ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 351–361

GEOMETRY OF THIRD ORDER ODE SYSTEMS

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Abstract. We compute cohomology spaces of Lie algebras that describe differential invariants of third order ordinary differential equations. We prove that the algebra of all differential invariants is generated by 2 tensorial invariants of order 2, one invariant of order 3 and one invariant of order 4. The main computational tool is a Serre-Hochschild spectral sequence and the representation theory of semisimple Lie algebras. We compute differential invariants up to degree 2 as application.

1. Introduction

The main aim of the article is to compute cohomology spaces which describe invariants of a system of 3rd order ODEs. Geometry of differential equations of finite type is studied in the paper [3] and is based on the interpretation of differential equation as first-order geometric structures on filtered manifolds.

A filtered manifold is a smooth manifold equipped with a filtration of the tangent bundle compatible with the Lie bracket of vector fields. The theory of geometric structures on filtered manifold was developed in the works of N. Tanaka [8, 9] and T. Morimoto [7]. We recall that a *system of a filtered manifold* is defined as a graded nilpotent Lie algebra associated with the filtration of the tangent space at a fixed point. It plays the key role in the study of structures on filtered manifolds. The symbol of a geometric structure is defined in a similar way. It consists of a graded Lie algebra $\mathfrak{g}$ with the negative part $\mathfrak{g}_- = \sum_{i<0} \mathfrak{g}_i$ equal to the symbol of a filtered manifold and of a certain subalgebra $\mathfrak{g}_0$ of $\mathrm{Der}_0(\mathfrak{g}_-)$ defined by the geometric structure.

If some additional conditions on the Lie algebra $\mathfrak{g}$ are satisfied, then every geometric structure on a filtered manifold with symbol $\mathfrak{g}$ can be equipped in a natural way with a so-called *normal Cartan connection*. In particular this allows to solve the problem of local equivalence of these structures and to describe differential invariants for each structure of this kind. It is known (see [10]) that the generators in the algebra of differential invariants are in the one-to-one correspondence with the positive part of the cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$. Thus, one of the major steps in study of geometric structures on filtered manifolds is the computation of this space.

2000 *Mathematics Subject Classification*: primary 34A26; secondary 17B56, 53B15.

Key words and phrases: geometry of ordinary differential equations, normal Cartan connections, cohomology of Lie algebras.

In this paper we compute explicitly cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ in the case, when the Lie algebra $\mathfrak{g}$ corresponds to an arbitrary system of 3rd order ODEs. The case of a single ordinary differential equation is studied in [1]. In the cases of $n \geq 1$ ordinary differential equations of second order and a single equation of third order the Lie algebra $\mathfrak{g}$ is simple, and the computation of cohomology spaces $H^2(\mathfrak{g}_-, \mathfrak{g})$ can be done with the help of Kostant theorem [6].

We consider the case of 3rd order ODEs with the number of equations two or more. In this case the symbol $\mathfrak{g}$ of a system of 3rd order ODEs is no longer semisimple. That is why Kostant theorem is not applicable. To compute $H^2(\mathfrak{g}_-, \mathfrak{g})$ space we apply Serre spectral sequence defined by the abelian ideal of $\mathfrak{g}$. This sequence stabilizes in the second term, so that we can obtain our results.

2. System of 3rd order ODEs symbol

A symbol $\mathfrak{g}$ of system, which consists of m equations of $(n+1)$-order, is equal to the semidirect product of the Lie algebra $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{gl}_m(\mathbb{R})$ and an abelian ideal V. The ideal V has form $V_n \otimes W$, where V_n is an irreducible $\mathfrak{sl}_2(\mathbb{R})$-module of dimension $n+1$ and W is the standard representation of $\mathfrak{gl}_m(\mathbb{R})$. Let us denote a Lie algebra $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{gl}_m(\mathbb{R})$ as $\mathfrak{a}$.

This article is dedicated to studying third order equation. We work with the following symbol:

$$\mathfrak{g} = (\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{gl}_m(\mathbb{R})) \leftthreetimes (V_2 \otimes W)\,.$$

Let us fix a basis of the algebra $\mathfrak{g}$ in the following way. Let x, y, h be the standart basis if an algebra $\mathfrak{sl}_2(\mathbb{R})$ with relations:

$$[x, y] = h\,, \quad [h, x] = 2x\,, \quad [h, y] = -2y\,.$$

In the matrix form this basis is the following:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let vectors e_0, e_1, e_2 define a basis of the module V_2, which is fixed by relations $xe_i = e_{i-1}$. We denote the standart basis of the vector space W and the induced basis of the Lie algebra $\mathfrak{gl}_m(\mathbb{R})$ as E_i and E^i_j. We have equality $E^i_j E_i = E_j$.

Let us define a grading of the Lie algebra $\mathfrak{g}$ in the following way:

$$\begin{aligned}
\mathfrak{g}_1 &= \langle y \rangle\,, \\
\mathfrak{g}_0 &= \langle h, E^i_j \rangle\,, \\
\mathfrak{g}_{-1} &= \langle x \rangle + \langle e_2 \otimes W \rangle\,, \\
\mathfrak{g}_{-2} &= \langle e_1 \otimes W \rangle\,, \\
\mathfrak{g}_{-3} &= \langle e_0 \otimes W \rangle\,.
\end{aligned}$$

According to results of [3], we have the correspondence between invariants of an ordinary differential equation and the positive part of the second cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$. We use Serre spectral sequence [4] to compute this cohomology space. Let $E^{p,q}_i$ be spectral sequence corresponding to the ideal V.

The following lemmas are either elementary or easy to prove by direct computation.

Lemma 1. *The space of cohomology classes $H^2(\mathfrak{g}_-, \mathfrak{g})$ is the direct sum of two spaces $E_2^{1,1}$ and $E_2^{0,2}$, where $E_2^{1,1}$ and $E_2^{0,2}$ have the following form:*

$$E_2^{1,1} = H^1\big(\mathbb{R}x, H^1(V, \mathfrak{g})\big)\,,$$
$$E_2^{0,2} = H^0\big(\mathbb{R}x, H^2(V, \mathfrak{g})\big)\,.$$

Let $V = V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_k}$ be the irreducible decomposition of an arbitrary $\mathfrak{sl}_2$-module V. It is easy to see that

$$H^n(\mathbb{R}x, V) = \oplus_{i=1}^k H^n(\mathbb{R}x, V_{n_i})\,.$$

Lemma 2. *Let V_m be an irreducible $\mathfrak{sl}_2$-module and v_0 and v_m be the vectors of V_m such that $x \cdot v_0 = 0$ and $y \cdot v_m = 0$. Then*

$$H^0(\mathbb{R}x, V_m) = \mathbb{R}v_0\,,$$
$$H^1(\mathbb{R}x, V_m) = \mathbb{R}x^* \otimes v_m\,.$$

In other words Lemma 2 is equivalent to the following:

$$H^0(\mathbb{R}x, V_m) = \mathrm{Inv}_x(V_m)\,,$$
$$H^1(\mathbb{R}x, V_m) = \mathrm{Inv}_y(V_m)\,.$$

We describe cohomology spaces $H^n(V, \mathfrak{g})$ by means of the Spenser operator $\mathcal{S}$:

$$\mathcal{S}^n\colon\ \mathrm{Hom}(\wedge^n V, \alpha) \to \mathrm{Hom}(\wedge^{n+1} V, V)\,,$$
$$\mathcal{S}^n(\varphi)(v_1 \wedge v_2 \wedge \cdots \wedge v_{n+1}) = \sum_{i=1}^{n+1} (-1)^i \varphi(v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_{n+1}) v_i\,.$$

Lemma 3. *We have $H^0(V, \mathfrak{g}) = V$ and*

$$H^n(V, \mathfrak{g}) = \ker \mathcal{S}^n \oplus \mathrm{Hom}(\wedge^n V, V)/\operatorname{im} \mathcal{S}^{n-1}\ .$$

Next lemma allows us to calculate the degree of cohomology space vectors.

Lemma 4. *For any cohomology class $[c] \in H^k(\mathfrak{g}_-, \mathfrak{g})$ we have*

$$h \cdot c = \alpha c\,, \quad z \cdot c = \beta c\,,$$

where $z = \sum_{i=1}^n E_i^i$. Then cohomology class $[c]$ is part of $H_p^k(\mathfrak{g}_-, \mathfrak{g})$ space, where

$$p = -2\beta - \frac{\alpha}{2}\,.$$

Lemma 5. *Map $\mathcal{S}^1$ is injective.*

Proof. Nagano [5] lists all linear reductive Lie algebras, for which map $\mathcal{S}^1$ has non-zero kernel. The algebra $\mathfrak{a} \subset \mathfrak{gl}(V)$ is not present there. This proves the lemma. □

3. Characterization of the space $E_2^{1,1}$

In this part we describe $E_2^{1,1}$ part of the cohomology space $H^2(\mathfrak{g}_-,\mathfrak{g})$.

Theorem 1. *The space $E_2^{1,1}$ has the following form*

$$x^*\otimes\left(\mathbb{R}y^2\otimes\mathfrak{gl}(W)+\mathbb{R}y\otimes\mathfrak{sl}(W)\right).$$

Elements $\varphi\in E_2^{1,1}$ which have form $\varphi\colon\mathbb{R}x\to\mathbb{R}y\otimes\mathfrak{sl}(W)$ have degree 2 and elements which have form $\varphi\colon\mathbb{R}x\to\mathbb{R}y^2\otimes\mathfrak{gl}(W)$ have degree 3.

Proof. We know that $E_2^{1,1}\cong H^1(\mathbb{R}x,H^1(V,\mathfrak{g}))$. The space $H^1(V,\mathfrak{g})$ is equal to:

$$\ker\mathcal{S}^1\oplus\operatorname{Hom}(V,V)/\operatorname{im}\mathcal{S}^0\ .$$

From Lemma 5 we have $\ker\mathcal{S}^1=0$. The operator $\mathcal{S}^0$ has the following form:

$$\mathcal{S}^0\colon\ \operatorname{Hom}(\mathbb{R},\mathfrak{a})\to\operatorname{Hom}(V,V)\,,\quad \mathcal{S}^0(\varphi)(v)=-\varphi(1)(v)\,.$$

Thus $\operatorname{im}\mathcal{S}^0=\mathfrak{a}$. Therefore:

$$E_2^{1,1}=H^1(\mathbb{R}x,\mathfrak{gl}(V)/\mathfrak{a})=\operatorname{Inv}_y\left(\mathfrak{gl}(V)\right).$$

The space $\operatorname{Inv}_y\left(\mathfrak{gl}(V)\right)$ has the following structure

$$\mathbb{R}y^2\otimes\mathfrak{gl}(W)\oplus\mathbb{R}y\otimes\mathfrak{sl}(W)\,.$$

This fact follows from the decomposition

$$\mathfrak{gl}(V)\cong V_2^*\otimes W^*\otimes V_2\otimes W\cong(V_4\oplus V_2\oplus V_0)\otimes\mathfrak{gl}(W)\,.$$

Applying Lemma 4 we get degree of these cohomology elements. This completes the proof. □

4. Characterization of the space $E_2^{2,0}$

Let us describe the space $\operatorname{Inv}_x H^2(V,\mathfrak{g})$. As we know,

$$H^2(V,\mathfrak{g})=\frac{\ker\mathcal{S}^2\oplus\operatorname{Hom}(V\wedge V,V)}{\operatorname{im}\mathcal{S}^1}\,.$$

The space $\ker\mathcal{S}^2$ is described by the following theorem.

Theorem 2. *The space $\ker\mathcal{S}^2$ is equal as an $\mathfrak{a}$-module to a space:*

$$\begin{aligned}&V_0\otimes\mathcal{S}^2(W^*)+V_2\otimes\mathrm{id}_W\,, &&\textit{if}\quad m=2\,,\\ &V_0\otimes\mathcal{S}^2(W^*)\,, &&\textit{if}\quad m\geq 3\,.\end{aligned}$$

Proof. We show that $V_0\otimes\mathcal{S}^2(W^*)\in\ker\mathcal{S}^2$. The module V_2 is equal to algebra $\mathfrak{sl}_2(\mathbb{R})$ as an $\mathfrak{sl}_2(\mathbb{R})$-module. The isomorphism has the following form:

$$v_0\to-2x,v_1\to h,v_2\to y\,. \tag{1}$$

Isomorphism (1) induces an isomorphism $\alpha\colon V\wedge V\to\mathfrak{sl}_2$:

$$\alpha(v_1\wedge v_2)=[v_1,v_2]\,.$$

For every bilinear form $\beta \in \mathcal{S}^2(W^*)$ define the cochain $\alpha_\beta \colon \wedge^2 V \to \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{gl}_m(\mathbb{R})$ in the following way:

$$\alpha_\beta(v \otimes v, v' \otimes w') = [v, v'] \cdot \beta(w, w') + K(v, v')\widehat{\beta}(w, w'),$$

where the form $\widehat{\beta} \in \S^2 W^* \otimes \mathfrak{gl}(W)$ is

$$\widehat{\beta}(w, w')w'' = \beta(w, w'')w' - \beta(w', w'')w'.$$

A form $K(v, v')$ is the Killing form of $\mathfrak{sl}_2(\mathbb{R})$. Let us verify that $\mathcal{S}^2\,\alpha = 0$. Denote a cyclic sum of three variables as $\{\ \}$. We have:

$$\begin{aligned}\mathcal{S}^2\,\alpha(v \otimes w, v' \otimes w', v'' \otimes w'') &= \{[[v, v'], v''] \otimes \beta(w, w')w''\} \\ &\quad + \{K(v, v')v'' \otimes (\beta(w, w'')w' - \beta(w', w'')w)\} \\ &= \{([[v, v'], v''] + K(v', v'')v - K(v, v'')v') \otimes \beta(w, w')w''\}.\end{aligned}$$

Moreover, it is easy to verify that:

$$[[v, v'], v''] + K(v', v'')v - K(v, v'')v' = 0.$$

Direct computation shows that $V_0 \otimes \mathcal{S}^2(W^*)$ is a unique cohomology class in case $m > 2$. In the case $m = 2$ computation shows that $\dim \ker \mathcal{S}^2 = 6$ and it has a structure given above. $\square$

Now we describe the structure of $E_2^{0,2} = \mathrm{Inv}_x\, H^2(V, \mathfrak{g})$.

Theorem 3. *The space $E_2^{0,2}$ has the following parts in the direct sum decomposition*

Space	*contained in*	*degree*
$V_6 \otimes \wedge^2(W^*) \otimes W$	$\mathrm{Hom}(\wedge^2 V, V)/\,\mathrm{im}\,\mathcal{S}^1$	0
$V_4 \otimes S_0^2(W^*) \otimes W$	$\mathrm{Hom}(\wedge^2 V, V)/\,\mathrm{im}\,\mathcal{S}^1$	0
$V_4 \otimes \wedge^2(W^*) \otimes W$	$\mathrm{Hom}(\wedge^2 V, V)/\,\mathrm{im}\,\mathcal{S}^1$	0
$V_2 \otimes \wedge_0^2 W^* \otimes W$	$\mathrm{Hom}(\wedge^2 V, V)/\,\mathrm{im}\,\mathcal{S}^1$	1
$V_0 \otimes S_0^2(W^*) \otimes W$	$\mathrm{Hom}(\wedge^2 V, V)/\,\mathrm{im}\,\mathcal{S}^1$	2
$V_0 \otimes S^2(W^*)$	$\ker \mathcal{S}^2 \subset \mathrm{Hom}(\wedge^2 V, \mathfrak{a})$	4
$V_2,\ m = 2$	$\ker \mathcal{S}^2 \subset \mathrm{Hom}(\wedge^2 V, \mathfrak{a})$	3

Here we denote traceless part of spaces $S^2(W^*) \otimes W$ and $\wedge^2(W^*) \otimes W$ as $S_0^2(W^*) \otimes W$ and $\wedge_0^2(W^*) \otimes W$ respectively.

Proof. The space $\mathrm{Hom}(\wedge^2 V, V)$ has the following $\mathfrak{sl}_2(\mathbb{R})$ decomposition:

$$\begin{aligned}\mathrm{Hom}(\wedge^2 V, V) &= \big(S^2(V_2^*) \otimes \wedge^2 W^* + \wedge^2 V_2^* \otimes S^2(W^*)\big) \otimes V_2 \otimes W \\ &= \big((V_4 \oplus V_0) \otimes V_2\big) \otimes \wedge^2 W^* \otimes W + V_2 \otimes V_2 \otimes S^2(W^*) \otimes W \\ &= (V_6 \oplus V_4 \oplus 2V_2) \otimes \wedge^2 W^* \otimes W + (V_4 \oplus V_2 \oplus V_0) \otimes S^2(W^*) \otimes W.\end{aligned}$$

Note that the space $\operatorname{im}\mathcal{S}^1$ is equal to $\operatorname{Hom}(V,\mathfrak{a})$, which has the following structure:

$$\begin{aligned}\operatorname{Hom}(V,\mathfrak{a}) = V_2^* \otimes W^* \otimes (V_2 + V_0 \otimes W^* \otimes W) = (V_4 \oplus V_2 \oplus V_0) \otimes W^* \\ + V_2 \otimes W^* \otimes W^* \otimes W\,.\end{aligned}$$

Let $\alpha \in \operatorname{Hom}(V,\mathfrak{a})$. Then the map $\mathcal{S}^1(\alpha) \in \operatorname{Hom}(V \wedge V, V)$ acts in the following way:

$$\mathcal{S}^1(\alpha)(v_1 \otimes w_1, v_2 \otimes w_2) = \alpha(v_1 \otimes w_1)\cdot v_2 \otimes w_2 - \alpha(v_2 \otimes w_2)\cdot v_1 \otimes w_1\,. \tag{2}$$

We see from the formula (2) that

$$\begin{aligned}&\operatorname{Hom}(\wedge^2 V, V)/(V_2^* \otimes W^* \otimes W^* \otimes W) = \\ &\qquad = (V_6 \oplus V_4 \oplus V_2) \otimes \wedge^2 W^* \otimes W \oplus (V_4 \oplus V_0) \otimes S^2(W^*) \otimes W.\end{aligned}$$

It follows from (**??**) that

$$\begin{aligned}V_2 \otimes W^* &\subset V_2 \otimes \wedge^2 W^* \otimes W\,,\\ V_0 \otimes W^* &\subset V_0 \otimes S^2(W^*) \otimes W\,.\end{aligned}$$

The space $V_2 \otimes \wedge^2 W^* \otimes W / V_2 \otimes W^*$ is identified with traceless part of $V_2 \otimes \wedge^2 W^* \otimes W$ space, the space $V_0 \otimes S^2(W^*) \otimes W / V_0 \otimes W^*$ is identified with traceless part of $V_0 \otimes S^2(W^*) \otimes W$. □

5. Normal Cartan connection

Consider a system of third-order ordinary differential equations of the form

$$(y^i)''' = f^i\big(x, y^j, (y^k)', (y^l)''\big)\,, \tag{3}$$

where i, $j = 1,\dots,m$ with $m \geq 2$. It determines a holonomic differential equation $\mathcal{E} \subset J^3(\mathbb{R}^{m+1}, 1)$ whose solutions are 1-dimensional submanifolds in $\mathbb{R}^{m+1}$. The functions

$$x, y_1, \dots, y_m, p_1 = y_1^1, \dots, p_m = y_m^1\,, \quad q_1 = y_1^1, \dots, q_m = y_m^1$$

form a coordinate system on the equation $\mathcal{E}$.

We choose the following co-frame on $\mathcal{E}$:

$$\begin{aligned}\theta^x &= dx\,;\\ \theta^i_{-1} &= dq^i - f^i(x,y,p,q)\,dx\,, && i = 1,\dots,m\,;\\ \theta^i_{-2} &= dp^i - q^i\,dx\,, && i = 1,\dots,m\,;\\ \theta^i_{-3} &= dy^i - p^i\,dx\,, && i = 1,\dots,m\,.\end{aligned}$$

We use below Einstein summation convention. For any function $F \in C^\infty(\mathcal{E})$, we have

$$dF = \frac{dF}{dx}\theta^x + \frac{\partial F}{\partial y^i}\theta^i_{-3} + \frac{\partial F}{\partial p^i}\theta^i_{-2} + \frac{\partial F}{\partial q^i}\theta^i_{-1}\,;$$

where

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + p^i \frac{\partial F}{\partial y^i} + q^i \frac{\partial F}{\partial p^i} + f^i \frac{\partial F}{\partial q^i} \,.$$

The coframe θ has the following structure equations:

$$\begin{aligned} d\theta^x &= 0 \\ d\theta^i_{-3} &= \theta^x \wedge \theta^i_{-2} \\ d\theta^i_{-2} &= \theta^x \wedge \theta^i_{-1} \\ d\theta^i_{-1} &= \frac{\partial f^i}{\partial y^j}\theta^x \wedge \theta^i_{-3} + \frac{\partial f^i}{\partial p^j}\theta^x \wedge \theta^i_{-2} + \frac{\partial f^i}{\partial q^j}\theta^x \wedge \theta^i_{-1}. \end{aligned}$$

From [3] we know that there is a Cartan connection on $\mathcal{E}$ with model G/H that is naturally associated with the equation (3). The group G is a semisimple product:

$$G = (SL(2,\mathbb{R}) \times GL(m,\mathbb{R})) \leftthreetimes (V(2) \otimes W) \;,$$

Let $\mathfrak{h}$ be the nonnegative part of $\mathfrak{g}$:

$$\mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_1 \,.$$

Assume that H is a subgroup of G with the Lie algebra $\mathfrak{h}$.

$$H = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \times A \,, \quad a \in \mathbb{R}^* \,,\; b \in \mathbb{R} \,,\; A \in \mathrm{GL}_m(\mathbb{R}) \,.$$

Let us define the family of Cartan connections adapted to the equation (3). We call a co-frame $\omega^x, \omega^i_{-1}, \omega^i_{-2}, \omega^i_{-3}$ adapted to equation (3) if the following conditions holds:

$$\begin{aligned} \langle \omega^i_{-3} \rangle &= \langle \theta^i_{-3} \rangle \,, \\ \langle \omega^i_{-3}, \omega^i_{-2} \rangle &= \langle \theta^i_{-3}, \theta^i_{-2} \rangle \,, \\ \langle \omega^i_{-3}, \omega^i_{-2}, \omega^i_{-1} \rangle &= \langle \theta^i_{-3}, \theta^i_{-2}, \theta^i_{-1} \rangle \,, \\ \langle \omega^i_{-3}, \omega^i_{-2}, \omega^x \rangle &= \langle \theta^i_{-3}, \theta^i_{-2}, \theta_x \rangle \,. \end{aligned}$$

We can express this conditions in coordinate free manner. First, there is a filtration

$$C = C^{-1} \subset C^{-2} \subset C^{-3} = T\mathcal{E} \,,$$

where $C^{-i-1} = [C^{-i}, C^{-1}]$. There is a one-dimensional distribution E whose integral curves are the lifts of solutions of equation (3). Let π^2_1 be the canonical projection from the second jet space $J^2(M)$ to the first jet space $J^1(M)$. We define kernel of the projection π^2_1 as F. This kernel F is an m-dimensional distribution. The distribution C is the direct sum of the distributions E and F. The distributions E, V have the form

$$\begin{aligned} E &= \Big\langle \frac{\partial}{\partial x} + p_i \frac{\partial}{\partial y_i} + q_i \frac{\partial}{\partial p_i} + f^i \frac{\partial}{\partial q_i} \Big\rangle \,; \\ V &= \Big\langle \frac{\partial}{\partial q_i} \Big\rangle \,, \end{aligned}$$

where $i, j = 1, \ldots, m$. We say that coframe is adapted to the equation (3) if

– the annihilator of forms ω^i_{-3}, ω^i_{-2}, ω^x is V;
– the annihilator of forms ω^i_{-3}, ω^i_{-2}, ω^i_{-1} is E;
– the annihilator of forms ω^i_{-1} is C^{-2}.

Let $\pi\colon P \to \mathcal{E}$ be a principle H-bundle. We say that a Cartan connection $\overline{\omega}$ on a principal H-bundle is adapted to equation (3), if for any local section s of π the set $\{s^*\overline{\omega}^x, s^*\overline{\omega}^i_{-1}, s^*\overline{\omega}^i_{-2}, s^*\omega^i_{-3}\}$ is an adapted co-frame on $\mathcal{E}$.

Now any adapted Cartan connection can be written down explicitly. Let $\overline{\omega}\colon TP \to \mathfrak{g}$ be the Cartan connection that we are looking for. We shall write $\overline{\omega}$ as follows:

$$\overline{\omega} = \overline{\omega}^i_{-3} v_0 \otimes e_i + \overline{\omega}^i_{-2} v_1 \otimes e_i + \overline{\omega}^i_{-1} v_2 \otimes e_i + \overline{\omega}^x x + \overline{\omega}^h h + \overline{\omega}^i_j e^j_i + \overline{\omega}^h h\,.$$

Lemma 6. *We can uniquely define a section $s\colon \mathcal{E} \to P$ by the following conditions*

$$\begin{aligned} s^*\overline{\omega}^i_{-3} &= \theta^i_{-3}\,,\\ s^*\overline{\omega}^h &\equiv 0 \mod \langle \theta^i_{-3}, \theta^i_{-2}, \theta^i_{-1}\rangle\,,\\ s^*\overline{\omega}^x &\equiv -\theta^x \mod \langle \theta^i_{-3}, \theta^i_{-2}, \theta^i_{-1}\rangle\,. \end{aligned}$$

Now we define $\omega\colon P \to \mathfrak{g}$ by the formula $\omega = s^*\overline{\omega}$. Let $\overline{\Omega}$ be the curvature tensor of $\overline{\omega}$, and let $\Omega = s^*\overline{\Omega}$. We see that

$$\begin{aligned} \Omega &= \Omega^i_{-3} v_0 \otimes e_i + \Omega^i_{-2} v_1 \otimes e_i + \Omega^i_{-1} v_2 \otimes e_i + \Omega^x x + \Omega^h h + \Omega^j_i e^i_j + \Omega^y y\\ &= (d\omega^i_{-3} + \omega^x \wedge \omega^i_{-2} + 2\omega^h \wedge \omega^i_{-3} + \omega^i_j \wedge \omega^j_{-3}) v_0 \otimes e_i\\ &\quad + (d\omega^i_{-2} + \omega^x \wedge \omega^i_{-1} + \omega^i_j \wedge \omega^j_{-2} + 2\omega^y \wedge \omega^i_{-3}) v_1 \otimes e_i\\ &\quad + (d\omega^i_{-1} - 2\omega^h \wedge \omega^i_{-1} + \omega^i_j \wedge \omega^j_{-1} + 2\omega^y \wedge \omega^i_{-2}) v_2 \otimes e_i\\ &\quad + (d\omega^x + 2\omega^h \wedge \omega^x) x + (d\omega^h + \omega^x \wedge \omega^y) h\\ &\quad + (d\omega^i_j + \omega^i_k \wedge \omega^k_j) e^j_i + (d\omega^y - 2\omega^h \wedge \omega^y) y\,. \end{aligned}$$

Now we define an arbitrary Cartan connection adapted to equation (3) as

$$\begin{aligned} \omega^i_{-3} &= \theta^i_{-3}\,;\\ \omega^i_{-2} &= \alpha^i_j \theta^j_{-2} + A^{i-3}_j \theta^j_{-3}\,;\\ \omega^i_{-1} &= \beta^i_j \theta^j_{-1} + B^{i-2}_j \theta^j_{-2} + C^{i-3}_j \theta^j_{-3}\,;\\ \omega^x &= -\theta^x + d_j \theta^j_{-2} + e^{-3}_j \theta^j_{-3}\,;\\ \omega^h &= F^{-1}_j \theta^j_{-1} + F^{-2}_j \theta^j_{-2} + F^{-3}_j \theta^j_{-3}\,;\\ \omega^i_j &= G^{i,x}_j \theta^x + G^{i,-1}_{jk} \theta^k_{-1} + G^{i,-2}_{jk} \theta^k_{-2} + G^{i,-3}_{jk} \theta^k_{-3}\,;\\ \omega^y &= H^x \theta^x + H^{-1}_j \theta^j_{-1} + H^{-2}_j \theta^j_{-2} + H^{-3}_j \theta^j_{-3}\,. \end{aligned}$$

We assign a degree to each of the coefficients in the above expressions by assuming that all these equalities are homogeneous. We will compute the normal Cartan connection by applying normal conditions on curvature degree by degree.

Degree 0. There are two following differentials in this degree:

$$d\omega^i_{-3} = \theta^x \wedge \theta^i_{-2}\,;$$
$$d\omega^i_{-2} = \theta^x \wedge \theta^i_{-1} + A^i_j\theta^x \wedge \theta^j_{-2} + dA^i_j \wedge \theta^j_{-3}\,.$$

We have only two nonzero components:

$$\Omega^i_{-3} \mod \langle \theta_{-2}\wedge\theta_{-2}, \theta_{-3}\rangle = \theta^x \wedge \theta^i_{-2} - \alpha^i_j\theta^x \wedge \theta^j_{-2}\,;$$
$$\Omega^i_{-2} \mod \langle \theta_{-2}, \theta_{-3}\rangle = \theta^x \wedge \theta^i_{-1} - \beta^i_j\theta^x \wedge \theta^j_{-1}\,.$$

We see from these equalities that we can make all Degree 0 coefficients equal to zero. Thus, we obtain $\alpha^i_j = \delta^i_j$ and $\beta^i_j = \delta^i_j$.

Degree 1. We will use one additional differential:

$$d\omega^i_{-1} = \frac{\partial f^i}{\partial q^j}\theta^x \wedge \theta^i_{-1} + \frac{\partial f^i}{\partial p^j}\theta^x \wedge \theta^i_{-2} + \frac{\partial f^i}{\partial y^j}\theta^x \wedge \theta^i_{-3}\,.$$

We have only 4 nonzero components. The first component is:

$$\Omega^i_{-3} \mod \langle \theta_{-2}\wedge\theta_{-3}, \theta_{-3}\wedge\theta_{-3}\rangle =$$
$$-\,\theta^x \wedge A^j_i\theta^j_{-3} + d_j\theta^j_{-2}\wedge\theta^i_{-2} + G^{i,x}_j\theta^x\wedge\theta^j_{-3} + G^{i,-1}_{jk}\theta^k_{-1}\wedge\theta^j_{-3} + 2F^{-1}_j\theta^j_{-1}\wedge\theta^i_{-3}\,.$$

We can make every component of Ω^i_{-3} equal to 0 and get

$$d_j = 0\,,\; G^{i,x}_j = A^i_j\,,\; G^{i,-1}_{jk} = -2\delta^j_i F^{-1}_k\,.$$

The second component is:

$$\Omega^i_{-2} \mod \langle \theta_{-2}\wedge\theta_{-2}, \theta_{-3}\rangle =$$
$$A^i_j\theta^x \wedge \theta^j_{-2} + d^{-2}_j\theta^j_{-2}\wedge\theta^i_{-1} - \theta^x\wedge B^i_j\theta^j_{-2} + G^{i,x}_j\theta^x \;\wedge\theta^j_{-2}$$
$$+\,G^{i,-1}_{jk}\theta^k_{-1}\wedge\theta^j_{-2} + 2H^{-1}_j\theta^j_{-1}\wedge\theta^i_{-2}\,.$$

Zero condition on this part of curvature implies:

$$B^i_j = 2A^i_j,\, G^{i,-1}_{jk} = 0, F^{-1}_k = 0\,.$$

The third component is:

$$\Omega^i_{-1} \mod \langle \theta_{-2}, \theta_{-3}\rangle =$$
$$\frac{\partial f^i}{\partial q^j}\theta^x\wedge\theta^j_{-1} + B^i_j\theta^x\wedge\theta^i_{-1} - 2F^{-1}_j\theta^j_{-1}\wedge\theta^i_{-1} + G^{i,x}_j\theta^x\wedge\theta^j_{-1} + G^{i,-1}_{jk}\theta^k_{-1}\wedge\theta^j_{-1}\,.$$

Now zero conditions give us:

$$A^i_j = -\frac{1}{3}\frac{\partial f^i}{\partial q^j}\,,\; G^{i,-1}_{jk} = 2F^{-1}_k\delta^i_j\,.$$

The last component $\Omega^x \mod \langle\theta_{-2},\theta_{-3}\rangle$ is equal to zero.

Let us summarize conditions on the Cartan connection we obtain in Degree 1:

$$A^i_j = G^{i,x}_j = \frac{1}{2}B^i_j = -\frac{1}{3}\frac{\partial f^i}{\partial q^j}\,,\quad F^{-1}_j = G^{i,-1}_{jk} = 0\,.$$

Degree 2. Starting from this degree calculation becomes more complicated. Therefore we will list only parts of calculation we need to obtain invariants and the normal form of the Cartan connection.

$$\begin{aligned}&\Omega^i_{-2} \quad \mathrm{mod}\ \langle \theta_{-2}\wedge\theta_{-3}, \theta_{-3}\wedge\theta_{-3}\rangle = \\ &\frac{dA^i_j}{dx}\theta^x\wedge\theta^j_{-3} + \frac{\partial A^i_j}{\partial q_k}\theta^k_{-1}\wedge\theta^j_{-3} - \theta^x\wedge C^i_j\theta^j_{-3} + e_j\theta^j_{-3}\wedge\theta^i_{-1} + G^{i,x}_j\theta^x\wedge A^j_k\theta^k_{-3} \\ &\qquad + 2H^x\theta^x\wedge\theta^i_{-3} + 2H^{-1}_j\theta^j_{-1}\wedge\theta^i_{-3} + G^{i,-2}_{j_k}\theta^k_{-2}\wedge\theta^j_{-2} + G^{i,-1}_{j_k}\theta^{-1}_k\wedge A^j_l\theta^{-3}_l\,.\end{aligned}$$

In coefficient $\theta^k_{-1}\wedge\theta^j_{-3}$ we obtain the first invariant. We denote it by I_2:

$$\frac{\partial A^i_j}{\partial q_k} - e_j\delta^i_k + 2H^{-1}_k\delta^i_j\,.$$

Explicitly, invariant I_2 will be the following:

$$I_2 = \mathrm{tr}_0\left(\frac{\partial^2 f^i}{\partial q_j\partial q_k}\right),$$

there by tr_0 we denote traceless part of the tensor. Coefficient $\theta^x\wedge\theta^j_{-3}$ we can make equal to zero and get:

$$C^i_j = \frac{dA^i_j}{dx} + A^i_kA^k_j + 2H^k\delta^i_j\,.$$

In coefficient $\theta_x\wedge\theta^j_{-2}$ of the curvature part Ω^i_{-1} we obtain the generalized Wilczynski invariant (see [2]). We denote it by W_2.

$$\begin{aligned}&\Omega^i_{-1} \quad \mathrm{mod}\ \langle \theta_{-2}\wedge\theta_{-2}, \theta_{-3}\rangle = \\ &\frac{\partial f^i}{\partial p_j}\theta^x\wedge\theta^j_{-2} + 2\frac{dA^i_j}{dx}\theta^x\wedge\theta^j_{-2} + 2\frac{\partial A^i_j}{\partial q_k}\theta^k_{-1}\wedge\theta^j_{-2} + C^i_j\theta^x\wedge\theta^i_{-2} - 2F^{-2}_j\theta^j_{-2}\wedge\theta^i_{-1} \\ &\qquad + G^{i,-2}_{jk}\theta^k_{-2}\wedge\theta^j_{-1} + 2H^x\theta^x\wedge\theta^i_{-2} + 2H^{-1}_j\theta^j_{-1}\wedge\theta^i_{-2} + G^{i,x}_k\theta^x\wedge B^k_j\theta^j_{-2}\,.\end{aligned}$$

The coefficient $\theta^x\wedge\theta^j_{-2}$ is the following:

$$\frac{\partial f^i}{\partial p_j} + 2\frac{dA^i_j}{dx} + C^i_j + 2H^x + 2A^i_kA^k_j = \frac{\partial f^i}{\partial p_j} + 3\frac{dA^i_j}{dx} + 3A^i_kA^k_j + 4H^x\delta^i_j\,.$$

The second degree generalized Wilczynski invariant is the following:

$$W_2 = tr_0\Big(\frac{\partial f^i}{\partial p_j} - \frac{d}{dx}\frac{\partial f^i}{\partial q_j} + \frac{1}{3}\frac{\partial f^i}{\partial q_k}\frac{\partial f^k}{\partial q_j}\Big)\,.$$

From Theorems 1 and 3 we know that I_2 and W_2 is are the only invariants in Degree 2.

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