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π-MAPPINGS IN ls-PONOMAREV-SYSTEMS

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Abstract. We use the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, where M is a locally separable metric space, to give a consistent method to construct a π-mapping (compact mapping) with covering-properties from a locally separable metric space M onto a space X. As applications of these results, we systematically get characterizations of certain π-images (compact images) of locally separable metric spaces.

1. Introduction

Finding characterizations of nice images of metric spaces is an interesting topic of general topology. Various kinds of characterizations have been obtained by means of certain networks [11], [18]. Recently, many authors were interested in finding characterizations of nice images of locally separable metric spaces under certain covering-mappings. The key to prove these results is to construct covering-mappings from a locally separable metric space onto a space. In [16], V. I. Ponomarev characterized open s-images of metric spaces by first-countable spaces. In [13], S. Lin and P. Yan generalized the Ponomarev's method, called the *Ponomarev-system*, to construct covering-mappings from a metric space onto a space with certain networks. In [2], the authors used the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_\lambda\})$ (here, the prefix "ls" is the abbreviation of "locally separable") to give necessary and sufficient conditions such that the mapping f is an s-mapping with covering-properties from a locally separable metric space M onto a space X. As applications of these results, characterizations of certain s-images of locally separable metric spaces have been obtained systematically. However, for an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_\lambda\})$, we do not know what conditions such that the mapping f is a π-mapping (compact mapping) with covering-properties from a locally separable metric space M onto a space X are. Take this problem into account, we are interested in finding a consistent method to construct a π-mapping (compact mapping) with covering-properties from a locally separable metric space M onto a space X.

In this paper, we use the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, where M is a locally separable metric space, to give a consistent method to construct a π-mapping (compact mapping) with covering-properties from a locally separable

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metric space M onto a space X. As applications of these results, we systematically get characterizations of certain π-images (compact images) of locally separable metric spaces. These results make the study of images of locally separable metric spaces more completely.

Throughout this paper, all spaces are T_1 and regular, all mappings are continuous and onto, a convergent sequence includes its limit point, $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \longrightarrow Y$ be a mapping, $x \in X$, and $\mathcal{P}$ be a family of subsets of X, we denote $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$, $st(x, \mathcal{P}) = \bigcup \mathcal{P}_x$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ converging to x in X is *eventually* in a subset U of X if $\{x_n : n \geq n_0\} \cup \{x\} \subset U$ for some $n_0 \in \mathbb{N}$, and it is *frequently* in U if $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset U$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$.

For terms are not defined here, please refer to [5] and [18].

2. Results

Definition 2.1. Let $\mathcal{P}$ be a family of subsets of a space X, and K be a subset of X.

(1) For each $x \in X$, $\mathcal{P}$ is a *network at x in X* [14], if $x \in \bigcap \mathcal{P}$ and if $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

$\mathcal{P}$ is a *network for X* [14], if $\mathcal{P}_x$ is a network at x in X for every $x \in X$.

(2) $\mathcal{P}$ is a *cfp-cover for K in X* [2], if for each compact subset H of K, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\}$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. If $K = X$, then a cfp-cover for K in X is a *cfp-cover for X* [20].

(3) $\mathcal{P}$ is a *cs-cover for K in X* (resp., *cs^*-cover for K in X*) [2], if for each convergent sequence S in K, S is eventually (resp., frequently) in some $P \in \mathcal{P}$. If $K = X$, then a cs-cover for K in X (resp., cs^*-cover for K in X) is a *cs-cover for X* [21] (resp., *cs^*-cover for X* [19]).

(4) $\mathcal{P}$ is a *wcs-cover for K in X* [2], if for each convergent sequence S converging to x in K, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}_x$ such that S is eventually in $\bigcup \mathcal{F}$. If $K = X$, then a wcs-cover for K in X is a *wcs-cover for X* [7].

Remark 2.2. (1) A cfp-cover (resp., cs-cover, wcs-cover, cs^*-cover) for X is abbreviated to a *cfp-cover* (resp., *cs-cover, wcs-cover, cs^*-cover*).

(2) For each subset K of X, if $\mathcal{P}$ is a cfp-cover (resp., cs-cover, wcs-cover, cs^*-cover), then $\mathcal{P}$ is a cfp-cover (resp., cs-cover, wcs-cover, cs^*-cover) for K in X.

The following lemma is clear.

Lemma 2.3. *Let $\mathcal{P}$ be a countable family of subsets of a space X. Then the following are equivalent for a convergent sequence S in X.*

(1) *$\mathcal{P}$ is a cfp-cover for S in X.*

(2) *$\mathcal{P}$ is a wcs-cover for S in X.*

(3) *$\mathcal{P}$ is a cs^*-cover for S in X.*

Definition 2.4. Let $f\colon X \longrightarrow Y$ be a mapping.

(1) f is a *compact-covering* mapping [15], if for each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.

(2) f is a *sequence-covering* mapping [17], if for each convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L) = S$.

(3) f is a *pseudo-sequence-covering* mapping [9], if for each convergent sequence S in Y, there exists a compact subset L of X such that $f(L) = S$.

(4) f is a *subsequence-covering* mapping [12], if for each convergent sequence S in Y, there exists a compact subset L of X such that $f(L)$ is a subsequence of S.

(5) f is a *sequentially-quotient* mapping [4], if for each convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(6) f is a *compact* mapping [3], if for each $y \in Y$, $f^{-1}(y)$ is compact subset of X.

(7) f is a *π-mapping* [3], if for each $y \in Y$ and each neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.

(8) f is an *s-mapping* [3], if for each $y \in Y$, $f^{-1}(y)$ is a separable subset of X.

(9) f is a *π-s-mapping* [10], if f is a π-mapping and an s-mapping.

The following lemma is well-known, where certain covers are preserved under covering-mappings.

Lemma 2.5. *Let $f\colon X \longrightarrow Y$ be a mapping, and $\mathcal{P}$ be a cover for X. Then the following hold.*

(1) *If $\mathcal{P}$ is a cs-cover for X and f is sequence-covering, then $f(\mathcal{P})$ is a cs-cover for Y.*

(2) *If $\mathcal{P}$ is a cfp-cover for X and f is compact-covering, then $f(\mathcal{P})$ is a cfp-cover for Y.*

(3) *If $\mathcal{P}$ is a wcs-cover for X and f is pseudo-sequence-covering, then $f(\mathcal{P})$ is a wcs-cover for Y.*

(4) *If $\mathcal{P}$ is a cs*-cover for X and f is sequentially-quotient, then $f(\mathcal{P})$ is a cs*-cover for Y.*

The next result concerning preservations of certain covers but there is no need to use covering-properties of mappings.

Lemma 2.6. *Let $f\colon X \longrightarrow Y$ be a mapping, and $\mathcal{P}$ be a cover for X. Then the following hold.*

(1) *If $\mathcal{P}$ is a cs-cover for a convergent sequence S in X, then $f(\mathcal{P})$ is a cs-cover for $f(S)$ in Y.*

(2) *If $\mathcal{P}$ is a cfp-cover for a compact subset K in X, then $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.*

(3) *If $\mathcal{P}$ is a wcs-cover for a convergent sequence S in X, then $f(\mathcal{P})$ is a wcs-cover for $f(S)$ in Y.*

(4) *If $\mathcal{P}$ is a cs^*-cover for a convergent sequence S in X, then $f(\mathcal{P})$ is a cs^*-cover for $f(S)$ in Y.*

Proof. (1). Let L be a convergent sequence in $f(S)$. Then $K = f^{-1}(L) \cap S$ is a convergent sequence in S satisfying that $f(K) = L$. Since $\mathcal{P}$ is a cs-cover for S in X, K is eventually in some $P \in \mathcal{P}$. This implies that L is eventually in $f(P)$. Therefore, $f(\mathcal{P})$ is a cs-cover for $f(S)$ in Y.

(2). Let L be a compact subset of $f(K)$. Then $H = f^{-1}(L) \cap K$ is a compact subset of K satisfying that $f(H) = L$. Since $\mathcal{P}$ is a cfp-cover for K in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\}$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. This implies that $f(\mathcal{F})$ is a finite subfamily of $f(\mathcal{P})$ such that $L \subset \bigcup\{f(C_F) : F \in \mathcal{F}\}$, where $f(C_F)$ is closed and $f(C_F) \subset f(F)$ for every $F \in \mathcal{F}$. Therefore, $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.

(3). Let L be a convergent sequence in $f(S)$ converging to y in Y. Then $K = f^{-1}(L) \cap S$ is a convergent sequence in S converging to some $x \in f^{-1}(y)$, and $f(K) = L$. Since $\mathcal{P}$ is a wcs-cover for S in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}_x$ such that K is eventually in $\bigcup \mathcal{F}$. Then $f(\mathcal{F})$ is a finite subfamily of $f(\mathcal{P})_y$ and L is eventually in $\bigcup f(\mathcal{F})$. It implies that $f(\mathcal{P})$ is a wcs-cover for $f(S)$ in Y.

(4). Let L be a convergent sequence in $f(S)$. Then $K = f^{-1}(L) \cap S$ is a convergent sequence in S satisfying that $f(K) = L$. Since $\mathcal{P}$ is a cs^*-cover for S in X, K is frequently in some $P \in \mathcal{P}$. Then L is frequently in $f(P)$. It implies that $f(\mathcal{P})$ is a cs^*-cover for $f(S)$ in Y. □

Definition 2.7. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers for a space X. $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is *a point-star network for X* [13], if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for every $x \in X$.

Definition 2.8. Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a point-star network for X. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, and endowed A_n with the discrete topology. Put

$$M = \Big\{a = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} A_n$, is a metric space, x_a is unique, and $x_a = \bigcap_{n\in\mathbb{N}} P_{\alpha_n}$ for every $a \in M$. Define $f\colon M \longrightarrow X$ by $f(a) = x_a$, then f is a mapping and $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev-system* [13].

Remark 2.9. There are two Ponomarev-systems in [13]. The Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ requires that $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X, and the Ponomarev-system $(f, M, X, \mathcal{P})$ requires that $\mathcal{P}$ is a strong network for X (i.e., for each $x \in X$, there exists $\mathcal{P}(x) \subset \mathcal{P}$ such that $\mathcal{P}(x)$ is a countable network at x in X). In this paper, we use the definition of Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, where $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X.

In [19, Lemma 2.2] and [8, Theorem 2.7], the authors have investigated the Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ and obtained conditions such that the mapping

f is a compact mapping (covering-mapping) from a metric space M onto a space X. In view of the proof of [8, Theorem 2.7], [19, Lemma 2.2(ii)], and Lemma 2.3, we get the following.

Lemma 2.10. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev-system. Then the following hold.*

(1) *For each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a cs-cover for a convergent sequence S in X if and only if there exists a convergent sequence L in M such that $S = f(L)$.*

(2) *For each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a cfp-cover for a compact set K in X if and only if there exists a compact subset L of M such that $K = f(L)$.*

(3) *For each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a wcs-cover for a convergent sequence S in X if and only if there exists a compact subset L of M such that $S = f(L)$.*

(4) *For each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a cs*-cover for a convergent sequence S in X if and only if there exists a convergent sequence L in M such that $f(L)$ is a subsequence of S.*

Definition 2.11. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a cover for a space X such that each X_λ has a sequence of covers $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cover* for X, if for each $\lambda \in \Lambda$, $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ consisting of countable covers $\mathcal{P}_{\lambda,n}$.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star π-cover* for X, if it is a double point-star cover for X, and $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X, where $\mathcal{P}_n = \bigcup\{\mathcal{P}_{\lambda,n} : \lambda \in \Lambda\}$ for every $n \in \mathbb{N}$. Note that, if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star π-cover for X, then $\{X_\lambda : \lambda \in \Lambda\}$ is a cover having *π-property* in the sense of [1].

(3) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is *point-finite* (resp., *point-countable*), if for each $\lambda \in \Lambda$ and $n \in \mathbb{N}$, both $\{X_\lambda : \lambda \in \Lambda\}$ and $\mathcal{P}_{\lambda,n}$ are point-finite (resp., point-countable).

Definition 2.12. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for X.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cs-cover* for X, if for each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is eventually in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a *cs*-cover for $S \cap X_\lambda$ in X_λ.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cfp-cover* for X, if for each compact subset K of X, there exists a finite subset Λ_K of Λ such that $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$ and, for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, K_λ is compact and $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ.

(3) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star wcs-cover* for X, if for each convergent sequence S in X, there exists a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is a convergent sequence and $\mathcal{P}_{\lambda,n}$ is a *wcs*-cover for S_λ in X_λ.

(4) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cs*-cover* for X, if for each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is frequently in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a *cs**-cover for a subsequence S_λ of S in X_λ.

(5) A double point-star cs-cover (resp., cfp-cover, wcs-cover, cs^*-cover) for X is a *double point-star π-cs-cover* (resp., *π-cfp-cover, π-wcs-cover, π-cs^*-cover*) for X if it is a double point-star π-cover for X.

Remark 2.13. (1) If $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cover (resp., cfp-cover, cs-cover, wcs-cover, cs^*-cover) for X, then $\{X_\lambda : \lambda \in \Lambda\}$ is a cover (resp., cfp-cover, cs-cover, wcs-cover, cs^*-cover) for X.

(2) Every point-finite double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ for X is a double point-star π-cover for X.

Definition 2.14. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for a space X, and $(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\})$ be the Ponomarev-system for every $\lambda \in \Lambda$. Since each $\mathcal{P}_{\lambda,n}$ is countable, M_λ is a separable metric space. Put $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, and $f = \bigoplus_{\lambda\in\Lambda} f_\lambda$. Then M is a locally separable metric space, and f is a mapping from a locally separable metric space M onto X. The system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an *ls-Ponomarev-system*.

Remark 2.15. The ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is based on a family of Ponomarev-systems $\{(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$. It is different from the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_\lambda\})$, which is based on a family of Ponomarev-systems $\{(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_\lambda\}) : \lambda \in \Lambda\}$, in [2].

In [8, Lemma 2.7], Y. Ge has proved a necessary and sufficient condition such that the mapping f in a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ is a compact mapping (s-mapping) from a metric space M onto a space X. The following result is a necessary and sufficient condition such that the mapping f is a compact mapping (s-mapping) from a locally separable metric space M onto a space X, where $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an ls-Ponomarev-system.

Proposition 2.16. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then the following hold.*

(1) *f is a compact mapping if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X.*

(2) *f is an s-mapping if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-countable double point-star cover for X.*

Proof. (1). *Necessity.* For each $x \in X$, since $f^{-1}(x)$ is compact, $\{\lambda \in \Lambda : f^{-1}(x) \cap M_\lambda \neq \emptyset\} = \{\lambda \in \Lambda : x \in X_\lambda\}$ is finite. Then $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite. For each $\lambda \in \Lambda$, since $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is compact, f_λ is a compact mapping. Then each $\mathcal{P}_{\lambda,n}$ is point-finite by [8, Theorem 2.7(1)]. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X.

Sufficiency. For each $x \in X$, since $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite, $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\}$ is finite. Since each $\mathcal{P}_{\lambda,n}$ is point-finite, $f_\lambda^{-1}(x)$ is compact by [8, Theorem 2.7(1)]. It implies that $f^{-1}(x) = \bigcup\{f_\lambda^{-1}(x) : \lambda \in \Lambda_x\}$ is compact. Then f is a compact mapping.

(2). In view of the proof of (1). □

Corollary 2.17. *A space X is a compact image of a locally separable metric space if and only if it has a point-finite double point-star cover.*

Proof. *Necessity.* Let $f\colon M \longrightarrow X$ be a compact mapping from a locally separable metric space M onto X. Since M is a locally separable metric space, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ where each M_λ is separable by [5, 4.4.F]. Since each M_λ is a separable metric space, M_λ has a sequence of open countable covers $\{\mathcal{B}_{\lambda,n} : n \in \mathbb{N}\}$ such that for every compact subset K of M_λ and any open set U in M_λ with $K \subset U$, there exists $n \in \mathbb{N}$ satisfying $st(K, \mathcal{B}_{\lambda,n}) \subset U$ by [5, 5.4.E]. Let $\mathcal{C}_{\lambda,n}$ be a locally finite open refinement of each $\mathcal{B}_{\lambda,n}$. Then, for each $\lambda \in \Lambda$, $\{\mathcal{C}_{\lambda,n} : n \in \mathbb{N}\}$ is a sequence of locally finite open countable covers for M_λ such that for every compact subset K of M_λ and any open set U in M_λ with $K \subset U$, there exists $n \in \mathbb{N}$ satisfying $st(K, \mathcal{C}_{\lambda,n}) \subset U$. For each $\lambda \in \Lambda$ and $n \in \mathbb{N}$, put $X_\lambda = f(M_\lambda)$, and $\mathcal{P}_{\lambda,n} = f(\mathcal{C}_{\lambda,n})$. We have the following claims (a)–(e).

(a) $\{X_\lambda : \lambda \in \Lambda\}$ is a cover for X.

(b) Each $\mathcal{P}_{\lambda,n}$ is countable.

(c) For each $\lambda \in \Lambda$, $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

Let $x \in U$ with U open in X_λ. Then $x \in V$ with V open in X and $V \cap X_\lambda = U$. Since f is compact, $f^{-1}(x)$ is compact. Then $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is a compact subset of M_λ and $f_\lambda^{-1}(x) \subset V_\lambda$ with $V_\lambda = f^{-1}(V) \cap M_\lambda$ open in M_λ. Therefore, there exists $n \in \mathbb{N}$ such that $st(f_\lambda^{-1}(x), \mathcal{C}_{\lambda,n}) \subset V_\lambda$. It implies that $st(x, \mathcal{P}_{\lambda,n}) \subset f(f^{-1}(V) \cap M_\lambda) \subset V \cap X_\lambda = U$. Then $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

(d) $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite.

For each $x \in X$, since f is compact, $f^{-1}(x)$ is compact. Then $f^{-1}(x)$ meets only finitely many M_λ's. It implies that $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite.

(e) Each $\mathcal{P}_{\lambda,n}$ is point-finite.

For each $x \in X_\lambda$, since f is compact, $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is a compact subset of M_λ. Then $f_\lambda^{-1}(x)$ meets only finitely many members of $\mathcal{C}_{\lambda,n}$ by locally finiteness of each $\mathcal{C}_{\lambda,n}$. It implies that x meets only finitely many members of each $\mathcal{P}_{\lambda,n}$. Then each $\mathcal{P}_{\lambda,n}$ is point-finite.

From (a)–(e) we get that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X.

Sufficiency. Let X be a space having a point-finite double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.16, X is a compact image of a locally separable metric space. □

For a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, it is well-known that f is a π-mapping. For an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, we give a sufficient condition such that the mapping f is a π-mapping as follows.

Proposition 2.18. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. If $\big\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\big\}$ is a double point-star π-cover for X, then f is a π-mapping.*

Proof. Let $x \in U$ with U open in X. Since $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X, there exists $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. For each $\lambda \in \Lambda$ with $x \in X_\lambda$

we find that $st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$ where $U_\lambda = U \cap X_\lambda$. If $a = (\alpha_i) \in M_\lambda$ such that $d(f^{-1}(x), a) < \dfrac{1}{2^n}$, there exists $b = (\beta_i) \in f_\lambda^{-1}(x)$ such that $d_\lambda(a, b) < \dfrac{1}{2^n}$, where d and d_λ are metrics on M and M_λ, respectively. Therefore, $\alpha_i = \beta_i$ if $i \leq n$. It implies that $x \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$, hence $a \in f_\lambda^{-1}(P_{\alpha_n}) \subset f_\lambda^{-1}(U_\lambda)$. This proves that $d_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq \dfrac{1}{2^n}$. Then

$$\begin{aligned} d(f^{-1}(x), M - f^{-1}(U)) &= \inf\{d(a,b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\} \\ &= \min\big\{1, \inf\{d_\lambda(a,b) : a \in f_\lambda^{-1}(x), b \in M_\lambda - f_\lambda^{-1}(U_\lambda), \lambda \in \Lambda\}\big\} \geq \frac{1}{2^n} > 0\,. \end{aligned}$$

It implies that f is a π-mapping. □

It is well-known that every compact mapping from a metric space is a π-mapping. Then the following example shows that the inverse implication of Proposition 2.18 does not hold.

Example 2.19. There exists an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ such that the following hold.

(1) f is a compact mapping.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is not a double point-star π-cover for X.

Proof. Let $X = \{x, y\}$ be a discrete space. Put $X_1 = X_2 = X$, and put $\mathcal{P}_{1,1} = \mathcal{P}_{2,2} = \{\{x\}, \{y\}\}$ and $\mathcal{P}_{1,n} = \{X\}$ if $n \neq 1$, $\mathcal{P}_{2,n} = \{X\}$ if $n \neq 2$. We find that $\bigcup\{\mathcal{P}_{1,n} : n \in \mathbb{N}\}$ is a point-star network for X_1, and $\bigcup\{\mathcal{P}_{2,n} : n \in \mathbb{N}\}$ is a point-star network for X_2. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists, where $\{X_\lambda : \lambda \in \Lambda\} = \{X_i : i \leq 2\}$.

(1). f is a compact mapping.

Clearly, $\{(X_i, \{\mathcal{P}_{i,n}\}) : i \leq 2\}$ is a point-finite double point-star cover for X. By Proposition 2.16, f is a compact mapping.

(2). $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is not a double point-star π-cover for X.

We find that $\mathcal{P}_1 = \mathcal{P}_2 = \{\{x\}, \{y\}, X\}$, and $\mathcal{P}_n = \{X\}$ if $n \geq 2$. Then $st(x, \mathcal{P}_n) = X$ for every $n \in \mathbb{N}$. This proves that $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is not a point-star network for X. Then $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is not a double point-star π-cover for X. □

Corollary 2.20. *The following hold for a space X.*

(1) *X is a π-image of a locally separable metric space if and only if it has a double point-star π-cover.*

(2) *X is a π-s-image of a locally separable metric space if and only if it has a point-countable double point-star π-cover.*

Proof. (1). *Necessity.* Let $f : M \longrightarrow X$ be a π-mapping from a locally separable metric space M onto X. As in the proof (1) $\Rightarrow$ (2) of [1, Proposition 2.4], we find that X has a double point-star π-cover.

Sufficiency. Let X be a space having a double point-star π-cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.18, X is a π-image of a locally separable metric space.

(2). *Necessity.* Combing the necessity of (1) with f being an s-mapping, we find that X has a point-countable double point-star π-cover.

Sufficiency. Let X be a space having a point-countable double point-star π-cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.16 and Proposition 2.18, X is a π-s-image of a locally separable metric space. $\square$

In [8] and [19], the authors have stated conditions such that the mapping f is a covering-mapping from a metric space M onto a space X, where $(f, M, X, \{\mathcal{P}_n\})$ is a Ponomarev-system. Next, we give necessary and sufficient conditions such that the mapping f is a covering-mapping from a locally separable metric space M onto a space X, where $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an ls-Ponomarev-system.

Theorem 2.21. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then the following hold.*

(1) *f is sequence-covering if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.*

(2) *f is compact-covering if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for a space X.*

(3) *f is pseudo-sequence-covering if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.*

(4) *f is sequentially-quotient if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.*

Proof. (1). *Necessity.* Let f be sequence-covering. For each convergent sequence S in X, $S = f(L)$ for some convergent sequence L in M. Then L is eventually in some M_λ. Therefore, S is eventually in X_λ. Put $S_\lambda = f_\lambda(L_\lambda)$, where $L_\lambda = L \cap M_\lambda$ is a convergent sequence. It follows from Lemma 2.10 that each $\mathcal{P}_{\lambda,n}$ is a cs-cover for S_λ in X_λ. Then each $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cs-cover for X. For each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is eventually in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ. It follows from Lemma 2.10 that there exists a convergent sequence L_λ in M_λ such that $S_\lambda = f_\lambda(L_\lambda) = f(L_\lambda)$. Since $S - S_\lambda$ is finite, $S - S_\lambda = f(F)$ for some finite subset F of M. Put $L = F \cup L_\lambda$, then L is a convergent sequence in M and $S = f(L)$. It implies that f is sequence-covering.

(2). *Necessity.* Let f be compact-covering. For each compact subset K of X, $K = f(L)$ for some compact subset L of M. Since L is compact, $\Lambda_K = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$ is a finite subset of Λ and each $L_\lambda = L \cap M_\lambda$ is compact. For each $\lambda \in \Lambda_K$, put $K_\lambda = f_\lambda(L_\lambda)$. Then K_λ is compact, $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$, and each $\mathcal{P}_\lambda$ is a cfp-cover for K_λ in X_λ by Lemma 2.10. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cfp-cover for X. For each compact subset K of X, there exists a finite subset Λ_K of Λ such that $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$ and, for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, K_λ is compact and $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ. It follows from Lemma 2.10 that there exists a compact subset L_λ of M_λ such that $K_\lambda = f_\lambda(L_\lambda) = f(L_\lambda)$. Put $L = \bigcup\{L_\lambda : \lambda \in \Lambda_K\}$. Then L is a compact subset of M and $K = f(L)$. It implies that f is compact-covering.

(3). *Necessity.* Let f be pseudo-sequence-covering. For each convergent sequence S in X, $S = f(L)$ for some compact subset L of M. Note that S is also a compact subset of X. Then, as in the proof of necessity of (2), there exists a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is compact and $\mathcal{P}_{\lambda,n}$ is a cfp-cover for S_λ in X_λ. For each $\lambda \in \Lambda_S$ and each $n \in \mathbb{N}$, we find that S_λ is a convergent sequence, and then, $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ by Lemma 2.3. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star wcs-cover for X. For each convergent sequence S in X, there exists a finite subset Λ_S of Λ such that $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is a convergent sequence and $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ. It follows from Lemma 2.10 that there exists a compact subset L_λ in M_λ such that $S_\lambda = f_\lambda(L_\lambda) = f(L_\lambda)$. Put $L = \bigcup\{L_\lambda : \lambda \in \Lambda_S\}$. Then L is a compact subset of M and $S = f(L)$. It implies that f is pseudo-sequence-covering.

(4). *Necessity.* Let f be sequentially-quotient. For each convergent sequence S in X, there exists some convergent sequence L of M such that $H = f(L)$ is a subsequence of S. Then, as in the proof necessity of (1), H is eventually in some X_λ and each $\mathcal{P}_{\lambda,n}$ is a cs-cover for $H \cap X_\lambda$ in X_λ. Therefore, S is frequently in X_λ and each $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a subsequence $S_\lambda = H \cap X_\lambda$ of S in X_λ. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cs^*-cover for X. For each convergent sequence S in X, there exists $\lambda \in \Lambda$ such that S is frequently in X_λ and, for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a subsequence S_λ of S in X_λ. It follows from Lemma 2.10 that there exists a convergent sequence L_λ in M_λ such that $f_\lambda(L_\lambda)$ is a subsequence of S_λ. Note that $f_\lambda(L_\lambda) = f(L_\lambda)$ is also a subsequence of S. It implies that f is sequentially-quotient. □

In [6] and [19], the authors have characterized compact images of locally separable metric spaces by means of certain point-star networks. From the above theorems, we systematically get characterizations of compact images of locally separable metric spaces under certain covering-mappings by means of double point-star covers as follows.

Corollary 2.22. *The following hold for a space X.*

(1) *X is a sequence-covering compact image of a locally separable metric space if and only if it has a point-finite double point-star cs-cover.*

(2) *X is a compact-covering compact image of a locally separable metric space if and only if it has a point-finite double point-star cfp-cover.*

(3) *X is a pseudo-sequence-covering compact image of a locally separable metric space if and only if it has a point-finite double point-star wcs-cover.*

(4) *X is a sequentially-quotient compact image of a locally separable metric space if and only if it has a point-finite double point-star cs^*-cover.*

Proof. (1). *Necessity.* Let $f\colon M \longrightarrow X$ be a sequence-covering compact mapping from a locally separable metric space M onto X. By using notations and arguments in the necessity of Corollary 2.17 again, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.

For each convergent sequence S in X, since f is sequence-covering, there exists a convergent sequence L in M such that $f(L) = S$. We find that L is eventually in some M_λ. Then S is eventually in X_λ. Since $L_\lambda = L \cap M_\lambda$ is a convergent sequence in M_λ and each $\mathcal{C}_{\lambda,n}$ is a cs-cover for L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S_\lambda = f(L_\lambda)$ in X_λ by Lemma 2.6. Then $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-finite double point-star cs-cover for X. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.16 and Theorem 2.21 (1), we find that X is a sequence-covering compact image of a locally separable metric space.

(2). *Necessity.* Let $f\colon M \longrightarrow X$ be a compact-covering compact mapping from a locally separable metric sapce M onto X. By using notations and arguments in the necessity of Corollary 2.17 again, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for X.

For each compact subset K of X, since f is compact-covering, there exists a compact subset L of M such that $f(L) = K$. Put $\Lambda_K = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$, then Λ_K is finite, and each $L_\lambda = L \cap M_\lambda$ is compact. For each $\lambda \in \Lambda_K$, put $K_\lambda = f(L_\lambda)$. Then $K = \bigcup\{K_\lambda : \lambda \in \Lambda_K\}$ and each K_λ is compact. For each $\lambda \in \Lambda_K$ and each $n \in \mathbb{N}$, since $\mathcal{C}_{\lambda,n}$ is a cfp-cover for L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ by Lemma 2.6. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cfp-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-finite double point-star cfp-cover for X. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.16 and Theorem 2.21 (2), we find that X is a compact-covering compact image of a locally separable metric space.

(3). *Necessity.* Let $f\colon M \longrightarrow X$ be a pseudo-sequence-covering compact mapping from a locally separable metric space M onto X. By using notations and arguments in the necessity of Corollary 2.17 again, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.

For each convergent sequence S in X, since f is pseudo-sequence-covering, there exists a compact subset L of M such that $f(L) = S$. Put $\Lambda_S = \{\lambda \in \Lambda : L \cap M_\lambda \neq \emptyset\}$, then Λ_S is finite, and each $L_\lambda = L \cap M_\lambda$ is compact. For each $\lambda \in \Lambda_S$, put $S_\lambda = f(L_\lambda)$, then $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$ and each S_λ is compact. Since S_λ is a compact subset of a convergent sequence S, S_λ is a convergent sequence. On the other hand, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, since $\mathcal{C}_{\lambda,n}$ is a cfp-cover for a compact subset L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for S_λ in X_λ by Lemma 2.6. Then $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ by Lemma 2.3. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-finite double point-star wcs-cover for X. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.16 and Theorem 2.21 (3), we find that X is a pseudo-sequence-covering compact image of a locally separable metric space.

(4). *Necessity.* Let $f\colon M \longrightarrow X$ be a sequentially-quotient compact mapping from a locally separable metric space M onto X. By using notations and arguments in the necessity of Corollary 2.17 again, it suffices to show that the double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.

For each convergent sequence S in X, since f is sequentially-quotient, there exists a convergent sequence L in M such that $f(L)$ is a subsequence of S. Since L is eventually in some M_λ, $L_\lambda = L \cap M_\lambda$ is a convergent sequence. Then $S_\lambda = f(L_\lambda)$ is a subsequence of S, and hence, S is frequently in X_λ. On the other hand, since each $\mathcal{C}_{\lambda,n}$ is a cs^*-cover for a convergent sequence L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for S_λ in X_λ by Lemma 2.6. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.

Sufficiency. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-finite double point-star cs^*-cover for X. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.16 and Theorem 2.21 (4), we find that X is a sequentially-quotient compact image of a locally separable metric space. $\square$

Remark 2.23. (1) Since subsequence-covering mappings and sequentially-quotient mappings are equivalent for metric domains, “sequentially-quotient” in Theorem 2.21 (4) and Corollary 2.22 (4) can be replaced by “subsequence-covering”.

(2) By Remark 2.13 (2), the prefix “*cs*-” (resp., “cfp-”, “wcs-”, “cs^*-”) in Corollary 2.22 can be replaced by “π-*cs*-” (resp., “π-cfp-”, “π-wcs-”, “π-cs^*-”).

In [6], Y. Ge proved that a space X is a sequentially-quotient compact image of a locally separable metric space if and only if X is a pseudo-sequence-covering compact image of a locally separable metric space. Next, we get this result again by using the following lemma.

Lemma 2.24. *Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for X such that $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite. Then the following are equivalent.*

(1) *$\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X.*

(2) *$\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cs^*-cover for X.*

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (1). Let S be a convergent sequence converging to x in X. Then there exists $\lambda \in \Lambda$ such that S is frequently in X_λ and each $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a

subsequence S_λ of S in X_λ. Put

$$\Lambda'_S = \big\{\lambda \in \Lambda : \text{ for every } n \in \mathbb{N}, \\ \mathcal{P}_{\lambda,n} \text{ is a } cs^*\text{-cover for some subsequence } S_\lambda \text{ of } S \text{ in } X_\lambda\big\}.$$

Since $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite, the limit point x of S meets only finitely many X_λ's. Then Λ'_S is finite. We shall prove that S is eventually in $\bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$. If not, there exists a subsequence L of S such that $L - \{x\} \subset S - \bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$. Since L is a convergent sequence in X, L is frequently in some X_α, and each $\mathcal{P}_{\alpha,n}$ is a cs^*-cover for some subsequence S_α of L. Since S_α is a subsequence of S, $\alpha \in \Lambda'_S$. It is a contradiction. Then S is eventually in $\bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$. Since $S - \bigcup\{S_\lambda : \lambda \in \Lambda'_S\}$ is finite, $S - \bigcup\{S_\lambda : \lambda \in \Lambda'_S\} = \bigcup\{S_\lambda : \lambda \in \Lambda''_S\}$, where Λ''_S is also a finite subset of Λ and each S_λ is a finite subset of X_λ. Put $\Lambda_S = \Lambda'_S \cup \Lambda''_S$, then $S = \bigcup\{S_\lambda : \lambda \in \Lambda_S\}$, where Λ_S is a finite subset of Λ and, for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, S_λ is a convergent sequence and $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for S_λ in X_λ. It follows from Lemma 2.3 that each $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ. Then $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wcs-cover for X. □

Corollary 2.25 (Theorem 2.2, [6]). *The following are equivalent for a space X.*

(1) *X is a pseudo-sequence-covering compact image of a locally separable metric space.*

(2) *X is a subsequence-covering compact image of a locally separable metric space.*

(3) *X is a sequentially-quotient compact image of a locally separable metric space.*

Proof. It is obvious from Corollary 2.22, Remark 2.23 (1), and Lemma 2.24. □

In [1], the authors have been characterized π-images of locally separable metric spaces by means of covers having π-property. From the above results, we systematically get characterizations of π-images (π-s-images) of locally separable metric spaces under certain covering-mappings by means of double point-star π-covers as follows.

Corollary 2.26. *The following hold for a space X.*

(1) *X is a sequence-covering π-image of a locally separable metric space if and only if it has a double point-star π-cs-cover.*

(2) *X is a compact-covering π-image of a locally separable metric space if and only if it has a double point-star π-cfp-cover.*

(3) *X is a pseudo-sequence-covering π-image of a locally separable metric space if and only if it has a double point-star π-wcs-cover.*

(4) *X is a sequentially-quotient π-image of a locally separable metric space if and only if it has a double point-star π-cs^*-cover.*

Proof. For the necessities, combining the necessity in the proof of Corollary 2.20 (1) and necessities in the proof of Corollary 2.22.

For the sufficiencies, let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star π-*cs*-cover (resp., π-*cfp*-cover, π-*wcs*-cover, π-*cs**-cover) for X. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists. By Proposition 2.18 and Theorem 2.21, f is a sequence-covering (resp., compact-covering, pseudo-sequence-covering, sequentially-quotient) π-mapping. It implies that X is a sequence-covering (resp., compact-covering, pseudo-sequence-covering, sequentially-quotient) π-image of a locally separable metric space. □

In view of the proof of Corollary 2.26, we get the following.

Corollary 2.27. *The following hold for a space X.*

(1) *X is a sequence-covering π-s-image of a locally separable metric space if and only if it has a point-countable double point-star π-cs-cover.*

(2) *X is a compact-covering π-s-image of a locally separable metric space if and only if it has a point-countable double point-star π-cfp-cover.*

(3) *X is a pseudo-sequence-covering π-s-image of a locally separable metric space if and only if it has a point-countable double point-star π-wcs-cover.*

(4) *X is a sequentially-quotient π-s-image of a locally separable metric space if and only if it has a point-countable double point-star π-cs*-cover.*

Proof. For necessities, combining necessities in the proof of Corollary 2.26 with f being an s-mapping, we find that X has a point-countable double point-star π-*cs*-cover (resp., π-*cfp*-cover, π-*wcs*-cover, π-*cs**-cover).

For sufficiencies, combining sufficiencies in the proof of Corollary 2.26 with Proposition 2.16. □

Take the above ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ and the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_\lambda\})$ in [2] into account, we pose the following question.

Question 2.28. *Find a general system to give a consistent method to construct s-mapping (π-mapping, compact mapping) with covering-properties from a locally separable metric space M onto a space X?*

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