_{v=1}^{n-1} \mathcal{M}_{a,k-a-1}^{(v)}\,.$$

To that end take

$$\phi = \sum_{a=0}^{k-1}\sum_{v=1}^{n-1} \phi_{a,k-a-1}^{(v)} \in \bigoplus_{a=0}^{k-1}\bigoplus_{v=1}^{n-1} \mathcal{M}_{a,k-a-1}^{(v)}$$

and define the polynomial

$$\Psi = \sum_{a=0}^{k-1} \sum_{v=1}^{n-1} \left(\frac{\underline{z}}{a+n-v} + \frac{\underline{z}^\dagger}{k-a-1+v} \right) \phi^{(v)}_{a,k-a-1} \, .$$

Then Ψ will belong to $\mathcal{M}_k$ and satisfy $\underline{\partial_{z}} \Psi = \underline{\partial_{z}}^\dagger \Psi = \phi$. □

Combining the above results we obtain the following Fischer decomposition, which, as mentioned in the introduction, was already obtained in [13] on the basis of group representation theory.

Theorem 1. *The space $\mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S})$ of spinor valued spherical monogenics of degree k may be decomposed into* $\mathrm{U}(n)$*-irreducibles as follows:*

$$\begin{aligned} (12) \quad \mathcal{M}_k = &\Big(\bigoplus_{a=0}^{k} \bigoplus_{v=0}^{n} \mathcal{M}^{(v)}_{a,k-a} \Big) \\ &\oplus \Big(\bigoplus_{a=0}^{k-1} \bigoplus_{v=1}^{n-1} \left(\frac{\underline{z}}{a+n-v} + \frac{\underline{z}^\dagger}{k-a-1+v} \right) \mathcal{M}^{(v)}_{a,k-a-1} \Big) \, . \end{aligned}$$

This last result means that, given a spinor valued spherical monogenic M_k, (9), there exist spinor valued spherical Hermitean monogenics $f^{(v)}_{a,k-a}$ and $g^{(v)}_{a,k-a-1}$ such that

$$M_k = \sum_{a=0}^{k} \sum_{v=0}^{n} f^{(v)}_{a,k-a} + \sum_{a=1}^{k} \sum_{v=2}^{n} \frac{\underline{z}\, g^{(v-1)}_{a-1,k-a}}{a+n-v} + \sum_{a=0}^{k-1} \sum_{v=0}^{n-2} \frac{\underline{z}^\dagger\, g^{(v+1)}_{a,k-a-1}}{k-a+v}$$

where the polynomials occuring in the respective projections from $\mathcal{M}_k$ onto the $\mathrm{U}(n)$-irreducibles involving spherical Hermitean monogenics, may be calculated as

$$\begin{aligned} f^{(v)}_{a,k-a} &= \left(\mathbf{1} - \frac{\underline{z}\underline{\partial_{z}}}{a+n-v} - \frac{\underline{z}^\dagger \underline{\partial_{z}}^\dagger}{k-a+v} \right) P^{(v)}_{a,k-a} \\ g^{(v-1)}_{a-1,k-a} &= \underline{\partial_{z}} P^{(v)}_{a,k-a} \\ g^{(v+1)}_{a,k-a-1} &= \underline{\partial_{z}}^\dagger P^{(v)}_{a,k-a} \end{aligned}$$

Now we are able to show explicitly how the Fischer decomposition (8) in terms of spherical Hermitean monogenics originates from the Fischer decomposition (7) in terms of standard spherical monogenics, by using the decomposition (12) of Theorem 1. First we have, according to (7):

$$\mathcal{H}_k(\mathbb{R}^{2n};\mathbb{S}) = \mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S}) \oplus (\underline{z} - \underline{z}^\dagger) \mathcal{M}_{k-1}(\mathbb{R}^{2n};\mathbb{S})$$

which, by means of (12), takes the form

$$\begin{aligned}
\mathcal{H}_k(\mathbb{R}^{2n};\mathbb{S}) &= \Big(\bigoplus_{a=0}^{k}\bigoplus_{v=0}^{n} \mathcal{M}^{(v)}_{a,k-a}\Big)\\
&\oplus \Big(\bigoplus_{a=0}^{k-1}\bigoplus_{v=1}^{n-1} \Big(\frac{\underline{z}}{a+n-v} + \frac{\underline{z}^\dagger}{k-a-1+v}\Big)\mathcal{M}^{(v)}_{a,k-a-1}\Big)\\
&\oplus (\underline{z}-\underline{z}^\dagger)\Big(\bigoplus_{a=0}^{k-1}\bigoplus_{v=0}^{n} \mathcal{M}^{(v)}_{a,k-a-1}\Big)\\
&\oplus (\underline{z}-\underline{z}^\dagger)\Big(\bigoplus_{a=0}^{k-2}\bigoplus_{v=1}^{n-1} \Big(\frac{\underline{z}}{a+n-v} + \frac{\underline{z}^\dagger}{k-a-2+v}\Big)\mathcal{M}^{(v)}_{a,k-a-2}\Big).
\end{aligned}$$

This means that for each spherical harmonic $H_k \in \mathcal{H}_k(\mathbb{R}^{2n};\mathbb{S})$ there exist spherical Hermitean monogenics

$$\begin{aligned}
f^{(v)}_{a,k-a} &\in \mathcal{M}^{(v)}_{a,k-a} && (a=0,\dots,k; v=0,\dots,n)\\
g^{(v)}_{a,k-a-1} &\in \mathcal{M}^{(v)}_{a,k-a-1} && (a=0,\dots,k-1;\ v=1,\dots,n-1)\\
h^{(v)}_{a,k-a-1} &\in \mathcal{M}^{(v)}_{a,k-a-1} && (a=0,\dots,k-1;\ v=0,\dots,n)\\
u^{(v)}_{a,k-a-2} &\in \mathcal{M}^{(v)}_{a,k-a-2} && (a=0,\dots,k-2;\ v=1,\dots,n-1)
\end{aligned}$$

such that

$$\begin{aligned}
H_k = &\Big(\sum_{a=0}^{k}\sum_{v=0}^{n} f^{(v)}_{a,k-a}\Big) + \Big(\sum_{a=0}^{k-1}\sum_{v=1}^{n-1} \Big(\frac{\underline{z}}{a+n-v} + \frac{\underline{z}^\dagger}{k-a-1+v}\Big) g^{(v)}_{a,k-a-1}\Big)\\
&+ \Big(\sum_{a=0}^{k-1}\sum_{v=0}^{n} (\underline{z}-\underline{z}^\dagger)\, h^{(v)}_{a,k-a-1}\Big)\\
&+ \Big(\sum_{a=0}^{k-2}\sum_{v=1}^{n-1} \Big(\frac{\underline{z}\,\underline{z}^\dagger}{k-a-2+v} - \frac{\underline{z}^\dagger\underline{z}}{a+n-v}\Big) u^{(v)}_{a,k-a-2}\Big).
\end{aligned}$$

Fixing a bidegree and a spinor-homogeneity degree the above decomposition yields

$$\begin{aligned}
H^{(v)}_{a,k-a} = f^{(v)}_{a,k-a} &+ \underline{z}\Big(\frac{g^{(v-1)}_{a-1,k-a}}{a+n-v} + h^{(v-1)}_{a-1,k-a}\Big) + \underline{z}^\dagger\Big(\frac{g^{(v+1)}_{a,k-a-1}}{k-a+v} - h^{(v+1)}_{a,k-a-1}\Big)\\
&+ \Big(\frac{\underline{z}\,\underline{z}^\dagger}{k-a-1+v} - \frac{\underline{z}^\dagger\underline{z}}{a-1+n-v}\Big) u^{(v)}_{a-1,k-a-1}
\end{aligned}$$

meaning that

$$\begin{aligned}
\mathcal{H}^{(v)}_{a,k-a} = \mathcal{M}^{(v)}_{a,k-a} &\oplus \underline{z}\ \mathcal{M}^{(v-1)}_{a-1,k-a} \oplus \underline{z}^\dagger\ \mathcal{M}^{(v+1)}_{a,k-a-1}\\
&\oplus \Big(\frac{\underline{z}\,\underline{z}^\dagger}{k-a-1+v} - \frac{\underline{z}^\dagger\underline{z}}{a-1+n-v}\Big)\mathcal{M}^{(v)}_{a-1,k-a-1}
\end{aligned}$$

which is precisely (8). Moreover we may now also determine the projection operators from $\mathcal{H}^{(v)}_{a,k-a}$ onto the U(n)-irreducibles involving spherical Hermitean monogenics. With the notations from above we successively obtain:

$$u^{(v)}_{a-1,k-a-1} = \frac{1}{k+n-1}\, \partial^\dagger_{\underline{z}} \wedge \partial_{\underline{z}}[H^{(v)}_{a,k-a}]$$

$$\Big(\frac{g^{(v+1)}_{a,k-a-1}}{k-a+v} - h^{(v+1)}_{a,k-a-1}\Big) = \frac{1}{k-a+v}\Big(\partial^\dagger_{\underline{z}} + \frac{\underline{z}}{a-1+n-v}\partial^\dagger_{\underline{z}} \wedge \partial_{\underline{z}}\Big)[H^{(v)}_{a,k-a}]$$

$$\Big(\frac{g^{(v-1)}_{a-1,k-a}}{a+n-v} + h^{(v-1)}_{a-1,k-a}\Big) = \frac{1}{a+n-v}\Big(\partial_{\underline{z}} - \frac{\underline{z}^\dagger}{k-a-1+v}\partial^\dagger_{\underline{z}} \wedge \partial_{\underline{z}}\Big)\,[H^{(v)}_{a,k-a}]$$

and

$$\begin{aligned} f^{(v)}_{a,k-a} &= \Big(\mathbf{1} - \frac{\underline{z}\partial_{\underline{z}}}{a+n-v} - \frac{\underline{z}^\dagger \partial^\dagger_{\underline{z}}}{k-a+v}\Big)[H^{(v)}_{a,k-a}] \\ &\quad - \Big(\frac{\underline{z}^\dagger \underline{z}(\partial^\dagger_{\underline{z}} \wedge \partial_{\underline{z}})}{(k-a+v)(k+n-1)} - \frac{\underline{z}\underline{z}^\dagger(\partial^\dagger_{\underline{z}} \wedge \partial_{\underline{z}})}{(a+n-v)(k+n-1)}\Big)\,[H^{(v)}_{a,k-a}] \end{aligned}$$

7. Dimensional analysis

Fischer decompositions of spaces of polynomials allow for dimension counting, which we will do in a systematic way in this section, first confirming well-known formulae for the spaces of spherical harmonics and spherical monogenics, and then establishing a dimension result for spaces of spherical Hermitean monogenics.

First recall that

$$\dim(\mathcal{P}_k(\mathbb{R}^m;\mathbb{C})) = D^k_m = \binom{m+k-1}{k}$$

D^k_m denoting the number of k-combinations of an m-element set, repetition being allowed. It follows that $\dim(\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S})) = 2^n D^k_{2n}$, since $\dim(\mathbb{S}) = 2^n$. In the same order of ideas we have

$$\dim(\mathcal{P}^{(v)}_{a,b}) = D^a_n D^b_n \ \dim(\mathbb{S}^{(v)}) = \binom{n}{v} D^a_n D^b_n$$

and observe that indeed

$$\sum_{a+b=k}\sum_{v=0}^{n} \dim(\mathcal{P}^{(v)}_{a,b}) = \left(\sum_{v=0}^{n}\binom{n}{v}\right)\Big(\sum_{a=0}^{k} D^a_n D^{k-a}_n\Big) = 2^n\, D^k_{2n} = \dim\big(\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S})\big)\,.$$

Next, from the Fischer decomposition (1) it follows that

$$\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S}) = \mathcal{H}_k(\mathbb{R}^{2n};\mathbb{S}) \oplus r^2\ \mathcal{P}_{k-2}(\mathbb{R}^{2n};\mathbb{S})$$

which yields for $k \geq 2$

$$\begin{aligned} h_k \equiv \dim\big(\mathcal{H}_k(\mathbb{R}^{2n};\mathbb{S})\big) &= \dim\big(\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S})\big) - \dim\big(\mathcal{P}_{k-2}(\mathbb{R}^{2n};\mathbb{S})\big) \\ &= 2^n(D^k_{2n} - D^{k-2}_{2n}) = 2^n\,\frac{2n+2k-2}{2n+k-2}\binom{2n+k-2}{k} \end{aligned}$$

while $h_0 = 2^n$, $h_1 = 2^{n+1}n$. In the same order of ideas we find for $a > 0$ and $b > 0$

$$\begin{aligned} h_{a,b}^{(v)} \equiv \dim\left(\mathcal{H}_{a,b}^{(v)}(\mathbb{R}^{2n};\mathbb{S})\right) &= \binom{n}{v}(D_n^a D_n^b - D_n^{a-1} D_n^{b-1}) \\ &= \binom{n}{v}\binom{n+a-2}{a}\binom{n+b-1}{b}\frac{n+a+b-1}{n+b-1} \end{aligned}$$

yielding

$$\begin{aligned} \sum_{a=0}^{k}\sum_{v=0}^{n}\dim\left(\mathcal{H}_{a,k-a}^{(v)}(\mathbb{R}^{2n};\mathbb{S})\right) &= 2^n\sum_{a=0}^{k}(D_n^a D_n^{k-a} - D_n^{a-1} D_n^{k-a-1}) \\ &= 2^n(D_{2n}^k - D_{2n}^{k-2}) = \dim\left(\mathcal{H}_k(\mathbb{R}^{2n};\mathbb{S})\right) \end{aligned}$$

as it should.

Now, from the Fischer decomposition (2) it follows that

$$\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S}) = \mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S}) \oplus \underline{X}\ \mathcal{P}_{k-1}(\mathbb{R}^{2n};\mathbb{S})$$

yielding for $k > 0$

$$\begin{aligned} m_k \equiv \dim\left(\mathcal{M}_k(\mathbb{R}^{2n};\mathbb{S})\right) &= \dim\mathcal{P}_k(\mathbb{R}^{2n};\mathbb{S}) - \dim\mathcal{P}_{k-1}(\mathbb{R}^{2n};\mathbb{S}) \\ &= 2^n\,(D_{2n}^k - D_{2n}^{k-1}) = 2^n\,D_{2n-1}^k = 2^n\binom{2n+k-2}{k} \end{aligned}$$

while $m_0 = h_0 = 2^n$. Note that

$$\dim(\mathcal{M}_k) + \dim(\mathcal{M}_{k-1}) = 2^n\,(D_{2n}^k - D_{2n}^{k-2}) = \dim(\mathcal{H}_k)$$

which is in accordance with (7).

Finally, putting $m_{a,b}^{(v)} = \dim(\mathcal{M}_{a,b}^{(v)})$, with $m_{a,b}^{(v)} = 0$ whenever $a < 0$ or $b < 0$ or $v < 0$ or $v > n$, we deduce from the Fischer decomposition (8) that

$$h_{a,k-a}^{(v)} = m_{a,k-a}^{(v)} + m_{a-1,k-a}^{(v-1)} + m_{a,k-a-1}^{(v+1)} + m_{a-1,k-a-1}^{(v)} \tag{13}$$

This means that the dimension of the spaces $\mathcal{M}_{a,b}^{(v)}$ of spherical Hermitean monogenics may be calculated recursively from the dimensions of the spaces of spherical harmonics:

$$\begin{aligned} m_{0,0}^{(v)} &= h_{0,0}^{(v)} \\ m_{0,1}^{(v)} &= h_{0,1}^{(v)} - h_{0,0}^{(v+1)} \\ m_{1,0}^{(v)} &= h_{1,0}^{(v)} - h_{0,0}^{(v-1)} \\ m_{0,2}^{(v)} &= h_{0,2}^{(v)} - h_{0,1}^{(v+1)} + h_{0,0}^{(v+2)} \\ m_{1,1}^{(v)} &= h_{1,1}^{(v)} - h_{1,0}^{(v+1)} - h_{0,1}^{(v-1)} + h_{0,0}^{(v)} \end{aligned}$$

$$m_{2,0}^{(v)} = h_{2,0}^{(v)} - h_{1,0}^{(v-1)} + h_{0,0}^{(v-2)}$$
$$m_{0,3}^{(v)} = h_{0,3}^{(v)} - h_{0,2}^{(v+1)} + h_{0,1}^{(v+2)} - h_{0,0}^{(v+3)}$$
$$m_{1,2}^{(v)} = h_{1,2}^{(v)} - h_{1,1}^{(v+1)} - h_{0,2}^{(v-1)} + h_{1,0}^{(v+2)} + h_{0,1}^{(v)} - h_{0,0}^{(v+1)}$$
$$m_{2,1}^{(v)} = h_{2,1}^{(v)} - h_{1,1}^{(v-1)} - h_{2,0}^{(v+1)} + h_{1,0}^{(v)} + h_{0,1}^{(v-2)} - h_{0,0}^{(v-1)}$$
$$m_{3,0}^{(v)} = h_{3,0}^{(v)} - h_{2,0}^{(v-1)} + h_{1,0}^{(v-2)} - h_{0,0}^{(v-3)}$$

etc. According to (12), these dimensions should satisfy

$$m_k = \sum_{a=0}^{k} \sum_{v=0}^{n} m_{a,k-a}^{(v)} + \sum_{a=0}^{k-1} \sum_{v=1}^{n-1} m_{a,k-a-1}^{(v)}$$

by means of which the correctness of the obtained results may be checked.

However, solving the recurrence relations (13) explicitly in order to obtain a closed form for $m_{a,k-a}^{(v)}$ turns out to be too complicated. Fortunately, the dimension of the spaces of spherical Hermitean monogenics may also be calculated in an alternative way. To this end we consider the Weyl dimension formula (see [21, p.301]) for the dimension of an irreducible finite dimensional representation of a simple Lie algebra $\mathfrak{g}$. This formula contains products over all positive roots of $\mathfrak{g}$, the number of which is increasing quickly, whence the formula is difficult to use in explicit calculations. Yet, in some cases significant simplifications occur. In particular in the present case, for representations of the algebra $\mathfrak{su}(n)$, a simplified formula may be used involving the so–called hook numbers, see [21, p.382]. Characterizing an irreducible representation by its highest weight λ, a Young (or Ferrers) diagram may be associated to it, which consists of left justified rows of boxes, each row containing as many boxes as indicated by the corresponding component of λ. Each box then has a hook number associated to its position in the diagram, which can be calculated following a simple rule: if there are x boxes in the diagram to the right of the considered one and y boxes below, then the hook number is $x + y + 1$. The Weyl dimension formula for the module with highest weight $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{n-1}]$ then takes the form

$$\frac{(\lambda_1 + n - 1)!}{(n-1)!} \frac{(\lambda_2 + n - 2)!}{(n-2)!} \cdots \frac{(\lambda_{n-1} + 1)!}{1!} \frac{1}{\Pi_{i,j\in\lambda} h_{i,j}}$$

where the product is taken over all hook numbers $h_{i,j}$ associated to all boxes in the diagram.

Now, the space $\mathcal{M}_{a,b}^{(v)}$ is a $\mathfrak{su}(n)$-module with highest weight

$$\lambda = [a + b + 1, b + 1, \ldots, b + 1, b, \ldots, b]$$

where the last $b + 1$ appears at the $(n - v)$-th place, see [13]. The corresponding Young diagram, with the hook numbers written in the corresponding boxes, is shown above, leading for $0 < v < n$ to the following expression for $\dim \mathcal{M}_{a,b}^{(v)}$:

$$m_{a,b}^{(v)} = \frac{a+b+n}{a+n-v} \binom{b+v-1}{b} \binom{b+n-1}{n-v-1} \binom{a+n-1}{a}$$

$a+b$ $+n-1$	$\cdots$	$a+n+1$	$a+n$	$a+n-v$	a	$\cdots$	2	1
$b+n-2$	$\cdots$	n	$n-1$	$n-v-1$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$b+v+1$	$\cdots$	$v+3$	$v+2$	2				
$b+v$	$\cdots$	$v+2$	$v+1$	1				
$b+v-2$	$\cdots$	v	$v-1$					
$\vdots$	$\vdots$	$\vdots$	$\vdots$					
$b+1$	$\cdots$	3	2					
b	$\cdots$	2	1					

FIG. 1: Ferrer diagram with hook numbers for $\mathcal{M}_{a,b}^{(v)}$

which has been checked to be in accordance with the recurrence relations (13).

As already mentioned above for $v = 0$ the spaces of spherical Hermitean monogenics are nothing else but the spaces of scalar valued holomorphic homogeneous polynomials in the variables $(z_1, \ldots, z_n)$, implying that $b = 0$. Hence $m_{a,0}^{(0)} = \dim \mathcal{P}_{a,0}^{(0)} = D_n^a = \binom{a+n-1}{a}$, which is confirmed by the Weyl dimension formula for the highest weight $[a, 0, \ldots, 0]$. Similarly, for $v = n$ we end up with anti-holomorphic homogeneous polynomials in the variables $(z_1^c, \ldots, z_n^c)$, implying that $a = 0$. Hence $m_{0,b}^{(n)} = \dim \mathcal{P}_{0,b}^{(n)} = D_n^b = \binom{b+n-1}{b}$, which is confirmed by the Weyl dimension formula for the highest weight $[b, b, \ldots, b]$.

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