ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 17–22

TANGENT DIRAC STRUCTURES OF HIGHER ORDER

P. M. Kouotchop Wamba, A. Ntyam, and J. Wouafo Kamga

Abstract. Let L be an almost Dirac structure on a manifold M. In [2] Theodore James Courant defines the tangent lifting of L on TM and proves that:
If L is integrable then the tangent lift is also integrable.
In this paper, we generalize this lifting to tangent bundle of higher order.

Introduction

Let M be a differential manifold ($\dim M = m > 0$). Consider the mapping ϕ_M defined by:

$$\begin{array}{rccc}\phi_M\colon & TM\oplus T^*M\times_M TM\oplus T^*M & \to & \mathbb{R}\\ & \big((X_1,\alpha_1),(X_2,\alpha_2)\big) & \mapsto & \frac{1}{2}\big(\langle X_1,\alpha_2\rangle_M+\langle X_2,\alpha_1\rangle_M\big)\end{array}$$

where $\langle\cdot\rangle_M$ is the canonical pairing defined by:

$$\begin{array}{ccc} TM\times_M T^*M & \to & \mathbb{R}\\ (X,\alpha) & \mapsto & \langle X,\alpha\rangle_M\end{array}$$

An almost Dirac structure on M, is a sub vector bundle L of the vector bundle $TM\oplus T^*M$, which is isotropic with respect to the natural indefinite symmetric scalar product ϕ_M (i.e $\forall (X_1,\alpha_1),(X_2,\alpha_2)\in\Gamma(L)$, $\phi_M((X_1,\alpha_1),(X_2,\alpha_2))=0$), and such that the rank of L is equal to the dimension of M.

We define on the set $\Gamma(TM\oplus T^*M)$ of sections of $TM\oplus T^*M$ a bracket by:

$$\forall (X_1,\alpha_1),(X_2,\alpha_2)\in\Gamma(TM\oplus T^*M)$$
$$[(X_1,\alpha_1),(X_2,\alpha_2)]_C=\big([X_1,X_2],\mathcal{L}_{X_1}\alpha_2-i_{X_2}d\alpha_1\big)\,.$$

This bracket is called Courant bracket. A Dirac structure (or generalized Dirac structure) is an almost Dirac structure such that:

$$\forall (X_1,\alpha_1),(X_2,\alpha_2)\in\Gamma(L)\,,\qquad [(X_1,\alpha_1),(X_2,\alpha_2)]\in\Gamma(L)\,.$$

This condition is called “integrability condition”.

2010 *Mathematics Subject Classification*: primary 53C15; secondary 53C75, 53D05.

Key words and phrases: Dirac structure, almost Dirac structure, tangent functor of higher order, natural transformations.

Received August 24, 2009, revised August 2010. Editor I. Kolář.

For $(X_3,\alpha_3)\in\Gamma(TM\oplus T^*M)$, in [2] is defined the 3-tensor $T_{TM\oplus T^*M}$ on the vector bundle $TM\oplus T^*M$ by:

$$T_{TM\oplus T^*M}((X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3))=\phi_M\big([(X_1,\alpha_1),(X_2,\alpha_2)],(X_3,\alpha_3)\big)\,.$$

We put $T_L=T_{TM\oplus T^*M}|_{\Gamma(L)\times\Gamma(L)\times\Gamma(L)}$. The integrability condition of L is determined by the vanishing of the 3-tensor T_L on the vector bundle L.

For all integer $r,k\geq 1$, we have the jet functor T_k^r of k-dimensional velocity of order r and, when $k=1$, this functor is denoted by T^r and is called tangent bundle of order r. When $r=1$, T^1 is a natural equivalence of tangent functor T.

The main results of this paper are theorems 2 and 3: giving an almost Dirac structure L on M, we construct an almost Dirac structure L^r on T^rM and we prove that: L is integrable if and only if L^r is integrable.

All manifolds and maps are assumed to be infinitely differentiable. r will be a natural integer ($r\geq 1$).

1. Other characterization of generalized Dirac structure

Let V be a real vector space of dimension m. We consider the map

$$\begin{array}{rccc}\phi_V\colon & V\oplus V^*\times V\oplus V^* & \to & \mathbb{R}\\ & \big((u,u^*),(v,v^*)\big) & \mapsto & \frac{1}{2}\big(\langle u,v^*\rangle+\langle v,u^*\rangle\big)\end{array}$$

where $\langle\cdot\rangle$ is the dual bracket $V\times V^*\to\mathbb{R}$.

Definition 1. A constant Dirac structure on V is a sub vector space L of dimension m of $V\oplus V^*$ such that:

$$\forall (u,u^*),(v,v^*)\in L\,,\qquad \phi_V\big((u,u^*),(v,v^*)\big)=0\,.$$

Theorem 1. *A constant Dirac structure L on V is determined by a pair of linear maps $a\colon\mathbb{R}^m\to V$ and $b\colon\mathbb{R}^m\to V^*$ such that:*

$$a^*\circ b+b^*\circ a=0 \tag{1}$$

$$\ker a\cap\ker b=\{0\} \tag{2}$$

Proof. Condition (1) is the isotropy of constant Dirac structure, and condition (2) is the maximality of the isotropy. □

Remark 1.

(1) We say that the constant Dirac structure L is determined by the linear maps a and b.

(2) An almost Dirac structure on a differential manifold M is a sub vector bundle of $TM\oplus T^*M$ such that: $\forall x\in M$, the fiber L_x of L over x is a constant Dirac structure on T_xM.

(3) An almost Dirac structure at a point $x\in M$ is determined by a pair of maps $a_x\colon\mathbb{R}^m\to T_xM$, $b_x\colon\mathbb{R}^m\to T_x^*M$ such that:

$$\begin{cases}a_x^*\circ b_x+b_x^*\circ a_x=0\\ \ker a_x\cap\ker b_x=\{0\}\end{cases}$$

Corollary. *An almost Dirac structure is determined in a neighbourhood U of a local trivialization $L|_U \approx U \times \mathbb{R}^m$ by a pair of vector bundle morphisms $a\colon U \times \mathbb{R}^m \to T_U M$, $b\colon U \times \mathbb{R}^m \to T^*_U M$ over U such that:*

$$\forall x \in U\,, \qquad \begin{cases} a^*_x \circ b_x + b^*_x \circ a_x = 0 \\ \ker a_x \cap \ker b_x = \{0\} \end{cases}$$

We denote by p_1 and p_2 the natural projections of $TM \oplus T^*M$ onto TM and T^*M respectively. Note that $a\colon L \to TM$ and $b\colon L \to T^*M$ are really globally defined and are nothing more than the projections p_1 and p_2.

Example 1. Let M be an m-dimensional manifold.

(1) Let ω be a differential form on M of degree 2.

$$\Gamma = \{(X, i_X\omega)\,, \quad X \in \mathfrak{X}(M)\}\,.$$

Γ is the set of differential sections of an almost Dirac structure on M. It is a Dirac structure if and only if ω is pre-symplectic form.

(2) Let Π be a bivector field on M.

$$\Gamma' = \{(i_\Pi\alpha, \alpha)\,, \quad \alpha \in \Omega^1(M)\}\,.$$

Γ' is the set of differential sections of an almost Dirac structure on M. It is a Dirac structure if and only if Π is a Poisson bivector.

We denote by $(x^i, \dot{x}^i)$ and (x^i, p_i) a local coordinates system of TM and T^*M respectively. Let L be an almost Dirac structure on M defined locally by:

$$a\colon U \times \mathbb{R}^m \to TM \quad \text{and} \quad b\colon U \times \mathbb{R}^m \to T^*M\,.$$

We have:

$$\begin{cases} a(x^i, e_j) = a^k_j \dfrac{\partial}{\partial x^k} \\ b(x^i, e_j) = b_{jk} dx^k \end{cases}$$

where (e_j) denote the canonical basis of $\mathbb{R}^m$. Locally the 3-tensor field T_L is:

$$T_L = \sum_{\text{cyclic},i,j,k} \left(a^p_i \frac{\partial b_{js}}{\partial x^p} a^s_k + a^p_i \frac{\partial a^s_j}{\partial x^p} b_{ks} \right).$$

2. Tangent Dirac structure of higher order

$\kappa^r_M\colon T^rTM \to TT^rM$ and $\alpha^r_M\colon T^*T^rM \to T^rT^*M$ denote the natural transformations defined in [1] and [7]. We have:

$$\langle \kappa^r_M(u), v^* \rangle_{T^rM} = \langle u, \alpha^r_M(v^*) \rangle'_{T^rM}\,, \qquad (u, v^*) \in T^rTM \times_{T^rM} T^*T^rM$$

where $\langle \cdot \rangle'_{T^rM} = \tau_r \circ T^r\langle \cdot \rangle$ and $\tau_r(j^r_0\varphi) = \dfrac{d^r\varphi}{dt^r}(t)|_{t=0}$.

We denote by ε^r_M the inverse map of α^r_M.

Consider the maps $a\colon U \times \mathbb{R}^m \to TM$ and $b\colon U \times \mathbb{R}^m \to T^*M$. We take their tangents of order r, to get:

$$T^ra\colon T^rU \times \mathbb{R}^{m(r+1)} \to T^rTM \qquad \text{and} \qquad T^rb\colon T^rU \times \mathbb{R}^{m(r+1)} \to T^rT^*M\,.$$

We apply natural transformations κ_M^r and ε_M^r respectively, to get the vector bundle maps over id_{T^rU} defined by:

$$a^r : T^rU \times \mathbb{R}^{m(r+1)} \to TT^rM \quad \text{and} \quad b^r : T^rU \times \mathbb{R}^{m(r+1)} \to T^*T^rM\,.$$

Theorem 2. *The pair of maps a^r and b^r determines a generalized almost Dirac structure L^r on T^rM, which we call the tangent lift of order r of the generalized almost Dirac structure on M determined by a and b.*

Proof. Firstly, we prove that: $(a^r)^* \circ b^r + (b^r)^* \circ a^r = 0$. Let $j_0^r\psi, j_0^r\varphi \in T^r(U\times\mathbb{R}^m)$, where φ, ψ: $\mathbb{R} \to U \times \mathbb{R}^m$ differentials. We have:

$$\begin{aligned}
\langle (a^r)^* \circ b^r(j_0^r\varphi), j_0^r\psi\rangle &= \langle b^r(j_0^r\varphi), a^r(j_0^r\psi)\rangle \\
&= \langle \varepsilon_M^r \circ T^r b, \kappa_M^r \circ T^r a(j_0^r\psi)\rangle \\
&= \langle T^r b(j_0^r\varphi), T^r a(j_0^r\psi)\rangle'_{T^rM} \\
&= \tau^r \circ j_0^r(\langle b\circ\varphi, a\circ\psi\rangle_M) \\
&= \tau^r \circ j_0^r(\langle a^*\circ b\circ\varphi, \psi\rangle_M)\,.
\end{aligned}$$

By the same way, we have:

$$\langle (b^r)^* \circ a(j_0^r\varphi), j_0^r\psi\rangle = \tau^r \circ j_0^r(\langle b^*\circ a\circ\varphi, \psi\rangle_M)$$

we deduce that:

$$\Big\langle ((a^r)^* \circ b^r + (b^r)^*\circ a)(j_0^r\varphi), j_0^r\psi\Big\rangle = \tau^r \circ j_0^r\big(\big\langle (a^*\circ b + b^*\circ a)\circ\varphi, \psi\big\rangle_M\big) = 0\,.$$

Secondly we prove that: $\ker a^r \cap \ker b^r = \{0\}$. We prove this case for $r = 2$. The proof for $r \geq 3$ is similar.

In the local coordinates system, we have:

$$\begin{array}{rcl} a : U\times\mathbb{R}^m & \to & U\times\mathbb{R}^m \\ (x,e) & \mapsto & (x, ae) \end{array} \quad \text{and} \quad \begin{array}{rcl} b : U\times\mathbb{R}^m & \to & U\times(\mathbb{R}^m)^* \\ (x,e) & \mapsto & (x, be) \end{array}$$

$$a^2(x,\dot{x},\ddot{x},e,\dot{e},\ddot{e}) = (x,\dot{x},\ddot{x},ae,\dot{a}e + a\dot{e},\ddot{a}e + \dot{a}\dot{e} + a\ddot{a})$$

$$b^2(x,\dot{x},\ddot{x},e,\dot{e},\ddot{e}) = (x,\dot{x},\ddot{x},\ddot{b}e + \dot{b}\dot{e} + b\ddot{e}, \dot{b}e + b\dot{e}, be)$$

$$a^2(e,\dot{e},\ddot{e}) = \begin{pmatrix} a & 0 & 0 \\ \dot{a} & a & 0 \\ \ddot{a} & \dot{a} & a \end{pmatrix}\begin{pmatrix} e \\ \dot{e} \\ \ddot{e}\end{pmatrix} \quad \text{and} \quad b^2(e,\dot{e},\ddot{e}) = \begin{pmatrix} \ddot{b} & \dot{b} & b \\ \dot{b} & b & 0 \\ b & 0 & 0 \end{pmatrix}\begin{pmatrix} e \\ \dot{e} \\ \ddot{e}\end{pmatrix}\,.$$

If $a^2(e,\dot{e},\ddot{e}) = b^2(e,\dot{e},\ddot{e}) = 0$, we have:

$$ae = 0 \quad be = 0 \quad \Rightarrow \quad e \in \ker a \cap \ker b = \{0\}\,.$$

and it follows that $e = 0$.

$$\left\{\begin{array}{l} b\dot{e} + \dot{b}e = 0 \\ a\dot{e} + \dot{a}e = 0 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{l} b\dot{e} = 0 \\ a\dot{e} = 0 \end{array}\right.$$

e and $\dot e$ are constant, it follows that $\dot e = 0$.

$$\begin{cases} b\ddot e = 0 \\ a\ddot e = 0 \end{cases} \quad \Rightarrow \quad \ddot e = 0\,.$$

Thus $\ker a^2 \cap \ker b^2 = \{0\}$. □

Theorem 3. *The almost Dirac structure L on M is integrable if and only if the almost Dirac structure L^r on T^rM is integrable.*

Proof. Consider the local coordinates system $\{x^1, \dots, x^m\}$ of M, we have:

$$a(x^i, e_j) = a^i_k \frac{\partial}{\partial x^k} \quad \text{and} \quad b(x^i, e_j) = b_{ik} dx^k\,.$$

We have:

$$a^r = \begin{pmatrix} a^i_j & \dots & 0 \\ \vdots & \dots & \vdots \\ \overset{(r)}{a^i_j} & \dots & a^i_j \end{pmatrix} \quad \text{and} \quad b^r = \begin{pmatrix} \overset{(r)}{b_{ij}} & \dots & b_{ij} \\ \vdots & \dots & \vdots \\ b_{ij} & \dots & 0 \end{pmatrix}\,.$$

We get $a^r = (A^i_j)_{1\le i,j\le m(r+1)}$ and $b^r = (B_{ij})_{1\le i,j\le m(r+1)}$. For $q, d = 0, 1, \dots r$, we have:

$$\forall (i,j) \in \{qm+1, \dots, m(q+1)\} \times \{dm+1, \dots, m(d+1)\}\,,$$

$$\begin{cases} A^i_j = (a^{i-mq}_{j-md})^{(q-d)} \\ \\ B_{ij} = (b_{i-mq,j-md})^{(r-q-d)} \end{cases} \quad \text{and}$$

We adopt the following notation:

$$\frac{\partial}{\partial x^p} = \frac{\partial}{\partial x^{p-m\alpha}_\alpha} = \Big(\frac{\partial}{\partial x^{p-m\alpha}}\Big)^{(\alpha)} \quad \big(\alpha m + 1 \le p \le \alpha(m+1)\big)\,.$$

The Courant tensor T_{ijk} of the almost Dirac structure is given by:

$$T_{ijk} = \sum_{\text{cyclic}, i,j,k} A^p_i \frac{\partial B_{js}}{\partial x^p} A^s_k + A^p_i \frac{\partial A^s_j}{\partial x^p} B_{ks}, \quad \text{we wish to verify that} \quad T_{ijk} = 0.$$

We take $hm+1 \le i \le m(h+1)$, $\ell m + 1 \le j \le m(\ell+1)$ and $tm+1 \le k \le m(t+1)$ for $h, \ell, t = 0, 1, \dots, r$. We have:

$$\begin{aligned} T_{ijk} &= \sum_{q=0}^{r}\sum_{d=0}^{r}\sum_{p=qm+1}^{q(m+1)}\sum_{s=dm+1}^{d(m+1)} \Big(A^p_i \frac{\partial B_{js}}{\partial x^p} A^s_k + A^p_i \frac{\partial A^s_j}{\partial x^p} B_{ks}\Big) \\ &= (a^{p-mq}_{i-mh})^{(q-h)} \frac{\partial (b_{j-m\ell,s-md})^{(r-\ell-d)}}{\partial x^{p-mq}_q} (a^{s-md}_{k-mt})^{(d-t)} \\ &\quad + (a^{p-mq}_{i-mh})^{(q-h)} \frac{\partial (a^{s-md}_{j-m\ell})^{(d-\ell)}}{\partial x^{p-mq}_q} (b_{k-mt,s-md})^{(r-d-t)} \end{aligned}$$

$$= (a^{p-mq}_{i-mh})^{(q-h)}\Big(\frac{\partial b_{j-m\ell,s-md}}{\partial x^{p-mq}}\Big)^{(r-\ell-d-q)}(a^{s-md}_{k-mt})^{(d-t)}$$
$$+ (a^{p-mq}_{i-mh})^{(q-h)}\Big(\frac{\partial a^{s-md}_{j-m\ell}}{\partial x^{p-mq}}\Big)^{(d-\ell-q)}(b_{k-mt,s-md})^{(r-d-t)}$$
$$= \Big(a^{p-md}_{i-mh}\frac{\partial b_{j-m\ell,s-md}}{\partial x^{p-mq}}a^{s-md}_{k-mt}\Big)^{(r-\ell-h-t)} + \Big(a^{p-mq}_{i-mh}\frac{\partial a^{s-md}_{j-m\ell}}{\partial x^{p-mq}}b_{k-mt,s-md}\Big)^{(r-\ell-h-t)}$$
$$= (a^{p-mq}_{i-mh}\frac{\partial b_{j-m\ell,s-md}}{\partial x^{p-mq}}a^{s-md}_{k-mt} + a^{p-mq}_{i-mh}\frac{\partial a^{s-md}_{j-m\ell}}{\partial x^{p-mq}}b_{k-mt,s-md})^{(r-\ell-h-t)}$$

the calculation above shows that $T_L = 0$ if and only if $T_{L^r} = 0$. □

Remark 2. This construction generalizes the tangent lifts of higher order of Poisson and pre-symplectic structure to tangent bundle of higher order.

References

[1] Cantrijn, F., Crampin, M., Sarlet, W., Saunders, D., *The canonical isomorphism between T^kT^* and T^*T^k*, C. R. Acad. Sci., Paris, Sér. II **309** (1989), 1509–1514.

[2] Courant, T., *Tangent Dirac Structures*, J. Phys. A: Math. Gen. **23** (22) (1990), 5153–5168.

[3] Courant, T., *Tangent Lie Algebroids*, J. Phys. A: Math. Gen. **27** (13) (1994), 4527–4536.

[4] Gancarzewicz, J., Mikulski, W., Pogoda, Z., *Lifts of some tensor fields and connections to product preserving functors*, Nagoya Math. J. **135** (1994), 1–41.

[5] Grabowski, J., Urbanski, P., *Tangent lifts of Poisson and related structures*, J. Phys. A: Math. Gen. **28** (23) (1995), 6743–6777.

[6] Kolář, I., Michor, P., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, 1993.

[7] Morimoto, A., *Lifting of some type of tensors fields and connections to tangent bundles of p^r-velocities*, Nagoya Math. J. **40** (1970), 13–31.

[8] Ntyam, A., Wouafo Kamga, J., *New versions of curvatures and torsion formulas of complete lifting of a linear connection to Weil bundles*, Ann. Pol. Math. **82** (3) (2003), 233–240.

Department of Mathematics, The University of Yaoundé 1,
P.O BOX, 812, Yaoundé, Cameroon
E-mail: wambapm@yahoo.fr

Department of Mathematics, ENS Yaoundé,
P.O BOX, 47 Yaoundé, Cameroon
E-mail: antyam@uy1-uninet.cm

Department of Mathematics, The University of Yaoundé 1,
P.O BOX, 812, Yaoundé Cameroon
E-mail: wouafoka@yahoo.fr