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SPECIAL EINSTEIN'S EQUATIONS ON KÄHLER MANIFOLDS

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Abstract. This work is devoted to the study of Einstein equations with a special shape of the energy-momentum tensor. Our results continue Stepanov's classification of Riemannian manifolds according to special properties of the energy-momentum tensor to Kähler manifolds. We show that in this case the number of classes reduces.

1. Introduction

The geometric properties of (pseudo-) Riemannian manifolds V_n, depending on the form the Einstein equations acquire in them, were studied by many authors. A large number of papers is devoted to the study of Einstein's equations with certain restrictions on the energy-momentum tensor and its first covariant derivatives [2, 5, 6, 7, 9].

S. E. Stepanov [8, 10] classified space-time manifolds according to certain relations among the first covariant derivatives of the energy-momentum tensor. He found three fundamental classes, related to geometrical assumptions about space-time. By combinations of the conditions determining the three fundamental classes he found three further classes. A seventh class is characterized by the vanishing of the covariant derivative of the energy-momentum tensor.

In the present paper we partially take over Stepanov's classification to Kähler spaces and investigate analogous, generalized classifying conditions. We show that for two out of the three fundamental classes space-time is Ricci symmetric and the energy-momentum tensor is covariantly constant.

In consequence, the energy-momentum tensor is covariantly constant also for the three classes derived from the fundamental ones. Thus for Kähler spaces the number of classes of Einstein equations reduces to one with covariantly constant and one with non-constant energy-momentum tensor. We study some of their properties and generalizations.

All geometric objects are formulated locally under the assumption of sufficient smoothness. Whereas S. E. Stepanov formulated his classifications by making use of bundles, for our purpose it is sufficient to write down the classifying relations in form of tensor equations.

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2. Einstein's equations

The equation $\mathrm{Ric} - \frac{1}{2}\, R\, g = T$ is called *Einstein's equation*. Here Ric is the Ricci tensor on the manifold V_n, g is the metric tensor, R is the scalar curvature, and T is the energy-momentum tensor. In components this reads as follows:

$$R_{ij} - \frac{1}{2}\, R\, g_{ij} = T_{ij}\,, \tag{1}$$

where R_{ij}, g_{ij} and T_{ij} are components of Ric, g and T.

From the Bianchi identities of the Ricci tensor follows $T_{\alpha i,\beta} g^{\beta\alpha} = 0$, (where the comma denotes the covariant derivative with respect to a connection on the manifold V_n), and g^{ij} are elements of the inverse matrix to g_{ij}.

Stepanov distinguishes the following three fundamental types of manifolds in terms of covariant derivatives of the energy-momentum tensor:

$$\Omega_1: \qquad T_{ij,k} + T_{jk,i} + T_{ki,j} = 0\,, \tag{2}$$

$$\Omega_2: \qquad T_{ij,k} - T_{ik,j} = 0\,, \tag{3}$$

$$\Omega_3: \qquad T_{ij,k} = a_k g_{ij} + b_i g_{jk} + b_j g_{ik}\,, \tag{4}$$

where a_k and b_i are arbitrary vectors.
In space-time manifolds of type Ω_1 the scalar curvature is covariantly constant and the Ricci tensor is a Killing tensor, i.e. $R_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$ is constant along geodesic curves with parameter s.

In the case Ω_2 the scalar curvature is constant, too, and the Levi-Civita connection of the metric, considered as a connection on the tangent bundle TM, satisfies the conditions of a Yang-Mills potential.

Ω_3 is a slight generalization in comparison with the condition in [8, 10] on $R_{ij,k}$, reformulated in terms of $T_{ij,k}$. The original conditions of Stepanov characterise manifolds with non-constant curvature that admit non-trivial geodesic mappings.

In [8, 10] three further classes are derived by simultanously imposing conditions Ω_1 and Ω_2, Ω_2 and Ω_3, and Ω_1 and Ω_3, respectively.

Using a generalized form of the introduced dependencies, we are going to study manifolds characterized by the following conditions:

$$\Omega_1^*: \quad T_{ij,k} + T_{jk,i} + T_{ki,j} = \lambda_k T_{ij} + \lambda_i T_{jk} + \lambda_j T_{ki} + \mu_k g_{ij} + \mu_i g_{jk} + \mu_j g_{ik}, \tag{5}$$

$$\Omega_2^*: \quad T_{ij,k} - T_{ik,j} = \rho_k T_{ij} - \rho_j T_{ik} + \sigma_k g_{ij} - \sigma_j g_{ik}, \tag{6}$$

$$\Omega_3^*: \quad T_{ij,k} = \phi_k T_{ij} + \gamma_i T_{jk} + \gamma_j T_{ki} + \eta_k g_{ij} + \chi_i g_{jk} + \chi_j g_{ik}, \tag{7}$$

where ϕ_i, λ_i, μ_i, ρ_i, γ_i, η_i, σ_i and χ_i are arbitrary vectors.

3. Kähler spaces

An n-dimensional (pseudo-)Riemannian manifold (M_n, g) is called a *Kähler space* K_n if besides the metric tensor g, a *structure* F, which is an affinor (i.e. a tensor field of type $(1,1)$), is given on M_n such that the following holds [3, 4, 11]:

$F^2 = -\operatorname{Id}$, $g(FX, X) = 0$ $(\forall X \in TK_n)$, $\nabla F = 0$, i.e. in coordinate form

$$F^h_\alpha F^\alpha_i = -\delta^h_i\,; \qquad F^\alpha_i g_{\alpha j} + F^\alpha_j g_{\alpha i} = 0\,; \qquad F^h_{i,j} = 0\,, \tag{8}$$

where δ^i_j is the Kronecker symbol.

Making use of this we can show that

$$g_{ij} = g_{\alpha\beta} F^\alpha_i F^\beta_j\,; \quad R_{ij} = R_{\alpha\beta} F^\alpha_i F^\beta_j\,. \tag{9}$$

Then due to (1), for the energy-momentum tensor the following relation holds

$$T_{ij} = T_{\alpha\beta} F^\alpha_i F^\beta_j\,; \qquad F^\alpha_i T_{\alpha j} + F^\alpha_j T_{\alpha i} = 0\,. \tag{10}$$

We prove the following theorem.

Theorem 1. *If in a Kähler space the condition Ω_2^* or Ω_3^* holds, then the energy-momentum tensor satisfies*

$$T_{ij,k} = \rho_k T_{ij} + \sigma_k g_{ij}\,. \tag{11}$$

Proof. Assume that in a Kähler space K_n the condition (6) holds, multiply it by $F^i_l F^j_h$, contract with respect to i and j and exchange l for i and h for j. We obtain

$$\begin{aligned} T_{\alpha\beta,k} F^\alpha_i F^\beta_j - T_{\alpha k,\beta} F^\alpha_i F^\beta_j &= \rho_k T_{\alpha\beta} F^\alpha_i F^\beta_j - \rho_\beta T_{\alpha k} F^\alpha_i F^\beta_j \\ &\quad + \sigma_k g_{\alpha\beta} F^\alpha_i F^\beta_j - \sigma_\beta g_{\alpha k} F^\alpha_i F^\beta_j\,. \end{aligned} \tag{12}$$

With the aid of (9) and (10) we can rewrite the last equation in the form

$$T_{ij,k} - T_{\alpha k,\beta} F^\alpha_i F^\beta_j = \rho_k T_{ij} - \rho_\beta T_{\alpha k} F^\alpha_i F^\beta_j + \sigma_k g_{ij} - \sigma_\beta g_{\alpha k} F^\alpha_i F^\beta_j\,. \tag{13}$$

After symmetrization of the indices i and k we get

$$T_{ij,k} + T_{jk,i} = \rho_k T_{ij} + \rho_i T_{jk} + \sigma_k g_{ij} - \sigma_i g_{jk}\,. \tag{14}$$

Exchanging the indices i and j we obtain

$$T_{ij,k} + T_{ik,j} = \rho_k T_{ij} + \rho_j T_{ik} + \sigma_k g_{ij} + \sigma_j g_{ik}\,. \tag{15}$$

Addition of (15) and (13) gives (11). Note that spaces satisfying Ω_3^* satisfy also the condition Ω_2^* as can be seen, when

$$\rho_i = \phi_i - \gamma_i; \quad \sigma_i = \eta_i - \chi_i \tag{16}$$

holds. □

By analyzing this result it is not difficult to prove

Theorem 2. *Kähler spaces K_n belonging to class Ω_2 or Ω_3 are characterized by the following conditions*

$$T_{ij,k} = 0\,, \qquad R_{ij,k} = 0\,. \tag{17}$$

From this theorem it follows immediately that for Kähler spaces also in the derived cases (Ω_1 and Ω_2, Ω_2 and Ω_3, Ω_1 and Ω_3) the energy-momentum tensor is covariantly constant. So all the classes of Einstein equations, with the exception of Ω_1, can be summarized under the characterization $T_{ij,k} = 0$. From this follows that for Kähler spaces K_n of class Ω_i (respectively Ω_i^*) only those fulfilling condition (2) (resp. (5)) are relevant.

In a further step of generalization we consider Kähler spaces characterized by the following conditions

$$\Omega_4^*: \quad T_{ij,k} - T_{ik,j} = \rho_k T_{ij} - \rho_j T_{ik} + \sigma_k g_{ij} - \sigma_j g_{ik} + \rho_\alpha T_{i\beta} F_k^\alpha F_j^\beta$$

$$- \rho_\beta T_{i\alpha} F_k^\alpha F_j^\beta + \sigma_\alpha g_{i\beta} F_k^\alpha F_j^\beta - \sigma_\beta g_{i\alpha} F_k^\alpha F_j^\beta \,. \tag{18}$$

$$\Omega_5^*: \quad T_{ij,k} = \phi_k T_{ij} + \gamma_i T_{jk} + \gamma_j T_{ki} + \eta_k g_{ij} + \chi_i g_{jk} + \chi_j g_{ik}$$

$$+ \gamma_\alpha T_{\beta k} F_i^\alpha F_j^\beta + \gamma_\beta T_{k\alpha} F_i^\alpha F_j^\beta + \chi_\alpha g_{\beta k} F_i^\alpha F_j^\beta + \chi_\beta g_{\alpha k} F_i^\alpha F_j^\beta \,. \tag{19}$$

Applying the methods used in the proof of Theorem 1 to (18) and taking into account (8), (9), (10) we convince ourselves that (18) acquires the form (19), this proves the next theorem.

Theorem 3. *There are no Kähler spaces K_n in the class Ω_4^* other than spaces belonging to Ω_5^*.*

In this way the Kähler spaces with non-constant energy-momentum tensor, considered in this paper, are divided into two essential classes: Ω_1^* and Ω_5^*.

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