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AN INTEGRAL FORMULA OF HYPERBOLIC TYPE FOR SOLUTIONS OF THE DIRAC EQUATION ON MINKOWSKI SPACE WITH INITIAL CONDITIONS ON A HYPERBOLOID

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Abstract. The Dirac equation for spinor-valued fields f on the Minkowski space of even dimension form a hyperbolic system of partial differential equations. In the paper, we are showing how to reconstruct the solution from initial data given on the upper sheet H^+ of the hyperboloid. In particular, we derive an integral formula expressing the value of f in a chosen point p as an integral over a compact cycle given by the intersection of the null cone with H^+ in the Minkowski space $\mathbb{M}$.

1. Introduction

Quaternionic analysis has a long history. It describes function theory, which is a generalization of complex function theory to four dimensions. Holomorphic functions are replaced by solutions of the Fueter equation (see [13] for an elegant review). A principal role in the theory is played by a higher-dimensional generalization of the Cauchy integral formula. It has also very close connection to physics. In fact, the Fueter equation itself can be identified with the Dirac equation on the four-dimensional Euclidean space. Applications of quaternionic analysis in physics were developed by Gürsey and Tze [9], applications to numerical methods and engineering problems can be found in [8]. Relations to twistor theory and integral formulae for massless fields developed by R. Penrose were described in [12].

In the 1960s Clifford analysis started to be studied in a systematic way by V. Inftimie and R. Delanghe independently as a generalization of quaternionic analysis into higher dimensions (see [1, 4]). It is a function theory for solutions of the Dirac equation on Euclidean space $\mathbb{R}^n$ (which are called monogenic functions). A generalization of the Cauchy integral formula is the central result of the theory. It states that the value of a spinor-valued monogenic function f at a point $p \in \mathbb{R}^n$ can be expressed as an integral of the function over a sphere around the point p multiplied with a suitable kernel (generalization of the standard Cauchy kernel for holomorphic functions). Integral formulae of this type are typical for elliptic (systems of) PDE's. Due to the fact that the (vector-valued) differential form in the

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formula is closed, there is a wide freedom to deform the hypersurface of integration without changing the value of the integral.

Solutions of the Dirac equation are real analytic and can be extended as holomorphic functions to (a suitable open subset of) the complexification $\mathbb{C}^n$. The cycle of integration in the formula can be hence deformed from the (real) Euclidean space into its complexification. Using this idea, it was possible to deduce from the Cauchy integral formula in Clifford analysis suitable integral formulae for higher dimensional generalization of massless fields of various spins on Minkowski space (see [5, 12]). They were then used in a description of the Penrose transform for these systems of PDE's (see [4, 2]).

The Fueter equation is invariant with respect to the group of rotations (in fact, even with respect to the conformal group in dimension four). As a consequence, finite dimensional representations of the group of rotations play an important role in the theory (see, e.g.,[3, 11]). It was recently stressed by I. Frenkel and M. Libine [7] that quaternionic analysis can be efficiently applied in the study of properties of infinite dimensional representations of the conformal group, its complexification and various related real versions. In particular, in [7] they recalled the Cauchy integral formula for solutions of the Fueter equation and employed it to develop a corresponding integral formula also for a hyperbolic case, i.e. for solutions of the Dirac equation on the Minkowski space. A two dimensional version of it can be found also in the paper [11].

There is, however, a substantial difference between formulae for solutions of the Dirac equation on Minkowski space developed in [7] and in [5, 12, 4, 2], respectively. In the former case, the dimension of the cycle of integration is three and it does not change with the point P (where the value of the field is being computed), and only values of the field on the contour of integration are needed. In the latter case, the dimension of the cycle of integration is two, the cycle of integration depends on the point P and values together with certain derivatives of the field are needed. This is the essential difference between integral formulae of elliptic and hyperbolic type (see [4]).

An important observation is that the integral formula from [7] has still a character of integral formulae for elliptic systems, while the integral formulae of hyperbolic type are already adapted to the case of hyperbolic systems. It is well known that the Huyghens principle holds for solutions of the wave equation in even dimensions. It means that the value of the wave f in a point p can be obtained from the values of the field and its derivatives on the intersection of the initial data surface and the null cone of the point p. As the initial data surface we can take one sheet of the hyperboloid. Notice that values of the field outside of the intersection are not needed for the reconstruction of the value $f(p)$. In the integral formula developed in [7], this is not obvious. Using integral formulae of hyperbolic type, it is also clear that no assumptions on behaviour of the field at infinity are needed.

The case treated in [7] is exactly the case when the value of the field is reconstructed from values of the field on (the upper sheet of) the hyperboloid. The aim of the paper is to use methods described in [5, 4] to compute explicitly how to reconstruct the value of the field in a point using initial data for the field given on the the upper sheet of the hyperboloid. In the Section 2, we summarize briefly the integral formula for solutions of the Dirac equation developed in [7]. The next section is devoted to a description of a general scheme that allows us to compute integral formulae for hyperbolic systems using the Leray residue and the Leray cobord (see [4]). Finally, in the Section 4, methods described in the Section 3 are applied to determine the integral formula for analytic solutions of the Dirac equation explicitly in any even dimension.

There is also a substantial work done on hyperbolic Clifford analysis by David Eelbode and Heinz Leutwiler. A lot of facts can be found in the thesis by D. Eelbode ([6]) and subsequent papers. He works in the so called ambient space which is a vector space with the Minkowski scalar product. The homogeneous functions on the forward null cone are being considered, i.e. densities on the projective version of the forward null cone. Hence the base space is the hyperbolic space and solutions of the Dirac equation there are solutions of an elliptic system on a Riemannian manifold. The theory for solutions of this system of equations is being developed systematically. Similarly, Leutwiler (see [10]) considers also solutions of the Dirac equation on the hyperbolic space but with vector values (the Riesz system).

We study the case where the base space is not the Riemannian manifold but the flat vector space with Minkowski scalar product (i.e. the flat pseudo-Riemannian manifold). The corresponding system of equations is then not elliptic but hyperbolic. This is the reason why integral formula discussed in the paper has the character of hyperbolic integral formulae (see the discussion above).

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2. An integral formula of elliptic type

In this part, we shall shortly review the formulation of the integral formula for solutions of the Fueter equation on the Minkowski space as developed in [7]. They use the following notation.

Let $\mathbb{H}_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{H}$ be the algebra of complexified quaternions. We can then realize the real quaternions and the Minkowski space as

$$\mathbb{H} = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^c = Z\} \tag{1}$$

$$\mathbb{M} = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^{c+} = -Z\} \tag{2}$$

where for $Z \in \mathbb{H}_{\mathbb{C}}$; $Z = z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3$ the involutions are defined as

$$Z^c = \overline{z^0} e_0 + \overline{z^1} e_1 + \overline{z^2} e_2 + \overline{z^3} e_3 \quad \text{complex conjugation} \tag{3}$$

$$Z^+ = z^0 e_0 - z^1 e_1 - z^2 e_2 - z^3 e_3 \quad \text{quaternionic conjugation} \tag{4}$$

If we identify the units e_0, e_1, e_2, e_3 with

$$e_0 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad e_2 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 \leftrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

we can think of $\mathbb{H}_{\mathbb{C}}$ as complex 2×2 matrices

$$\mathbb{H}_{\mathbb{C}} = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix};\ z_{ij} \in \mathbb{C} \right\}, \tag{5}$$

where $z_{11} = z^0 - iz^3$ and so on. We observe that Z^{c+} is the matrix adjoint Z^*.

It is easy to see that for any $Z \in \mathbb{H}_{\mathbb{C}}$ the expression $N(Z) = ZZ^+$ is a quadratic form over $\mathbb{C}$, in particular, $N(Z) \in \mathbb{C}$. Hence we can define the inverse of an element $Z \in \mathbb{H}_{\mathbb{C}}$, $N(Z) \neq 0$ by

$$Z^{-1} = \frac{Z^+}{N(Z)}.$$

The group U(2) is then realized as

$$\mathrm{U}(2) = \{\mathrm{Z} \in \mathbb{H}_{\mathbb{C}};\ \mathrm{Z}^* = \mathrm{Z}^{-1}\}. \tag{6}$$

Fix an open subset $U \subset \mathbb{M}$ and let f be a differentiable function on U with values in $\mathbb{H}_{\mathbb{C}}$. For $Y \in \mathbb{M}$ write

$$Y = -ie_0 y^0 + e_1 y^1 + e_2 y^2 + e_3 y^3. \tag{7}$$

Definition 1. The function f is called left-regular in U if it satisfies

$$\partial_{\mathbb{M}} f(Y) = ie_0 \frac{\partial f}{\partial y^0}(Y) + e_1 \frac{\partial f}{\partial y^1}(Y) + e_2 \frac{\partial f}{\partial y^2}(Y) + e_3 \frac{\partial f}{\partial y^3}(Y) = 0, \qquad \forall\, Y \in U. \tag{8}$$

Lemma 1. *Consider an element* $\gamma = \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{H}_{\mathbb{C}})$ *with* $\gamma^{-1} = \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix}$. *The fractional linear map on* $\mathbb{H}_{\mathbb{C}}$

$$\pi_l(\gamma)\colon Z \mapsto -i(Z+1)(Z-1)^{-1} \tag{9}$$

maps $\mathbb{M}$ *onto an open dense subset of* U(2). *Conversely,*

$$\pi_l(\gamma^{-1})\colon Z \mapsto (Z-i)(Z+i)^{-1} \tag{10}$$

maps an open dense subset of $U(2)$ *onto* $\mathbb{M}$. *In such a way,* U(2) *can be regarded as a compactification of* $\mathbb{M}$.

For the function $h\colon \mathbb{H}_{\mathbb{C}} \to \mathbb{H}_{\mathbb{C}}$ define the meromorphic action of an element $g \in \mathrm{GL}(2, \mathbb{H}_{\mathbb{C}})$ by

$$\pi_l(g)\colon h(Z) \mapsto (\pi_l(g)h)(Z) = \frac{(cZ+d)^{-1}}{N(cZ+d)} h((aZ+b)(cZ+d)^{-1}),$$

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{H}_{\mathbb{C}}).$$

Definition 2. The space of all left-regular functions f on $\mathbb{M}$ such that $\pi_l(\gamma^{-1})f$ extends holomorphically to an open neighbourhood of U(2) in $\mathbb{H}_{\mathbb{C}}$ will be denoted by $\nu(\mathbb{M})$. Holomorphic here means holomorphic in every component.

In other words, an element of the space $\nu(\mathbb{M})$ is a left-regular function on $\mathbb{M}$ that is also regular at infinity.

For $Y \in \mathbb{M}$ with $N(Y) < 0$ and $\epsilon \in \mathbb{R}$, we define

$$Y^\epsilon = \begin{cases} (1+i\epsilon)Y & \text{if} \quad i\operatorname{Tr} Y > 0\,; \\ (1-i\epsilon)Y & \text{if} \quad i\operatorname{Tr} Y < 0\,. \end{cases} \tag{11}$$

Let $R > 0$, and consider a two-sheeted hyperboloid $H_R = \{Y \in \mathbb{M};\, N(Y) = -R^2\}$. For $\epsilon \in \mathbb{R}$ we can define a deformed hyperboloid in $\mathbb{H}_\mathbb{C}$ as

$$H_R^\epsilon = \{Y^\epsilon;\, Y \in \mathbb{M},\, N(Y) = -R^2\}\,. \tag{12}$$

Theorem 1. *Let $f \in \nu(\mathbb{M})$ and let*

$$Dz = e_0 dz^1 \wedge dz^2 \wedge dz^3 - e_1 dz^0 \wedge dz^2 \wedge dz^3 \\ + e_2 dz^0 \wedge dz^1 \wedge dz^3 - e_3 dz^0 \wedge dz^1 \wedge dz^2\,.$$

Then, for $\epsilon > 0$ sufficiently small,

$$f(Y_0) = \frac{1}{2\pi^2} \int_{H_R^\epsilon - H_{R'}^{-\epsilon}} \frac{(Z-Y_0)^{-1}}{N(Z-Y_0)}\, Dz\, f(Z)\,, \quad \forall\, Y_0 \in \mathbb{M},\, N(Y_0) < 0, \tag{13}$$

for any $R, R' > 0$. In particular, the integral converges absolutely. Note that the multiplication is in the algebra of complexified quaternions.

3. Integral formulae of hyperbolic type

In this section we will review the theory of integral formulae in Clifford analysis as developed in [4]. All theorems quoted here can be found there together with their proofs.

We shall now describe the Cauchy integral formula for solutions of the Dirac equation in Clifford analysis in its homological form. Let S be a left module over the complex Clifford algebra $\mathbb{C}_{2n}$ and let $\Omega \subset \mathbb{C}^{2n}$ be a domain. We shall consider smooth maps $f\colon \Omega \to S$ satisfying the complex Dirac equation

$$\partial_{\underline{z}} f = 0\,, \qquad \partial_{\underline{z}} = \sum_{i=1}^{2n} e_i \frac{\partial}{\partial z_i} \quad \text{and} \quad e_i^2 = -1,\ i = 1, \dots, 2n\,. \tag{14}$$

The Cauchy kernel has similar form as in the complex case

$$\frac{-1}{A_{2n-1}} \frac{Q}{|Q|^{2n}}\,,$$

where A_{2n-1} is the area of S^{2n-2}, $Q \in \mathbb{C}^{2n}$ and

$$|Q|^2 = (Q, Q) = \sum_{i=1}^{2n} -Q_i^2\,. \tag{15}$$

The index of a $(2n-1)$-dimensional cycle Σ with respect to a point P is defined by

$$\operatorname{Ind}_\Sigma P = \frac{-1}{A_{2n-1}} \int_\Sigma \frac{(Q-P)}{|Q-P|^{2n}} DQ,$$

where $DQ = \sum_{j=1}^{2n} (-1)^{j+1} e_j dQ_1 \wedge \cdots \wedge \widehat{dQ_j} \wedge \cdots \wedge dQ_{2n}$, multiplication is the Clifford multiplication and $\{e_j\}_{j=1}^{2n}$ is the basis of $\mathbb{R}^{2n}$.

Theorem 2. *Let Σ be a $(2n-1)$-dimensional cycle in $\mathbb{C}^{2n}$ and let $\mathbb{C}N_P$ denote the complex null cone with vertex P. Then*

(i) $\mathrm{Ind}_\Sigma P$ *does not depend on the choice of a cycle representing an element in $H_{2n-1}(\mathbb{C}^{2n} \setminus \mathbb{C}N_P, \mathbb{Z})$.*

(ii) $\mathrm{Ind}_\Sigma P$ *is an integer.*

(iii) *The function* $\mathrm{Ind}_\Sigma P$ *is locally constant on its domain of definition.*

Theorem 3 (Elliptic Integral Formula)**.** *Let f be a solution of the Dirac equation on a domain $\Omega \subset \mathbb{C}^{2n}$ with values in S and suppose that P is a point in Ω such that $\Omega \cap \mathbb{C}N_P$ is star-shaped with respect to P. Then for each $(2n-1)$-dimensional cycle Σ in $\Omega \setminus \mathbb{C}N_P$ which is homologically trivial in Ω*

$$\mathrm{Ind}_\Sigma P\ f(P) = \int_\Sigma \omega\,, \tag{16}$$

where ω is given by

$$\omega = \omega(P, Q, f) = \frac{-1}{A_{2n-1}} \frac{(Q-P)}{|Q-P|^{2n}} \cdot DQ \cdot f(Q)\,, \tag{17}$$

and the $\cdot$ here means the action of a Clifford algebra $\mathbb{C}_{2n}$ on the left $\mathbb{C}_{2n}$-module S or on the algebra $\mathbb{C}_{2n}$ itself.

The key tool making the transformation of this formula into a formula of hyperbolic type possible, is the concept of the Leray residue and the Leray cobord. It can be summarized for our purposes as follows (see [4]).

Definition 3. Let M be a manifold of dimension m and $S \subset M$ a submanifold of dimension n. An oriented tubular neighbourhood of S in M is a pair (f, ξ), where $\xi = (\pi, E, S)$ is an oriented vector bundle over S where $\pi\colon E \to S$ is a projection from the total space E to the base space S, and $f\colon E \to M$ is an embedding such that

(i) f preserves orientations

(ii) $f|_S = 1_S$ (S being identified with the zero section of E)

(iii) $f(E)$ is an open neighbourhood of S in M.

Definition 4. An oriented closed tubular neighbourhood of radius $\epsilon > 0$ of S in M is the image $f(D_\epsilon)$ determined by a tubular neighbourhood (f, ξ) together with the norm $\|\cdot\|$ (coming from the positive definite scalar product on fibers, depending smoothly on a point) where $D_\epsilon = \{x \in E; \|x\| \le \epsilon\}$ is the disk subbundle of ξ with radius ϵ.

Definition 5. Let S and M be as before and let $U = f(D_1)$ be an oriented closed tubular neighbourhood given by a choice of $(f, \xi, \|.\|)$. Let $\pi\ :\ \partial U \to S$

be the induced projection and let i denote the inclusion $\partial U \subset M \setminus S$. Then for $p \geq m-n-1$ the map

$$\operatorname{Res} = \pi_* \circ i^* \; : \; H^p(M \setminus S) \to H^{p-(m-n-1)}(S) \tag{18}$$

is called the Leray-Norguet residue.

The cohomology and homology groups being dual to each other, there exists a dual map δ to the Leray-Norguet residue map called the Leray-Norguet cobord. Duality being expressed by the next theorem.

Theorem 4 (Leray Residue Theorem). *Let $0 \leq p \leq n$. Then for each closed $(p+m-n-1)$-form ω on $M \setminus S$ and for each p-dimensional cycle C on S we have*

$$\int_{\delta C} \omega = \int_C \operatorname{Res} \omega \,. \tag{19}$$

And we can formulate the integral formula of the hyperbolic type.

Theorem 5 (Hyperbolic Integral Formula). *Let f be a solution of the complex Dirac equation on a domain $\Omega \subset \mathbb{C}^{2n}$ and let P be a point in Ω such that $\Omega \cap \mathbb{C}N_P$ is star-shaped with respect to P. Then for each $(2n-2)$-dimensional cycle C in $\Omega \cap \mathbb{C}N_P$ we have*

$$\operatorname{Ind}_{\delta C} P \; f(P) = \int_C \operatorname{Res} \omega \,, \tag{20}$$

where $\operatorname{Res}\omega$ denotes the Leray-Norguet residue of the form

$$\omega = \omega(P, Q, f) = \frac{-1}{A_{2n-1}} \frac{(Q-P)}{|Q-P|^{2n}} \cdot DQ \cdot f(Q) \,,$$

A_{2n-1} is the area of S^{2n-2} and $DQ = \sum_{j=1}^{2n} (-1)^{j+1} e_j dQ_1 \wedge \cdots \wedge \widehat{dQ_j} \wedge \cdots \wedge dQ_{2n}$.

Example. Consider S^{2n-2} as an intersection of the complex null cone $\mathbb{C}N_P$ and the Euclidean space $\mathbb{R}^{2n}$ with $P \notin \mathbb{R}^{2n}$. Then $\operatorname{Ind}_{\delta S^{2n-2}} P = \pm 1$ depending on the orientation chosen.

Theorem 6. *Let f be a solution of the Dirac equation on a domain Ω and let P be a point in Ω. Let c be a smooth vector field on $U = \Omega \cap (\mathbb{C}N_P \setminus \{P\})$ such that $(c, c) = 0$ and $(b, c) = 1/2$ where $b = Q - P$. Then for $Q \in U$ and each $t_i \in T_Q \mathbb{C}N_P$, $i = 1, \ldots, 2n-2$,*

$$\begin{aligned} &\operatorname{res} \omega[t_1, \ldots, t_{2n-2}] = \\ &= \frac{-i}{\pi^{n-1}} \cdot \frac{\partial^{n-1}}{\partial \tau^{n-1}} \Big((b+\tau c) DQ[c, \nabla_{t_1}(b+\tau c), \ldots, \nabla_{t_{2n-2}}(b+\tau c)] f(Q+\tau c) \Big) \Big|_{\tau=0} \end{aligned} \tag{21}$$

is a representative of the cohomology class $\operatorname{Res}\omega$.

Lemma 2. *Suppose that the same conditions as in the Theorem 6 hold and that, moreover, all $t_i \in T_Q \mathbb{C}N_P$, $i = 1, \ldots, 2n-2$ are orthogonal to the vector $c(Q)$ for any $Q \in U$. If $\nabla_{t_i} b = t_i$ and $\tau \in \mathbb{C}$ then*

$$\begin{aligned} &(b+\tau c) DQ\big[c, \nabla_{t_1}(b+\tau c), \ldots, \nabla_{t_{2n-2}}(b+\tau c)\big] \\ &2bcdQ\big[b, c, \nabla_{t_1}(b+\tau c), \ldots, \nabla_{t_{2n-2}}(b+\tau c)\big] \end{aligned} \tag{22}$$

where the multiplication is the Clifford multiplication and $dQ = dQ_1 \wedge \cdots \wedge dQ_{2n}$.

4. A new integral formula for data given on a hyperboloid

We will compute a hyperbolic integral formula in the special case where the cycle of integration is an intersection of the upper sheet of a hyperboloid of two sheets H_R and a null cone N_p in the Minkowski space with the vertex $p \in \mathbb{M}$, where p has coordinates $p = (p_1, \dots, p_{2n})$ in the standard basis. In order to have the intersection with the upper sheet of the hyperboloid we have necessarily $(p,p) < 0$ and $p_{2n} > 0$, where (p,p) is the Minkowski inner product with the signature $(+, \dots, +, -)$.

First, we need to describe the cycle of integration. The hyperboloid of two sheets, is given by

$$\sum_{i=1}^{2n-1} x_i^2 - x_{2n}^2 = (x,x) = -R^2 \tag{23}$$

where for $x \in \mathbb{M}$ we write $x = (x_1, \dots, x_{2n})$ in the standard basis. The null cone in the Minkowski space has to satisfy the equation

$$\sum_{i=1}^{2n-1} (x_i - p_i)^2 - (x_{2n} - p_{2n})^2 = (x-p, x-p) = 0\,. \tag{24}$$

We further assume that $p \notin H_R$. Then the intersection is a real, codimension 2 hypersurface in the Minkowski space, it is moreover compact and we denote it by C.

Now, having determined the cycle of integration we continue by computing the residue of the form

$$\omega = \omega(p,x,f) = \frac{1}{A_{2n-1}} \frac{-(x-p)}{|x-p|^{2n}} Dx \cdot f(x) \tag{25}$$

As we want to integrate the residue form on the cycle C it will be enough to compute the residue only on the tangent space of the cycle C. Since the hyperboloid and a null cone are transversal (their tangent spaces are in general position at intersection points) the tangent space to the cycle C is given by two normal vectors (one for each hypersurface). The normal vector to the hyperboloid at a point x is given by

$$n_{H_R}(x) = (x_1, x_2, \dots, x_{2n}) \tag{26}$$

in the standard basis of the tangent space of $\mathbb{M}$ at a point x. We will abuse the notation and write x also for the normal n_{H_R}. A second normal vector, the one corresponding to the null cone is described as

$$n_{N_p}(x) = (x_1 - p_1, x_2 - p_2, \cdots, x_{2n} - p_{2n}) \tag{27}$$

and we will similarly denote it by $x - p$.

Take a transversal vector field c on C, i.e. for any $x \in C$ the vector $c(x)$ does not lie in T_xC. According to [4] each transversal vector field c on C induces a tubular neighbourhood which then allows us to compute the value of a representative $\operatorname{res}\omega$ in the cohomology class $\operatorname{Res}\omega$ on the vectors $t_i \in T_xC$. Any choice of the vector field c is possible but some are more convenient than others. If we choose c that satisfies the following conditions, it will simplify the computations significantly.

$$(c(x), t) = 0\,, \quad \forall\, t \in T_x C \tag{28}$$

$$(c(x), c(x)) = 0 \tag{29}$$

$$(c(x), x-p) = \frac{1}{2} \tag{30}$$

for any $x \in C$. The first relation ensures that c will be orthogonal to all vectors in the tangent space of the cycle C. The second condition is an isotropy condition and the third condition is a normalization with respect to $x-p$. It is easy to see that $x-p$ solves the first two conditions but fails to solve the third one.

Lemma 3. *A vector field c on $C = N_p \cap H_R$ that satisfies conditions* (28), (29) *and* (30) *at every point of C is given by the following formula*

$$c(x) = \frac{R^2}{[R^2 + (p,p)]^2}(x-p) - \frac{1}{R^2 + (p,p)}x\,. \tag{31}$$

Proof. To satisfy (28) we see that $c(x)$ has to have the form

$$c(x) = \alpha(x)(x-p) + \beta(x)x$$

for some real-valued functions α, β on C (because the linear combination might depend on the point considered). Since x and $x-p$ are two linearly independent vectors orthogonal to all $t \in T_x C$, the vector field c has to be a linear combination of those. Combining with (30) we arrive at

$$\begin{aligned} \frac{1}{2} &= (c(x), x-p) \\ &= (\alpha(x)(x-p) + \beta(x)x, x-p) \\ &= \beta(x)(x, x-p) \end{aligned}$$

We use it when we work out the condition (29) below

$$\begin{aligned} 0 &= (c(x), c(x)) \\ &= (\alpha(x)(x-p) + \beta(x)x, \alpha(x)(x-p) + \beta(x)x) \\ &= \alpha(x) + \beta(x)^2(-R^2) \end{aligned}$$

where we also used (23) and (24). And we will need those also in the last step of the proof when rewriting the equation for the hyperboloid H_R

$$-R^2 = (x,x) = (x-p+p, x-p+p) = 2(x-p,p) + (p,p)\,,$$

hence

$$-R^2 - (p,p) = 2(x-p,p) = 2(x-p, p-x+x) = 2(x-p,x)$$

which completes the proof. □

We would like to know how the vector field c changes when we move along the cycle C. We can do that explicitly if we have the standard covariant derivative at our disposal.

Lemma 4. *Let $x \in C = N_p \cap H_R$, $\tau \in \mathbb{C}$ and let $t \in T_x C$. Denote by $\nabla_v \Phi$ the standard (flat) covariant derivative of the vector field Φ in the direction v. Then for the vector field c that satisfies conditions* (28), (29) *and* (30) *at every point of C we have*

$$\nabla_t \left(x - p + \tau c(x)\right) = \Big(1 + \tau \frac{-(p,p)}{[R^2 + (p,p)]^2}\Big) t \,. \tag{32}$$

Proof. We start by computing $\nabla_{e_i} x$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, that is, all zeros with 1 at the i-th place.

$$\nabla_{e_i} x = \frac{\partial}{\partial x_i} x = e_i \,.$$

Straightforward computation then gives

$$\begin{aligned}\nabla_t \left(x - p + \tau c(x)\right) &= \left(1 + \tau\Big(\frac{R^2}{[R^2 + (p,p)]^2} - \frac{1}{R^2 + (p,p)}\Big)\right) t \\ &= \Big(1 + \tau \frac{-(p,p)}{[R^2 + (p,p)]^2}\Big) t\end{aligned}$$

□

We will now fix the embedding of $\mathbb{M}$ into $\mathbb{C}^{2n}$. The only requirement that we have to impose is that when restricting $\mathbb{C}^{2n}$ to the Minkowski space $\mathbb{M}$ the bilinear forms on $\mathbb{C}^{2n}$ and on $\mathbb{M}$ are equal. Recall that the bilinear form on $\mathbb{C}^{2n}$ is given by the quadratic form (15). This gives the following embedding

$$x = (x_1, \ldots, x_{2n}) \mapsto X = (ix_1, \ldots, ix_{2n-1}, x_{2n}) \tag{33}$$

where $x \in \mathbb{M}$ and $X \in \mathbb{C}^{2n}$.

Lemma 5. *Let $p \in \mathbb{M} \setminus H_R$, $(p,p) < 0$, $p_{2n} > 0$ and denote by K the compact set given as the union of line segments $\overline{px}$ where and $x \in C = N_p \cap H_R$ over all C. Let f be a real analytic function on an open domain $\Omega \subset \mathbb{M}$ where $K \subset \Omega$. Then f can be extended to a holomorphic function on a domain $\Omega_{\mathbb{C}} \subset \mathbb{C}^{2n}$ which satisfies the following conditions*

(1) $K \subset \Omega_{\mathbb{C}}$,
(2) $\Omega_{\mathbb{C}} \cap \mathbb{C}N_p$ *is star-shaped with respect to the point p.*

Proof. We will provide a constructive proof. For any point $x \in K$ denote by $r(x)$ the maximal radius of the ball

$$B(x, r(x)) = \{y \in \mathbb{M}, \|x - y\|^2 = \sum_{i=1}^{2n} (x_i - y_i)^2 \leq r(x)^2\} \subset \mathbb{M}$$

which is contained in Ω and on which the function f equals its Taylor series (in fact, we can left out the second condition and nothing changes). With the embedding (33) in mind, we can consider the complex ball around the point $x \in K$

$$B(x, r(x)) = \{Y \in \mathbb{C}^{2n}, \|X - Y\|^2 = \sum_{i=1}^{2n} |X_i - Y_i|^2 \leq r(x)^2\} \subset \mathbb{C}^{2n}$$

where $X = (ix_1, \ldots, ix_{2n-1}, x_{2n})$ is the image of x under the embedding. We use the same notation because no confusion can arise. Simply by replacing the real variables in the Taylor series of f by the complex ones we get a Taylor series of the holomorphic extension of the function f on $B(x, r(x)) \subset \mathbb{C}^{2n}$. Now take the union of the complex balls over the compact K

$$\Omega'_{\mathbb{C}} = \bigcup_{x \in K} B\Big(x, \frac{r(x)}{2}\Big) \subset \mathbb{C}^{2n}.$$

Since K is compact we can choose a finite subcovering, let us call it $\mathcal{U}$. Denote by

$$\mu = \min_{\mathcal{U}} \frac{r(x)}{2}$$

and take again the union of the complex balls over the compact K with the fixed radius μ

$$\Omega_{\mathbb{C}} = \bigcup_{x \in K} B(x, \mu) \subset \mathbb{C}^{2n}.$$

Clearly, $K \subset \Omega_{\mathbb{C}}$ holds and also $\Omega_{\mathbb{C}}$ is star-shaped with respect to p, hence $\Omega_{\mathbb{C}} \cap \mathbb{C}N_p$ is star-shaped with respect to the point p. The last thing left to check is that f extends holomorphically to $\Omega_{\mathbb{C}}$. The key to that is to show

$$\Omega_{\mathbb{C}} \subset \bigcup_{x \in K} B\big(x, r(x)\big)\,.$$

But for any $x \in K$ there exists $y \in K$ with $B\big(y, \frac{r(y)}{2}\big) \in \mathcal{U}$ and $x \in B\big(y, \frac{r(y)}{2}\big)$ which implies that there exists a complex ball $B(x, \mu)$ around x where f can be holomorphically extended because of the following inclusions

$$B(x, \mu) \subset B\Big(x, \frac{r(y)}{2}\Big) \subset B\big(y, r(y)\big)\,,$$

and the first inclusion holds due to the minimality of μ on $\mathcal{U}$. □

The following lemma shows how the solution of the Dirac equation on $\Omega \subset \mathbb{M}$ can be extended to a solution of the complex Dirac equation on $\Omega_{\mathbb{C}} \subset \mathbb{C}^{2n}$.

Lemma 6. *Let $\Omega \subset \mathbb{M}$ be an open domain and let f be a real analytic solution of the Dirac equation*

$$\sum_{j=1}^{2n-1} -ie_j \frac{\partial f}{\partial x_j} + e_{2n} \frac{\partial f}{\partial x_{2n}} = 0 \tag{34}$$

on the domain Ω. Let $\Omega_{\mathbb{C}} \subset \mathbb{C}^{2n}$ be a domain where exists a holomorphic extension of f. Then the holomorphic extension of f is a solution of the complex Dirac equation (14) *on $\Omega_{\mathbb{C}}$.*

Proof. We can write the real analytic function f as a Taylor series (without the loss of generality and for the simplicity we write the Taylor series in a neighbourhood of origin)

$$f(x) = \sum_{j_1, \ldots, j_{2n}=1}^{\infty} a_{j_1, \ldots, j_{2n}} x_1^{j_1} \ldots x_{2n}^{j_{2n}}\,,$$

where the coefficients have values in S and $(x_1, \ldots, x_{2n}) \in \Omega \subset \mathbb{M}$. The Taylor series of the holomorphic extension of f to $\Omega_{\mathbb{C}} \subset \mathbb{C}^{2n}$ then looks like (with respect to the embedding (33))

$$f(X) = \sum_{j_1,\ldots,j_{2n}=1}^{\infty} a_{j_1,\ldots,j_{2n}} (-iX_1)^{j_1} \ldots (-iX_{2n-1})^{j_{2n-1}} X_{2n}^{j_{2n}} ,$$

where the coefficients are the same and $(-iX_1, \ldots, -iX_{2n-1}, X_{2n}) \in \Omega_{\mathbb{C}}$.

We need to check that this extension still solves the complex Dirac equation (note that the complex Dirac equation reduces to the Dirac equation (34) on $\mathbb{M}$ under the given embedding). The Dirac equation for the Taylor series of f on Ω is given by

$$\begin{aligned} 0 = \sum_{k=1}^{2n-1} \sum_{j_1,\ldots,j_{2n}=1}^{\infty} & -ie_k j_k a_{j_1,\ldots,j_{2n}} x_1^{j_1} \ldots x_k^{j_k-1} \ldots x_{2n}^{j_{2n}} \\ & + e_{2n} j_{2n} a_{j_1,\ldots,j_{2n}} x_1^{j_1} \ldots x_{2n-1}^{j_{2n-1}} x_{2n}^{j_{2n}-1} . \end{aligned} \tag{35}$$

Similarly, the complex Dirac equation for the Taylor series of f on $\Omega_{\mathbb{C}}$ is given by

$$\begin{aligned} 0 = \sum_{k=1}^{2n-1} \sum_{j_1,\ldots,j_{2n}=1}^{\infty} & -ie_k j_k a_{j_1,\ldots,j_{2n}} (-iX_1)^{j_1} \ldots (-iX_k)^{j_k-1} \ldots (-iX_{2n-1})^{j_{2n-1}} X_{2n}^{j_{2n}} \\ & + e_{2n} j_{2n} a_{j_1,\ldots,j_{2n}} (-iX_1)^{j_1} \ldots (-iX_{2n-1})^{j_{2n-1}} X_{2n}^{j_{2n}-1} . \end{aligned} \tag{36}$$

The systems of equations for coefficients $a_{j_1,\ldots,j_{2n}}$ for the equation (35) and (36) are equivalent, hence the complex Dirac equation for f on $\Omega_{\mathbb{C}}$ holds whenever the Dirac equation for f on Ω holds. $\square$

Lemma 7. *Let $p \in \mathbb{M} \setminus H_R$, $(p,p) < 0$, $p_{2n} > 0$ and denote by K the compact set given as the union of line segments $\overline{px}$ where and $x \in C = N_p \cap H_R$ over all C. Let $\Omega \subset \mathbb{M}$ be an open domain such that $K \subset \Omega$. Let f be a real analytic solution of the Dirac equation* (34) *on the domain Ω and let c be a smooth vector field on $C = N_p \cap H_R$ such that the conditions* (28), (29) *and* (30) *are satisfied at every point $x \in C$. Let $\gamma = \frac{-(p,p)}{[R^2+(p,p)]^2}$ and denote by dS the pullback of the volume form on C. Then (for $x \in C$)*

$$\operatorname{res}\omega|_C = \pm \frac{(x-p)\cdot c(x)}{\pi^{n-1}} \sum_{j=0}^{n-1} \frac{(2n-2)!}{(n-1+j)!} \gamma^{n-1+j} \left. \frac{d^j}{d\tau^j} f\big(x + \tau c(x)\big) \right|_{\tau=0} dS \tag{37}$$

Proof. We will prove the lemma by evaluating both forms on any $t_i \in T_xC$ for $i = 1, \ldots, 2n-2$. Be virtue of Lemma 4

$$dx\big[x-p, c(x), \nabla_{t_1}\big(x-p+\tau c(x)\big), \ldots, \nabla_{t_{2n-2}}\big(x-p+\tau c(x)\big)\big] = F_0\,(1+\tau\gamma)^{2n-2}$$

where $F_0 = \det[x-p, c(x), t_1, \ldots, t_{2n-2}]$. From the Gram matrix and orthogonality relations we can compute (up to an orientation)

$$F_0 = \pm\frac{i}{2} \det[t_1, \ldots, t_{2n-2}] = \pm\frac{i}{2} dS[t_1, \ldots, t_{2n-2}] .$$

To apply the Theorem 6 we have to extend the function f from $\Omega \subset \mathbb{M}$ to the holomorphic solution of the complex Dirac equation (14) on some $\Omega^{\mathbb{C}} \subset \mathbb{C}^{2n}$. But this is treated in Lemma 5 and Lemma 6. Therefore, we apply Theorem 6 to the extension of f on $\Omega_{\mathbb{C}} \subset \mathbb{C}^{2n}$ and compute the residue of the form ω restricted to the cycle C and use Lemma 2 to obtain the final form.

$$\operatorname{res} \omega|_C \, [t_1, \ldots, t_{2n-2}] = \pm \frac{(x-p) \cdot c(x)}{\pi^{n-1}} \times \frac{d^{n-1}}{d\tau^{n-1}} \Big[(1+\tau\gamma)^{2n-2} f\big(x + \tau c(x)\big) \Big] \Big|_{\tau=0} dS[t_1, \ldots, t_{2n-2}]$$

Working out the derivative finishes the proof. □

Lemma 8. *Let $p \in \mathbb{M} \setminus H_R$ and let $C = H_R \cap N_p$. Then*

$$\operatorname{Ind}_{\delta C} p = \pm 1 \tag{38}$$

where δ is the cobord map.

Proof. We start the proof by writing the hyperbolic integral formula for the constant function on the cycle C

$$\operatorname{Ind}_{\delta C} p = \int_C \operatorname{Res} \omega$$

and we know that $\operatorname{Res} \omega$ is a closed form. It is clearly possible to deform the cycle C along the generators of the null cone to the sphere, explicitly the homotopy looks like

$$H_{N_p}(x, t, \varepsilon) = p + \varepsilon \frac{x-p}{x_{2n} - p_{2n}} + t\Big(x - p - \varepsilon \frac{x-p}{x_{2n} - p_{2n}} \Big)$$

where for $t = 0$ we get the sphere S^{2n-2} and for $t = 1$ we have the cycle C and we can always take ε small enough so that the sphere and the cycle do not intersect. From the Stokes' Theorem and example we computed above we conclude that

$$\operatorname{Ind}_{\delta C} p = \operatorname{Ind}_{\delta S^{2n-2}} p = \pm 1$$

and the orientation is the orientation induced by the choice of the standard basis on $\mathbb{M}$. □

Theorem 7. *Let $p \in \mathbb{M} \setminus H_R$ such that $(p,p) < 0$ and $p_{2n} > 0$ and let Ω be an open neighbourhood of the compact set K which is given as the union of all line segments joining points of $C = N_p \cap H_R$ with p. Let f be a real analytic solution of the Dirac equation* (34) *on a domain Ω and let c be a smooth vector field on C such that the conditions* (28), (29) *and* (30) *are satisfied at every point $x \in C$. Let $\gamma = \frac{-(p,p)}{[R^2 + (p,p)]^2}$ and denote by dS the pullback of the volume form on C. Then*

$$f(p) = \frac{1}{\pi^{n-1}} \sum_{j=0}^{n-1} \frac{(2n-2)!}{(n-1+j)!} \gamma^{n-1+j} \int_C (x-p) \cdot c(x) \, \frac{d^j}{d\tau^j} f(x + \tau c(x)) \Big|_{\tau=0} dS \tag{39}$$

holds.

Proof. We first use Lemma 5 and Lemma 6 to extend the function f holomorphically to the solution of the complex Dirac equation on $\Omega_{\mathbb{C}}$ (see proof of Lemma 5). Then we put together Lemma 7, Lemma 8 and Theorem 5 to get the desired result. □

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