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STRATONOVICH-WEYL CORRESPONDENCE FOR DISCRETE SERIES REPRESENTATIONS

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Abstract. Let $M=G/K$ be a Hermitian symmetric space of the noncompact type and let π be a discrete series representation of G holomorphically induced from a unitary character of K. Following an idea of Figueroa, Gracia-Bondìa and Vàrilly, we construct a Stratonovich-Weyl correspondence for the triple (G,π,M) by a suitable modification of the Berezin calculus on M. We extend the corresponding Berezin transform to a class of functions on M which contains the Berezin symbol of $d\pi(X)$ for X in the Lie algebra $\mathfrak{g}$ of G. This allows us to define and to study the Stratonovich-Weyl symbol of $d\pi(X)$ for $X\in\mathfrak{g}$.

1. Introduction

The notion of Stratonovich-Weyl correspondence was introduced in [35] as a generalization of the classical Weyl correspondence [1]. The systematic study of the Stratonovich-Weyl correspondences began with the work of J. M. Gracia-Bondìa, J. C. Vàrilly and their co-workers (see [22], [19], [17] and [21]).

Definition 1.1 ([21])**.** Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and let μ be a (suitably normalized) G-invariant measure on M. Then a Stratonovich-Weyl correspondence for the triple (G,π,M) is an isomorphism W from a vector space of operators on $\mathcal{H}$ to a space of (generalized) functions on M satisfying the following properties:

(1) W maps the identity operator of $\mathcal{H}$ to the constant function 1;

(2) the function $W(A^*)$ is the complex-conjugate of $W(A)$;

(3) Covariance: we have $W(\pi(g)\,A\,\pi(g)^{-1})(x)=W(A)(g^{-1}\cdot x)$;

(4) Traciality: we have

$$\int_M W(A)(x)W(B)(x)\,d\mu(x)=\mathrm{Tr}(AB)\,.$$

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For example, if G is the $(2n+1)$-dimensional Heisenberg group H_n which acts on $\mathbb{R}^{2n}$ by translations and π is the Schrödinger representation of H_n on $L^2(\mathbb{R}^n)$ then the classical Weyl correspondence gives a Stratonovich-Weyl correspondence for the triple $(H_n, \pi, \mathbb{R}^{2n})$ [20], [21].

When G is a compact semisimple Lie group, π a unitary irreducible representation of G on a finite dimensional Hilbert space $\mathcal{H}$ and M the coadjoint orbit of G which is associated with π by the Kostant-Kirillov method of orbits [26], a Stratonovich-Weyl correspondence for (G, π, M) was constructed in [19] by a suitable modification of the Berezin calculus on M (see also [12] and [15]).

Let us also mention that, in [17], a Stratonovich-Weyl correspondence for the massive representations of the Poincaré group was constructed. Another examples of Stratonovich-Weyl correspondences can be found in [5] and [6]. A generalization of the notion of Stratonovich-Weyl correspondence was introduced in [9].

In the present paper, we consider a connected semisimple noncompact real Lie group G with finite center. Let K be a maximal compact subgroup of G. We assume that the center of K has positive dimension. Then $M = G/K$ is a Hermitian symmetric space of the noncompact type which is diffeomorphic to a bounded symmetric domain $\mathcal{D}$. Let π_χ be a discrete series representation of G holomorphically induced from a unitary character χ of K. The representation π_χ can be realized on a Hilbert space $\mathcal{H}_\chi$ of holomorphic functions on $\mathcal{D}$. The domain $\mathcal{D}$ can be quantized by the general method of quantization introduced by Berezin [7], [8]. In [14], we gave explicit formulas for the Berezin symbols of $\pi_\chi(g)$ for $g \in G$ and $d\pi_\chi(X)$ for X in the Lie algebra $\mathfrak{g}$ of G (see also [13]). The Berezin symbol of $\pi_\chi(g)$ plays a central role in the Fourier theory for G [4], [38]. On the other hand, the Berezin symbol of $d\pi_\chi(X)$ is related to the coadjoint orbit of G associated with π_χ by the Kirillov-Kostant method of orbits (see [14, Proposition 5.5]; also, see [13, Proposition 3.3]). However, for the Fourier theory of G and for physical applications, it is convenient to use Stratonovich-Weyl symbols instead of Berezin symbols [19].

Berezin quantization on $\mathcal{D}$ gives an isomorphism S_χ from the space of Hilbert--Schmidt operators on $\mathcal{H}_\chi$ (endowed with the Hilbert-Schmidt norm) onto $L^2(\mathcal{D}, \mu)$ where μ is a G-invariant measure on $\mathcal{D}$. Here, we construct a Stratonovich-Weyl correspondence W_χ for the triple $(G, \pi_\chi, \mathcal{D})$ as in the compact case [19]. In fact, if we revisit [19] in the light of [3], [2], [30], [18] and [32], then we see that W_χ is the isometric part in the polar decomposition of S_χ, that is, $W_\chi = B_\chi^{-1/2} S_\chi$ where $B_\chi = S_\chi S_\chi^*$ is the so-called Berezin transform. Note that Berezin transforms for weighted Bergman spaces on bounded symmetric domains and their spectral decompositions have been intensively studied (see for instance [36], [32], [39] and [18]).

Here, in contrast to the compact case, the operator $d\pi_\chi(X)$ is generally not of the Hilbert-Schmidt type and then $W_\chi(d\pi_\chi(X))$ is not defined a priori. In this paper, we show how to extend B_χ to a class of functions on $\mathcal{D}$ which contains the Berezin symbols $S_\chi(d\pi_\chi(X))$ for $X \in \mathfrak{g}$. This allows us to define $W_\chi(d\pi_\chi(X))$. More precisely, we show that there exists a constant $a_\chi > 0$ such that $W_\chi(d\pi_\chi(X)) = a_\chi S_\chi(d\pi_\chi(X))$ for each $X \in \mathfrak{g}$. This result is similar to that obtained in the compact case, see [15, Proposition 5.2], and it implies that W_χ is generally not an adapted Weyl correspondence in the sense of [11].

This paper is organized as follows. In Section 2, we introduce the representation π_χ, the Berezin calculus on $\mathcal{D}$ and we review some results from [14]. In Section 3, we construct a Stratonovich-Weyl correspondence W_χ for $(G, \pi_\chi, \mathcal{D})$ as mentioned above. In Section 4, we show how to extend the Berezin transform to functions of the form $S_\chi(d\pi_\chi(u))$ where $u \in \mathcal{U}(\mathfrak{g})$. As an application, we extend W_χ to the operators $d\pi_\chi(X)$ $(X \in \mathfrak{g})$ and we determine the form of $W_\chi(d\pi_\chi(X))$ (Section 5). Finally, in Section 6, we study the case of the holomorphic discrete series of $G = SU(1,1)$.

2. Berezin quantization for discrete series representations

In this section, we first review some well-known facts on Hermitian symmetric spaces of the noncompact type and on holomorphic discrete series representations. Our main references are [23, Chapter VIII], [27, Chapter 6], [29, Chapter XII] and [34, Chapter II].

Let G be a connected semisimple noncompact real Lie group with finite center and let K be a maximal compact subgroup of G. We assume that the center of the Lie algebra of K is non trival. Then the homogeneous space G/K is a Hermitian symmetric space of the noncompact type.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathfrak{g}^c$ and $\mathfrak{k}^c$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ and G^c, K^c the corresponding complex Lie groups containing G and K, respectively. We denote by β the Killing form of $\mathfrak{g}^c$, that is, $\beta(X,Y) = \mathrm{Tr}(\operatorname{ad} X \operatorname{ad} Y)$ for X, $Y \in \mathfrak{g}^c$. Let $\mathfrak{p}$ be the ortho-complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to β. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{h}^c$ the complexification of $\mathfrak{h}$. Let H the connected subgroup of K with Lie algebra $\mathfrak{h}$. Let Δ be the root system of $\mathfrak{g}^c$ relative to $\mathfrak{h}^c$ and let $\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}^c$. Then we have the direct decompositions $\mathfrak{k}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta_c} \mathfrak{g}_\alpha$ and $\mathfrak{p}^c = \sum_{\alpha\in\Delta_n} \mathfrak{g}_\alpha$ where $\mathfrak{p}^c$ denotes the complexification of $\mathfrak{p}$ and Δ_c (resp. Δ_n) denotes the set of compact (resp. noncompact) roots. We choose an ordering on Δ as in [23, p. 384], and we denote by Δ^+, Δ_c^+ and Δ_n^+ the corresponding sets of positive roots, positive compact roots and positive noncompact roots, respectively. We set $\mathfrak{p}^+ = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \sum_{\alpha\in\Delta_n^+} \mathfrak{g}_{-\alpha}$. Then we have $[\mathfrak{k}^c, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$ and $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subspaces [23, Proposition 7.2.]. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we also have $[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}^c$. We denote by P^+ and P^- be the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$, respectively.

For each $\mu \in (\mathfrak{h}^c)^*$, we denote by H_μ the element of $\mathfrak{h}^c$ satisfying $\beta(H, H_\mu) = \mu(H)$ for all $H \in \mathfrak{h}^c$. Note that if μ is real-valued on $i\mathfrak{h}$ then $iH_\mu \in \mathfrak{g}$. For μ, $\nu \in (\mathfrak{h}^c)^*$, we set $(\mu, \nu) := \beta(H_\mu, H_\nu)$.

Let θ denotes conjugation over the real form $\mathfrak{g}$ of $\mathfrak{g}^c$. For $X \in \mathfrak{g}^c$, we set $X^* = -\theta(X)$. We denote by $g \to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \to X^*$ to G^c. Recall that the multiplication map $(z, k, y) \to zky$ is a diffeomorphism from $P^+ \times K^c \times P^-$ onto an open submanifold of G^c containing G [23, Lemma 7.9]. Following [29], we introduce the projections

$\zeta\colon P^+K^cP^- \to P^+$, $\kappa\colon P^+K^cP^- \to K^c$ and $\eta\colon P^+K^cP^- \to P^-$. Then the map $gK \to \log\zeta(g)$ from G/K to $\mathfrak{p}^+$ induces a diffeomorphism from G/K onto a bounded domain $\mathcal{D} \subset \mathfrak{p}^+$ [23, p. 392]. The natural action of G on G/K corresponds to the action of G on $\mathcal{D}$ given by $g\cdot Z = \log\zeta(g\exp Z)$. The G-invariant measure on $\mathcal{D}$ is $d\mu(Z) = \chi_0(\kappa(\exp Z^*\exp Z))\,d\mu_L(Z)$ where χ_0 is the character on K^c defined by $\chi_0(k) = \mathrm{Det}_{\mathfrak{p}^+}(\mathrm{Ad}\,k)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathcal{D}$ [29]. To simplify the notation, we set $k(Z) := \kappa(\exp Z^*\exp Z)$ for $Z \in \mathcal{D}$.

We introduce the holomorphic discrete series representations of scalar type of G as follows. Let χ be a unitary character of K. We also denote by χ the extension of χ to K^c. Let us introduce the Hilbert space $\mathcal{H}_\chi$ of holomorphic functions on $\mathcal{D}$ such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2\chi(k(Z))\,c_\chi d\mu(Z) < +\infty$$

where the constant c_χ is defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} (\chi\cdot\chi_0)\big(k(Z)\big)\,d\mu_L(Z)\,.$$

Note that $\chi(k(Z)) > 0$ for all $Z \in \mathcal{D}$. Indeed, for each $Z \in \mathcal{D}$ there exists $g_Z \in G$ such that $g_Z\cdot 0 = Z$. Writing $g_Z = \exp Zky$ with $k \in K^c$ and $y \in P^-$, we have $k(Z) = (k^*)^{-1}k^{-1}$ which gives $\chi(k(Z)) = \overline{\chi(k)}^{-1}\chi(k) = |\chi(g_Z^{-1}\exp Z)|^2 > 0$.

Proposition 2.1 ([31], [27])**.** *Let* $\lambda := d\chi|_{\mathfrak{h}^c}$ *and* $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. *Then* $\mathcal{H}_\chi$ *is nonzero if and only if* $(\lambda+\delta,\alpha) < 0$ *for every noncompact positive root* α. *In that case,* $\mathcal{H}_\chi$ *contains all polynomials. Moreover, the action of* G *on* $\mathcal{H}_\chi$ *defined by*

$$\pi_\chi(g)f(Z) = \chi\big(\kappa(g^{-1}\exp Z)\big)^{-1} f(g^{-1}\cdot Z)$$

is a unitary representation of G *which belongs to the holomorphic discrete series of* G.

In the rest of the paper, we assume that χ satisfies the preceding condition. Note that $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, we have the reproducing property $f(Z) = \langle f, e_Z\rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$, where the coherent states $e_Z \in \mathcal{H}_\chi$ are defined by $e_Z(W) = \chi(\kappa(\exp Z^*\exp W))^{-1}$ (see [29], Chapter XII for instance). Here $\langle\cdot,\cdot\rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

Now we introduce the Berezin calculus on $\mathcal{D}$ as follows. Consider an operator (not necessarily bounded) A on $\mathcal{H}_\chi$ whose domain contains e_Z for each $Z \in \mathcal{D}$. The Berezin (covariant) symbol of A is the function defined on $\mathcal{D}$ by

$$S_\chi(A)(Z) = \frac{\langle A\,e_Z\,,\,e_Z\rangle_\chi}{\langle e_Z\,,\,e_Z\rangle_\chi}\,.$$

From the equality

$$\pi_\chi(g)\,e_Z = \overline{\chi\big(\kappa(g\exp Z)\big)}^{-1}\,e_{g\cdot Z} \tag{2.1}$$

for $g \in G$ and $Z \in \mathcal{D}$ (see [14, Proposition 2.2]), we deduce that, for each $Z \in \mathcal{D}$, e_Z is a smooth vector for π_χ and hence the Berezin symbol of $d\pi_\chi(X)$ ($X \in \mathfrak{g}$) is well-defined.

Also, note that if A is an operator on $\mathcal{H}_\chi$ whose domain contains the coherent states e_Z ($Z \in \mathcal{D}$) then, for each $g \in G$, the domain of $\pi_\chi(g^{-1})A\pi_\chi(g)$ also contains e_Z for each $Z \in \mathcal{D}$ and we have

$$S_\chi(\pi_\chi(g^{-1})A\pi_\chi(g))(Z) = S(A)(g \cdot Z) \tag{2.2}$$

for each $g \in G$ and $Z \in \mathcal{D}$.

In [14], we gave explicit expressions for the derived representation $d\pi_\chi$, for the Berezin symbols of $\pi_\chi(g)$ and $d\pi_\chi(X)$. In the rest of this section, we recall some results from [14].

If L is a Lie group and X is an element of the Lie algebra of L then we denote by X^+ the right invariant vector field on L generated by X, that is, $X^+(h) = \frac{d}{dt}(\exp tX)h|_{t=0}$ for $h \in L$.

Let $p_{\mathfrak{p}^+}$, $p_{\mathfrak{k}^c}$ and $p_{\mathfrak{p}^-}$ be the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\mathfrak{k}^c$ and $\mathfrak{p}^-$ associated with the direct decomposition $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$. By differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+K^cP^-$, we can easily prove the following result.

Lemma 2.1 ([14]). *Let $X \in \mathfrak{g}^c$ and $g = z\,k\,y$ where $z \in P^+$, $k \in K^c$ and $y \in P^-$. We have*

1. $d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\, p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\, X))^+(z)$.
2. $d\kappa_g(X^+(g)) = (p_{\mathfrak{k}^c}(\mathrm{Ad}(z^{-1})\, X))^+(k)$.
3. $d\eta_g(X^+(g)) = (\mathrm{Ad}(k^{-1})\, p_{\mathfrak{p}^-}(\mathrm{Ad}(z^{-1})\, X))^+(y)$.

From this lemma, we deduce the following propositions (see [14] and also [29, Proposition XII.2.1]).

Proposition 2.2. *For $X \in \mathfrak{g}^c$ and $f \in \mathcal{H}_\chi$, we have*

$$d\pi_\chi(X)f(Z) = d\chi(p_{\mathfrak{k}^c}(\mathrm{Ad}((\exp Z)^{-1})\, X))\, f(Z) - (df)_Z\big(p_{\mathfrak{p}^+}(e^{-\operatorname{ad} Z}\, X)\big)\,.$$

In particular, we have

1. *If $X \in \mathfrak{p}^+$ then $d\pi_\chi(X)f(Z) = -(df)_Z(X)$.*
2. *If $X \in \mathfrak{k}^c$ then $d\pi_\chi(X)f(Z) = d\chi(X)f(Z) + (df)_Z([Z,X])$.*
3. *If $X \in \mathfrak{p}^-$ then $d\pi_\chi(X)f(Z) = -d\chi([Z,X])f(Z) - \frac{1}{2}(df)_Z([Z,[Z,X]])$.*

Proposition 2.3.

1. *Let $g \in G$. We have*
$$S_\chi\big(\pi_\chi(g)\big)(Z) = \chi\big(\kappa(\exp Z^* g^{-1} \exp Z)^{-1}\kappa(\exp Z^* \exp Z)\big)\,.$$
2. *Let $X \in \mathfrak{g}^c$. We have*
$$S_\chi\big(d\pi_\chi(X)\big)(Z) = d\chi\big(p_{\mathfrak{k}^c}(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)\, X\big)\,.$$
In particular, for $X \in \mathfrak{k}^c$, we have
$$S_\chi\big(d\pi_\chi(X)\big)(Z) = d\chi\big(X + [\log \eta((\exp Z^* \exp Z), [X,Z]]\big)\,.$$

(3) *We can write*

$$S\big(d\pi_\chi(X)\big)(Z) = i\beta\big(\psi_\chi(Z), X\big)$$

where the map ψ_χ *defined by*

$$\psi_\chi(Z) := \mathrm{Ad}\big(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)\big)\,(-iH_\lambda)$$

is a diffeomorphism from $\mathcal{D}$ *onto the orbit* $\mathcal{O}_\chi$ *of* $-iH_\lambda \in \mathfrak{g}$ *for the adjoint action of* G.

3. Berezin transform and Stratonovich-Weyl correspondence

We retain the notation from Section 2. Also, we denote by $\mathcal{L}_2(\mathcal{H}_\chi)$ the space of the Hilbert-Schmidt operators on $\mathcal{H}_\chi$ and by μ_χ the G-invariant measure on $\mathcal{D}$ defined by $d\mu_\chi(Z) = c_\chi d\mu_0(Z) = c_\chi \chi_0(k(Z)) d\mu_L(Z)$. Then the map S_χ is a bounded operator on $\mathcal{L}_2(\mathcal{H}_\chi)$ into $L^2(\mathcal{D}, \mu_\chi)$ which is one-to-one and has dense range [33], [36]. It is not hard to verify that the adjoint operator $S_\chi^*\colon L^2(\mathcal{D}, \mu_\chi) \to \mathcal{L}_2(\mathcal{H}_\chi)$ is given by

$$S_\chi^* F = \int_{\mathcal{D}} F(Z)\, P_Z\, d\mu_\chi(Z) \tag{3.1}$$

where P_Z is the orthogonal projection operator of $\mathcal{H}_\chi$ on the line generated by e_Z. The Berezin transform $B_\chi = S_\chi S_\chi^*$ is then the operator on $L^2(\mathcal{D}, \mu_\chi)$ given by

$$B_\chi F(Z) = \int_{\mathcal{D}} F(W)\, \frac{|\langle e_Z, e_W\rangle|_\chi^2}{\langle e_Z, e_Z\rangle_\chi \langle e_W, e_W\rangle_\chi}\, d\mu_\chi(W) \tag{3.2}$$

(see, for instance, [7], [36], [39]). Note that B_χ commute with $\rho(g)$ $(g \in G)$ where ρ denotes the left-regular representation of G on $L^2(\mathcal{D}, \mu_\chi)$.

Now, we introduce the polar decomposition of S_χ: $S_\chi = (S_\chi S_\chi^*)^{1/2} W = B_\chi^{1/2} W_\chi$ where $W_\chi := B_\chi^{-1/2} S_\chi$ is a unitary operator from $\mathcal{L}_2(\mathcal{H}_\chi)$ onto $L^2(\mathcal{D}, \mu_\chi)$. The following proposition is analogous to Theorem 3 of [19].

Proposition 3.1. *The map* $W_\chi\colon \mathcal{L}_2(\mathcal{H}_\chi) \to L^2(\mathcal{D}, \mu_\chi)$ *is a Stratonovich-Weyl correspondence for the triple* $(G, \pi_\chi, \mathcal{D})$.

Proof. We have to verify that the properties (1), (2) and (3) of Definition 1.1 are satisfied. Property (1) follows from the fact that $B_\chi 1 = 1$. Since we have the properties $S_\chi(A^*) = \overline{S_\chi(A)}$ and $S_\chi^*(\overline{F}) = (S_\chi^* F)^*$, we see that B_χ hence $B_\chi^{-1/2}$ commute with complex conjugation. This gives Property (2). Finally, Property (3) is a consequence of Equality (2.2). $\square$

In the rest of this section, we show that the Stratonovich-Weyl correspondence W_χ is related to the operator Q introduced in [32] as a natural generalization of the Weyl transform.

Let $A \in \mathcal{L}_2(\mathcal{H}_\chi)$. For $Z \in \mathcal{D}$, we have

$$\begin{aligned} A\,f(Z) &= \langle A\,f\,,\,e_Z\rangle_\chi = \langle f\,,\,A^*\,e_Z\rangle_\chi \\ &= \int_{\mathcal{D}} f(W)\overline{A^*\,e_Z(W)}\,\langle e_W, e_W\rangle_\chi^{-1}\,d\mu_\chi(W) \\ &= \int_{\mathcal{D}} f(W)\overline{\langle A^*\,e_Z, e_W\rangle}_\chi\,\langle e_W, e_W\rangle_\chi^{-1}\,d\mu_\chi(W) \\ &= \int_{\mathcal{D}} f(W)\langle A\,e_W, e_Z\rangle_\chi\,\langle e_W, e_W\rangle_\chi^{-1}\,d\mu_\chi(W)\,. \end{aligned}$$

This shows that the kernel of A is the function

$$k_A(Z,W) = \langle A\,e_W, e_Z\rangle_\chi \tag{3.3}$$

which is holomorphic in the variable Z and anti-holomorphic in the variable W.

Now, let $\mathcal{H}_\chi^-$ be the Hilbert space conjugate to $\mathcal{H}_\chi$, that is, the elements of $\mathcal{H}_\chi^-$ are the functions $\overline{f}$ where $f \in \mathcal{H}_\chi$ and the Hilbert norm on $\mathcal{H}_\chi^-$ is defined by $\|\overline{f}\|_{\mathcal{H}_\chi^-} = \|f\|_\chi$. We form the Hilbert space tensor product $\mathcal{H}_\chi \otimes \mathcal{H}_\chi^-$ which can be identified with $\mathcal{L}_2(\mathcal{H}_\chi)$ endowed with the Hilbert-Schmidt norm by means of the map $\mathcal{K} : A \to k_A$. In [32], the authors introduced the restriction operator $D : \mathcal{H}_\chi \otimes \mathcal{H}_\chi^- \to L^2(\mathcal{D}, \mu_\chi)$

$$k(Z,W) \to k(Z,Z)\langle e_Z, e_Z\rangle_\chi^{-1}$$

and its polar decomposition $D = |D|Q$. Then, by using (3.3), we see immediately that $S_\chi = D \circ \mathcal{K}$. Hence we can conclude that $W_\chi = Q \circ \mathcal{K}$.

4. Extension of the Berezin transform

We introduce some additional notation. Let $(E_\alpha)_{\alpha\in\Delta_n^+}$ be a basis for $\mathfrak{p}^+$ as in [23, Chapter VIII, Corollary 7.6]. In particular, we have $E_\alpha \in \mathfrak{g}_\alpha$ and $[E_\alpha, E_{-\alpha}] = \frac{2}{\alpha(H_\alpha)}H_\alpha$ for each $\alpha \in \Delta_n^+$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an enumeration of Δ_n^+. We write $Z = \sum_{k=1}^n z_k E_{\alpha_k}$ for the decomposition of $Z \in \mathfrak{p}^+$ in the basis (E_{α_k}). If f is a holomorphic function on $\mathcal{D}$, then we denote by $\partial_k f$ the partial derivative of f with respect to z_k. We say that a function $f(Z)$ on $\mathcal{D}$ is a polynomial of degree q in the variable Z if $f(\sum_{k=1}^n z_k E_{\alpha_k})$ is a polynomial of degree q in the variables $z_1, z_2, \ldots, z_n$. For Z, $W \in \mathcal{D}$, we set $l_Z(W) := \log\eta(\exp Z^* \exp W) \in \mathfrak{p}^-$. We first establish some technical lemmas.

Lemma 4.1. (1) *For Z, $W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have*

$$\frac{d}{dt}\,e_Z(W+tV)\Big|_{t=0} = -e_Z(W)\,d\chi([l_Z(W), V])\,.$$

(2) *For Z, $W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have*

$$\frac{d}{dt}\,l_Z(W+tV)\Big|_{t=0} = \frac{1}{2}[l_Z(W), [l_Z(W), V]]\,.$$

(3) *The function* $(\partial_{k_1}\partial_{k_2}\ldots\partial_{k_q}\,e_Z)(W)$ *is of the form* $e_Z(W)P(l_Z(W))$ *where* P *is a polynomial of degree* $\leq q$.

(4) *For each* $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, *the operator* $d\pi_\chi(X_1X_2\ldots X_q)$ *is a sum of terms of the form* $P(Z)\partial_{k_1}\partial_{k_2}\ldots\partial_{k_q}$ *where* P *is a polynomial in* Z *of degree* $\leq 2q$.

Proof. By (2) of Lemma 2.1, we have

$$\begin{aligned}\frac{d}{dt}\,e_Z(W+tV)\big|_{t=0} &= \frac{d}{dt}\,\chi^{-1}\big(\kappa(\exp Z^*\exp W\exp tV)\big)\big|_{t=0}\\ &= d\chi^{-1}_{\kappa(\exp Z^*\exp W)}d\kappa_{\exp Z^*\exp W}\big(\big(\operatorname{Ad}(\exp Z^*\exp W)\,V\big)^+(\exp Z^*\exp W)\big)\\ &= -\chi^{-1}(\kappa(\exp Z^*\exp W))\,d\chi\big(p_{\mathfrak{k}^c}\big(\operatorname{Ad}\big(\kappa(\exp Z^*\exp W)\eta(\exp Z^*\exp W)\big)V\big)\big)\,.\end{aligned}$$

Since $d\chi(p_{\mathfrak{k}^c}(\operatorname{Ad}(k)X)) = d\chi(\operatorname{Ad}(k)p_{\mathfrak{k}^c}(X)) = d\chi(p_{\mathfrak{k}^c}(X))$ for each $k \in K^c$ and each $X \in \mathfrak{g}^c$, we obtain

$$\begin{aligned}\frac{d}{dt}\,e_Z(W+tV)\big|_{t=0} &= -e_Z(W)d\chi\left(p_{\mathfrak{k}^c}\big(\operatorname{Ad}\big(\eta(\exp Z^*\exp W)\big)V\big)\right)\\ &= -e_Z(W)d\chi\left([\log\eta(\exp Z^*\exp W), V]\right)\,.\end{aligned}$$

Then Statement (1) is proved. Similarly, by using (3) of Lemma 2.1, we have

$$\begin{aligned}&\frac{d}{dt}\,l_Z(W+tV)\big|_{t=0}\\ &\quad= d\log_{\eta(\exp Z^*\exp W)}d\eta_{\exp Z^*\exp W}\left((\operatorname{Ad}(\exp Z^*)V)^+(\exp Z^*\exp W)\right)\\ &\quad= \operatorname{Ad}\kappa(\exp Z^*\exp W)^{-1}\,p_{\mathfrak{p}^-}\left(\operatorname{Ad}\big(\zeta(\exp Z^*\exp W)^{-1}\exp Z^*\big)V\right)\\ &\quad= p_{\mathfrak{p}^-}\left(\operatorname{Ad}\big(\eta(\exp Z^*\exp W)\big)V\right)\\ &\quad= \frac{1}{2}\,[\log\eta(\exp Z^*\exp W), [\log\eta(\exp Z^*\exp W), V]]\end{aligned}$$

and hence we have proved (2). Now, by induction on q, we easily obtain (3). Finally, (4) is a consequence of Proposition 2.3. □

The following lemma is an immediate consequence of Lemma 4.1 (see also [16]).

Lemma 4.2. *Each holomorphic differential operator on* $\mathcal{D}$ *with polynomial coefficients has Berezin symbol. In particular, for each* $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, $S_\chi(d\pi_\chi(X_1X_2\ldots X_q))$ *is well-defined and is a sum of terms of the form* $P(Z)Q(l_Z(Z))$ *where* P *is a polynomial of degree* $\leq 2q$ *and* Q *is a polynomial of degree* $\leq q$.

Lemma 4.3. *Let* $\gamma_1, \gamma_2, \ldots, \gamma_r$ *be a subset of* Δ_n^+ *consisting of strongly orthogonal roots.*

(1) *Let* $\tilde\chi$ *be a character (non necessarily unitary) on* K *and* $\tilde\lambda = d\tilde\chi|_{\mathfrak{h}^c}$. *Then* $(\tilde\lambda, \gamma_k)$ *does not depend on* $k = 1, 2, \ldots, r$.

(2) *In particular, let* $\lambda_0 := d\chi_0|_{\mathfrak{h}^c}$. *Then* $q_\chi = -2\frac{(\lambda_0+\lambda,\gamma_k)}{(\gamma_k,\gamma_k)}$ *does not depend on* $k = 1, 2, \ldots, r$.

Proof. (1) By [28, Lemma 2.1], each γ_r is of the form $\gamma_r=\mu_1+\sum_{i\geq 2}n_i\mu_i$ where μ_1 is the unique noncompact simple root and the μ_i $(i\geq 2)$ are the compact simple roots. Since $(\tilde\lambda,\mu_i)=0$ for each $i\geq 2$, we have $(\tilde\lambda,\gamma_k)=(\tilde\lambda,\mu_1)$ for each k.
(2) By [28, Theorem 2], (γ_k,γ_k) does not depend on k. The result then follows from (1). ☐

We are now in position to extend the Berezin transform to a class of Berezin symbols of unbounded operators. Note that, by fixing an Iwasawa decomposition $G=NAK$, we get a smooth section $G/K\to NA\subset G$ and then we obtain a smooth section $\mathcal{D}\to G$, $Z\to g_Z$.

Proposition 4.1. *If $q\leq q_\chi$ then for each $X_1,X_2,\ldots,X_q\in\mathfrak{g}^c$, the Berezin transform of $S_\chi(d\pi_\chi(X_1X_2\ldots X_q))$ is well-defined.*

Proof. First, note that if we change variables $W\to g_Z\cdot W$ in the integral (3.2) then by (2.1) we obtain

$$\begin{aligned}(B_\chi F)(Z)&=\int_{\mathcal{D}}F(g_Z\cdot W)\langle e_W,e_W\rangle_\chi^{-1}\,d\mu_\chi(W)\\ &=\int_{\mathcal{D}}F(g_Z\cdot W)\,c_\chi(\chi\cdot\chi_0)(k(W))\,d\mu_L(W)\,. \qquad (4.1)\end{aligned}$$

In particular, if $F(W)=S_\chi(d\pi_\chi(X_1X_2\ldots X_q))(W)$ then, by (2.2), we have $F(g_Z\cdot W)=S_\chi(d\pi_\chi(Y_1Y_2\ldots Y_q))(W)$ where $Y_k:=\mathrm{Ad}(g_Z^{-1})X_k$ for $k=1,2,\ldots,q$.

We will show that, under the condition that $q\leq q_\chi$, the function

$$W\to S_\chi\big(d\pi_\chi(Y_1Y_2\ldots Y_q)\big)(W)(\chi\cdot\chi_0)\big(k(W)\big)$$

is bounded hence integrable on $\mathcal{D}$. Recall that $S_\chi(d\pi_\chi(Y_1Y_2\ldots Y_q))(W)$ is the sum of terms of the form $P(W)Q(\log\eta(\exp W^*\exp W))$ where P is a polynomial and Q is a polynomial of degree $\leq q$.

Let $\gamma_1,\gamma_2,\ldots,\gamma_r$ as in Lemma 4.3. Then each $W\in\mathcal{D}$ can be written as $W=\mathrm{Ad}(k)\big(\sum_{k=1}^r t_sE_{\gamma_s}\big)$ for $k\in K$ and $-1<t_s<1$, $1\leq s\leq r$ (see for instance [23, Chapter VIII]). From matrix calculations in the group $SL(2,\mathbb{C})$ and strongly orthogonality of the roots γ_s, we have

$$\log\eta(\exp W^*\exp W)=\mathrm{Ad}(k)\Big(-\sum_{s=1}^r\frac{t_s}{1-t_s^2}\,E_{-\gamma_s}\Big) \qquad (4.2)$$

and

$$k(W)=\kappa(\exp W^*\exp W)=k\,\exp\Big(\sum_{s=1}^r\log\frac{1}{1-t_s^2}\,[E_{\gamma_s},E_{-\gamma_s}]\Big)\,k^{-1}\,. \qquad (4.3)$$

Then

$$(\chi\cdot\chi_0)\big(k(W)\big)=\prod_{s=1}^r(1-t_s^2)^{-(\lambda+\lambda_0)([E_{\gamma_s},E_{-\gamma_s}])}$$

and, since we have

$$-(\lambda+\lambda_0)([E_{\gamma_s},E_{-\gamma_s}])=-2\,\frac{(\lambda+\lambda_0)(H_{\gamma_s})}{\gamma_s(H_{\gamma_s})}=-2\frac{(\lambda_0+\lambda,\gamma_s)}{(\gamma_s,\gamma_s)}=q_\chi\,,$$

we obtain

$$(\chi \cdot \chi_0)\big(k(W)\big) = \prod_{s=1}^{r} (1 - t_s^2)^{q_\chi}\,. \tag{4.4}$$

Hence we see that the condition $q \leq q_\chi$ guarantees that the functions

$$W \to P(W)\, Q\big(l_W(W)\big)(\chi \cdot \chi_0)\big(k(W)\big)$$

are bounded on $\mathcal{D}$. This finishes the proof. $\square$

Remarks.

(1) Since we have

$$\int_{\mathcal{D}} (\chi \cdot \chi_0)\big(k(W)\big)\, d\mu_L(W) < +\infty\,,$$

we see immediately from (4.4) that $q_\chi \geq 0$.

(2) By [13, Lemma 5.2], we have $\chi(k(Z)) \leq 1$ for each $Z \in \mathcal{D}$ with equality if and only if $Z = 0$. This implies that $-\lambda([E_{\gamma_s}, E_{-\gamma_s}]) > 0$ for each $s = 1, 2, \ldots, r$.

(3) An extension of the Berezin transform to another class of functions on $\mathcal{D}$ is given in [36].

5. Stratonovich-Weyl symbols of derived representation operators

When $q_\chi \geq 1$, the Berezin transform of $S_\chi(d\pi_\chi(X))$ $(X \in \mathfrak{g}^c)$ is well-defined by Proposition 4.1. In this section, we determine the form of $B_\chi S_\chi(d\pi_\chi(X))$ and we show how to extend the Stratonovich-Weyl correspondence to the operators $d\pi_\chi(X)$ $(X \in \mathfrak{g}^c)$. To this aim, we first study the linear form b_χ defined on $\mathfrak{g}^c$ by

$$b_\chi(X) := B_\chi S_\chi\big(d\pi_\chi(X)\big)(0) = \int_{\mathcal{D}} S_\chi\big(d\pi_\chi(X)\big)(Z)\chi\big(k(Z)\big)\, d\mu_\chi(Z)\,. \tag{5.1}$$

Proposition 5.1. *There exists a real number $a_\chi \geq 1$ such that $b_\chi(X) = a_\chi \lambda(p_{\mathfrak{h}^c}(X))$ for each $X \in \mathfrak{g}^c$. Here $p_{\mathfrak{h}^c}$ denotes the projection operator from $\mathfrak{g}^c$ onto $\mathfrak{h}^c$ associated with the decomposition $\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$.*

Proof. For each $k \in K$ and each $Z \in \mathcal{D}$, we have $\pi_\chi(k)e_Z = \chi(k)e_{k\cdot Z}$ and then $\langle e_{k\cdot Z}, e_{k\cdot Z}\rangle_\chi = \langle e_Z, e_Z\rangle_\chi$. Thus, by changing variables $Z \to k^{-1}\cdot Z$ in the integral (5.1) and by using the fact that

$$S_\chi\big(d\pi_\chi(X)\big)(k^{-1}\cdot Z) = S_\chi\big(d\pi_\chi(\mathrm{Ad}(k)X)\big)(Z)\,,$$

we get

$$b_\chi(X) = b_\chi\big(\,\mathrm{Ad}(k)X\big) \tag{5.2}$$

for each $k \in K$ and each $X \in \mathfrak{g}^c$. Specializing to $X = E_\alpha$ $(\alpha \in \Delta)$ and $k = \exp Y$ where $Y \in \mathfrak{h}$ and noting that $\mathrm{Ad}(k)E_\alpha = e^{\alpha(Y)}E_\alpha$, we find that $b_\chi(E_\alpha) = 0$ for each $\alpha \in \Delta$.

On the other hand, observe that, for each $X \in \mathfrak{g}$,

$$\overline{b_\chi(X)} = B_\chi S_\chi\big(d\pi_\chi(X^*)\big)(0) = b_\chi(X^*) = -b_\chi(X)$$

and then $b_\chi(X)\in i\mathbb{R}$. Now, introduce the element $H_{b_\chi}\in\mathfrak{k}^c$ satisfying $b_\chi(Y)=\beta(Y,H_{b_\chi})$ for each $Y\in\mathfrak{k}^c$. Then $H_{b_\chi}\in i\mathfrak{k}$. By (5.2), we have $\mathrm{Ad}(k)H_{b_\chi}=H_{b_\chi}$ for each $k\in K$. This implies that iH_{b_χ} lies in the center of $\mathfrak{k}$. Since the center of $\mathfrak{k}$ is one-dimensional (see for instance [24]) and contains iH_λ, there exists a real number a_χ such that $iH_{b_\chi}=a_\chi iH_\lambda$. Thus we have $b_\chi=a_\chi\lambda$ on $\mathfrak{h}^c$. Hence, we have obtained that $b_\chi(X)=a_\chi\lambda(p_{\mathfrak{h}^c}(X))$ for each $X\in\mathfrak{g}^c$. It remains to show that $a_\chi\geq 1$. To this goal, we consider the function φ_χ defined on $\mathcal{D}$ by $\varphi_\chi(Z)=S_\chi(d\pi_\chi(H_\lambda))(Z)$. By Proposition 2.3, we have

$$\varphi_\chi(Z)=\lambda(H_\lambda)+\lambda([\log\eta(\exp Z^*\exp Z),[H_\lambda,Z]])\,.$$

Moreover, since iH_λ is central in $\mathfrak{k}$, we have $\varphi_\chi(\mathrm{Ad}(k)Z)=\varphi_\chi(Z)$ for each $k\in K$ and $Z\in\mathcal{D}$.

As in the proof of Proposition 4.1 we write each $Z\in\mathcal{D}$ as $Z=\mathrm{Ad}(k)\left(\sum_{s=1}^r t_sE_{\gamma_s}\right)$ with $k\in K$ and $-1<t_s<1$ for $s=1,2,\ldots,r$. Then, for each $Z\in\mathcal{D}$, we have

$$\begin{aligned}\varphi_\chi(Z)&=\lambda(H_\lambda)+\lambda\Big(\Big[-\sum_{s=1}^r\frac{t_s}{1-t_s^2}E_{-\gamma_s},[H_\lambda,\sum_{s=1}^r t_sE_{\gamma_s}]\Big]\Big)\\&=\lambda(H_\lambda)+2\sum_{s=1}^r\frac{t_s^2}{1-t_s^2}\frac{(\gamma_s,\lambda)^2}{(\gamma_s,\gamma_s)}\geq\lambda(H_\lambda)\end{aligned}$$

Thus

$$a_\chi\lambda(H_\lambda)=b_\chi(H_\lambda)=\int_{\mathcal{D}}\varphi_\chi(Z)\chi\big(k(Z)\big)\,d\mu_\chi(Z)\geq\lambda(H_\lambda)\,.$$

Hence $a_\chi\geq 1$. □

Proposition 5.2. *With the notation of Proposition 5.1, for each $X\in\mathfrak{g}^c$, we have $B_\chi S_\chi(d\pi_\chi(X))=a_\chi S_\chi(d\pi_\chi(X))$.*

Proof. Applying successively Equality (4.1), Proposition 5.1, Proposition 2.3 and Equality (2.2), we have

$$\begin{aligned}B_\chi S_\chi(d\pi_\chi(X))(Z)&=B_\chi S_\chi\big(d\pi_\chi(\mathrm{Ad}(g_Z^{-1})X)\big)(0)\\&=a_\chi\lambda\big(p_{\mathfrak{h}^c}(\mathrm{Ad}(g_Z^{-1})X)\big)=a_\chi S_\chi\big(d\pi_\chi(\mathrm{Ad}(g_Z^{-1})X)\big)(0)\\&=a_\chi S_\chi\big(d\pi_\chi(X)\big)(g_Z\cdot 0)=a_\chi S_\chi\big(d\pi_\chi(X)\big)(Z)\end{aligned}$$

for each $Z\in\mathcal{D}$ and each $X\in\mathfrak{g}^c$. □

Consequently, we can define $B_\chi^{-1/2}$ on the space of functions of the form $S_\chi(d\pi_\chi(X))$ and W_χ on the space $\{d\pi_\chi(X)\,:\,X\in\mathfrak{g}^c\}$. Moreover, we have $W_\chi(d\pi_\chi(X))=a_\chi^{-1/2}S_\chi(d\pi_\chi(X))$ for each $X\in\mathfrak{g}^c$.

In [14], we showed that S_χ is adapted to π_χ in the sense that the linear form $X\to -iS_\chi(d\pi_\chi(X))$ lies in the coadjoint orbit of G associated with π_χ by the method of orbits (see also Proposition 2.3). In general, we have $a_\chi\neq 1$ (see for example Section 6) and then W_χ is not adapted to π_χ. However, the following proposition shows that W_χ is ‘asymptotically adapted’.

Proposition 5.3. *We have* $\lim_{m\to+\infty}a_{\chi^m}=1$.

Proof. Here we use the same notation as in the proofs of Proposition 4.1 and Proposition 5.1. We have

$$a_{\chi^m}=\frac{1}{(m\lambda,m\lambda)}\int_{\mathcal{D}}\varphi_{\chi^m}(Z)(\chi^m\cdot\chi_0)\big(k(Z)\big)\,c_{\chi^m}d\mu_L(Z)\,.$$

Then

$$a_{\chi^m}-1=\int_{\mathcal{D}}\frac{\varphi_{\chi^m}(Z)-(m\lambda,m\lambda)}{(m\lambda,m\lambda)}\,(\chi^m\cdot\chi_0)\big(k(Z)\big)\,c_{\chi^m}d\mu_L(Z)\,.$$

Changing variables $Z\to Z/\sqrt{m}$ in this integral, we get

$$a_{\chi^m}-1=m^{-n}c_{\chi^m}\int_{\sqrt{m}\mathcal{D}}I_m(Z)\,d\mu_L(Z)$$

where we have put

$$I_m(Z):=\frac{\varphi_{\chi^m}(Z/\sqrt{m})-(m\lambda,m\lambda)}{(m\lambda,m\lambda)}\,(\chi^m\cdot\chi_0)\big(k(Z/\sqrt{m})\big)\,.$$

By [13, Lemma 5.3], we have $\lim_{m\to+\infty}m^{-n}c_{\chi^m}=\pi^{-n}$. On the other hand, we have

$$\begin{aligned}I_m(Z)=&\Big(\sum_{s=1}^{r}2\frac{(\gamma_s,\lambda)^2}{(\lambda,\lambda)(\gamma_s,\gamma_s)}\,\frac{(t_s/\sqrt{m})^2}{1-(t_s/\sqrt{m})^2}\Big)\\&\times\prod_{s=1}^{r}(1-(t_s/\sqrt{m})^2)^{-(\lambda_0+m\lambda)([E_{\gamma_s},E_{-\gamma_s}])}\end{aligned}$$

where $|t_s|<\sqrt{m}$ for each s and we see that $\lim_{m\to+\infty}I_m(Z)=0$. In order to obtain the desired result, it suffices to verify that the Lebesgue dominated convergence theorem can be applied. This can be done as follows. Recall that we have $-\lambda([E_{\gamma_s},E_{-\gamma_s}])>0$ for each $s=1,2,\ldots,r$. Then we fix m_0 so that we have

$$-m\lambda([E_{\gamma_s},E_{-\gamma_s}])-1\geq-\frac{m}{2}\lambda([E_{\gamma_s},E_{-\gamma_s}])$$

for each $m\geq m_0$ and each $s=1,2,\ldots,r$. Thus for each $m\geq m_0$ and each $Z\in\sqrt{m}\mathcal{D}$, we have

$$\begin{aligned}I_m(Z)&\leq\sum_{s=1}^{r}2\frac{(\gamma_s,\lambda)^2}{(\lambda,\lambda)(\gamma_s,\gamma_s)}\,\prod_{s=1}^{r}(1-(t_s/\sqrt{m})^2)^{-(\lambda_0+m\lambda)([E_{\gamma_s},E_{-\gamma_s}])-1}\\&\leq C\prod_{s=1}^{r}(1-(t_s/\sqrt{m})^2)^{-\frac{m}{2}\lambda([E_{\gamma_s},E_{-\gamma_s}])}\\&\leq C\exp\Big(\sum_{s=1}^{r}\frac{1}{2}\lambda([E_{\gamma_s},E_{-\gamma_s}])t_s^2\Big)\end{aligned}$$

where $C>0$ is a constant which does not depend on m. Hence we obtain the estimate

$$I_m(Z)\leq Ce^{-D|Z|^2}$$

where $D>0$ is a constant and $|\cdot|$ is an Euclidean norm on $\mathcal{P}^+$. This ends the proof. $\square$

6. Example

In this section, we consider the case of the holomorphic discrete series of $SU(1,1)$ (see [10]). We take

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1, \quad a, b \in \mathbb{C} \right\}$$

and

$$K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}$ of G has basis

$$u_1 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and its complexification $\mathfrak{g}^c$ is $sl(2,\mathbb{C})$. Then we have $G^c = SL(2,\mathbb{C})$ and

$$K^c = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \quad a \in \mathbb{C} \setminus (0) \right\}.$$

The conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$ is given by

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\bar{a} & \bar{c} \\ \bar{b} & -\bar{d} \end{pmatrix}$$

and we have $X^* = -\theta(X)$ for $X \in \mathfrak{g}^c$.

The root system of $\mathfrak{g}^c = sl(2,\mathbb{C})$ relative to $\mathfrak{k}^c$ consists in the two noncompact roots α and $-\alpha$ where $\alpha(iu_3) = 1$. The corresponding root spaces are

$$\mathfrak{g}_\alpha = \mathbb{C} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-\alpha} = \mathbb{C} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We say that a root is positive if it is positive on $iu_3 \in i\mathfrak{h}$. Then α is the positive root and $\mathfrak{p}^+ = \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \mathfrak{g}_{-\alpha}$. The corresponding groups are

$$P^+ = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \; : \; z \in \mathbb{C} \right\}, \qquad P^- = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \; : \; z \in \mathbb{C} \right\}.$$

In the rest of this section, we identify $\mathfrak{p}^+$ to $\mathbb{C}$ by means of the map

$$z \to Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}.$$

Each element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ such that $d \neq 0$ has the following $P^+K^cP^-$-decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix}.$$

In particular we have $G \subset P^+K^cP^-$.

The map $gK = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} K \in G/K \to \log\zeta(g) = b/\bar{a}$ is then a diffeomorphism from G/K onto the unit disk $D = \{z \in \mathbb{C} \, : \, |z| = 1\}$ and we can verify that the natural action of G on G/K corresponds to the action of G on D by fractional linear transformations defined by

$$g \cdot z = \frac{az+b}{\bar{b}z+\bar{a}}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \, z \in D.$$

Note that the map

$$z \to g_z := \frac{1}{\sqrt{1-z\bar{z}}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}$$

is a section for the action of G on D, that is, we have $g_z \cdot 0 = z$ for each $z \in D$. One can easily verify that a G-invariant measure on D is $d\mu(z) = (1-z\bar{z})^{-2}\, d\mu_L(z)$ where $d\mu_L(z) := dx\, dy$ denotes the Lebesgue measure on D ($z = x+iy$, $x, y \in \mathbb{R}$).

Now, we fix an integer m and we consider the unitary character χ_m of K defined by

$$\chi_m \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{-im\theta}\,.$$

We denote also by χ_m the extension of χ_m to K^c. We obtain immediately

$$\chi_m\big(\kappa(\exp Z^* \exp Z)\big) = (1-z\bar{z})^m\,.$$

The space $\mathcal{H}_{\chi_m}$ is the Hilbert space of holomorphic functions f such that

$$\|f\|_m^2 := \int_D |f(z)|^2\, (1-z\bar{z})^{m-2}\, \tfrac{m-1}{\pi}\, dxdy < +\infty\,. \tag{6.1}$$

Let $\lambda_m = d\chi_m$. By Proposition 2.1, $\mathcal{H}_{\chi_m}$ is nonzero if and only if the condition

$$(\lambda_m + \tfrac{1}{2}\alpha, \alpha) > 0$$

holds. Since $\lambda_m = -\frac{m}{2}\alpha$, this condition reads $\frac{1-m}{2}\,(\alpha,\alpha) < 0$ and, as the restriction of β to $i\mathfrak{k}$ is positive definite, it is equivalent to $m \geq 2$.

Also, note that the normalization of the measure in (6.1) is taken so that $\|1\|_m = 1$.

For each $m \geq 2$, the representation π_m of $G = SU(1,1)$ corresponding to m is realized in $\mathcal{H}_{\chi_m}$ as

$$\begin{aligned}\big(\pi_m(g)\big)f(z) &= \chi_m^{-1}\big(\kappa(g^{-1}\exp Z)\big)\, f(g^{-1}\cdot z) \\ &= (-\bar{b}z + a)^{-m}\, f(g^{-1}\cdot z)\end{aligned}$$

for $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G$, $f \in \mathcal{H}_{\chi_m}$ and $z \in D$.

One can easily show that the family $f_p(z) := \binom{m+p-1}{p}^{1/2} z^p$ is an orthonormal basis for $\mathcal{H}_{\chi_m}$ (see [29, p. 11], for instance). From this, we see that the coherent states

$$e_z(w) = \chi_m(\kappa(\exp Z^* \exp W)^{-1}) = (1-\bar{z}w)^{-m} = \sum_{p\geq 0} \overline{f_p(z)} f_p(w)$$

satisfy the reproducing property $\langle f, e_z^m\rangle_m = f(z)$ for each $f \in \mathcal{H}_{\chi_m}$ and each $z \in D$.

Here we obtain the following formula for the Berezin symbol of $\pi_m(g)$ for $g \in G$

$$S_m\big(\pi_m(g)\big)(z) = \frac{(\pi_m(g)e_z)(z)}{e_z(z)} = \frac{(1-z\bar{z})^m}{(a-\bar{b}z+b\bar{z}-\bar{a}z\bar{z})^m}, \qquad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}.$$

Moreover, since $d\pi_m$ is given by

$$\begin{aligned} d\pi_m\,(u_1)f(z) &= \frac{m}{2}\,i\,z\,f(z) + \frac{1}{2}\,i\,(z^2+1)\,f'(z) \\ d\pi_m\,(u_2)f(z) &= \frac{m}{2}\,z\,f(z) + \frac{1}{2}\,(z^2-1)\,f'(z) \\ d\pi_m\,(u_3)f(z) &= \frac{m}{2}\,i\,f(z) + izf'(z) \end{aligned}$$

we get

$$\begin{aligned} S_m\big(d\pi_m\,(u_1)\big)\,(z) &= i\frac{m}{2}\,\frac{z+\bar{z}}{1-z\bar{z}} \\ S_m\big(d\pi_m\,(u_2)\big)\,(z) &= \frac{m}{2}\,\frac{z-\bar{z}}{1-z\bar{z}} \\ S_m\big(d\pi_m\,(u_3)\big)\,(z) &= i\frac{m}{2}\,\frac{1+z\bar{z}}{1-z\bar{z}}\,. \end{aligned}$$

From this we deduce that $S_m(d\pi_m(X))(z) = i\beta(X, \psi_m(z))$ where the map ψ_m is defined by

$$\psi_m(z) := \frac{m}{8}i\begin{pmatrix} \frac{1+z\bar{z}}{1-z\bar{z}} & -\frac{2z}{1-z\bar{z}} \\ \frac{2\bar{z}}{1-z\bar{z}} & -\frac{1+z\bar{z}}{1-z\bar{z}} \end{pmatrix}.$$

Note that $\psi_m(0) = -iH_m$ where H_m is the coroot vector of λ_m and that $\psi_m(z) = \mathrm{Ad}(g_z)\,(-iH_m)$. Then ψ_m is a diffeomorphism from D onto the orbit of $-iH_m$ under the adjoint action of G.

Now, we turn to the Berezin transform B_m. Here we have

$$B_m(f)(z) = \int_D F(w)\,\frac{|1-\bar{z}w|^4}{(1-z\bar{z})^2}\,(1-w\bar{w})^{m-2}\,\tfrac{m-1}{\pi}d\mu_L(w)\,. \tag{6.2}$$

Let us compute q_{χ_m} (see Section 4). We have

$$q_{\chi_m} = -2\,\frac{(d\chi_0+\lambda_m,\alpha)}{(\alpha,\alpha)} = -2(1-\frac{m}{2}) = m-2$$

and Proposition 4.1 asserts that if $q \le q_{\chi_m}$ then for each $X_1, X_2, \ldots, X_q$ in $\mathfrak{g}^c$, the Berezin transform of $S_m(d\pi_m(X_1X_2\ldots X_q))$ is well-defined. Here, this can be directly verified as follows. By using the formulas for $d\pi_m$ given above, we immediately see that $d\pi_m(X_1X_2\ldots X_q)$ is a linear combination of the differential operators $D_{p,r} := z^p(\frac{d}{dz})^r$ where $r \le q$. By differentiating $e_z(w) = (1-\bar{z}w)^{-m}$, we get

$$S_m(D_{p,r})(w) = m(m+1)\ldots(m+r-1)\,w^p\bar{w}^r(1-w\bar{w})^{-r}\,.$$

Taking formula (6.2) into account, we see that the Berezin transform of $S_m(D_{p,r})$ is well-defined. Hence the result.

Now, we want to compute the constant a_{χ_m} for $m>2$ (see Section 5). To this aim, we apply the equality $(B_mF)(0)=a_{\chi_m}F(0)$ to the function

$$F(Z):=S_m\big(d\pi_m(iu_3)\big)(z)=-\frac{m}{2}\,\frac{1+z\bar{z}}{1-z\bar{z}}\,.$$

We then obtain $(B_mF)(0)=-m^2/2(m-2)$ and hence we find that $a_{\chi_m}=m/m-2$. In particular, we have $\lim_{m\to+\infty}a_{\chi^m}=1$, in accordance with Proposition 5.3.

Finally, let us mention that the computation of a_{χ_m} can be performed similarly when $G=SU(p,q)$, $K=S(U(p)\times U(q))$ and χ_m is the unitary character of K defined by

$$\chi_m\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}=(\mathrm{Det}\ A)^{-m}\,.$$

In that case, by adapting some methods from [25], we find that $a_{\chi_m}=m/m-p-q$ for $m>p+q$.

References

[1] Ali, S. T., Englis, M., *Quantization methods: a guide for physicists and analysts*, Rev. Math. Phys. **17** (4) (2005), 391–490.

[2] Arazy, J., Upmeier, H., *Invariant symbolic calculi and eigenvalues of invariant operators on symmeric domains*, Function spaces, interpolation theory and related topics, Lund, de Gruyter, Berlin, 2002, pp. 151–211.

[3] Arazy, J., Upmeier, H., *Weyl Calculus for Complex and Real Symmetric Domains*, Harmonic analysis on complex homogeneous domains and Lie groups (Rome, 2001), vol. 13 (3–4), Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 2002, pp. 165–181.

[4] Arnal, D., Cahen, M., Gutt, S., *Exponential and holomorphic discrete series*, Bull. Soc. Math. Belg. Sér. B **41** (1989), 207–227.

[5] Arratia, O., Del Olmo, M. A., *Moyal quantization on the cylinder*, Rep. Math. Phys. **40** (1997), 149–157.

[6] Ballesteros, A., Gadella, M., Del Olmo, M. A., *Moyal quantization of* $2+1$*–dimensional Galilean systems*, J. Math. Phys. **33** (1992), 3379–3386.

[7] Berezin, F. A., *Quantization*, Math. USSR–Izv. **8** (1974), 1109–1165, Russian.

[8] Berezin, F. A., *Quantization in complex symmetric domains*, Math. USSR–Izv. **9** (1975), 341–379.

[9] Brif, C., Mann, A., *Phase–space formulation of quantum mechanics and quantum–state reconstruction for physical systems with Lie–group symmetries*, Phys. Rev. A **59** (2) (1999), 971–987.

[10] Cahen, B., *Contraction de* $SU(1,1)$ *vers le groupe de Heisenberg*, Mathematical works, Part XV, Luxembourg: Université du Luxembourg, Séminaire de Mathématique, 2004, pp. 19–43.

[11] Cahen, B., *Weyl quantization for semidirect products*, Differential Geom. Appl. **25** (2007), 177–190.

[12] Cahen, B., *Berezin quantization on generalized flag manifolds*, Math. Scand. **105** (2009), 66–84.

[13] Cahen, B., *Contraction of discrete series via Berezin quantization*, J. Lie Theory **19** (2009), 291–310.

[14] Cahen, B., *Berezin quantization for discrete series*, Beiträge Algebra Geom. **51** (2010), 301–311.

[15] Cahen, B., *Stratonovich–Weyl correspondence for compact semisimple Lie groups*, Rend. Circ. Mat. Palermo (2) **59** (2010), 331–354.

[16] Cahen, M., Gutt, S., Rawnsley, J., *Quantization on Kähler manifolds IV*, Lett. Math. Phys. **34** (1995), 159–168.

[17] Cariñena, J. F., Gracia-Bondìa, J. M., Vàrilly, J. C., *Relativistic quantum kinematics in the Moyal representation*, J. Phys. A **23** (1990), 901–933.

[18] Davidson, M., Òlafsson, G., Zhang, G., *Laplace and Segal–Bargmann transforms on Hermitian symmetric spaces and orthogonal polynomials*, J. Funct. Anal. **204** (2003), 157–195.

[19] Figueroa, H., Gracia-Bondìa, J. M., Vàrilly, J. C., *Moyal quantization with compact symmetry groups and noncommutative analysis*, J. Math. Phys. **31** (1990), 2664–2671.

[20] Folland, B., *Harmonic Analysis in Phase Space*, Princeton Univ. Press, 1989.

[21] Gracia-Bondìa, J. M., *Generalized Moyal quantization on homogeneous symplectic spaces*, Deformation theory and quantum groups with applications to mathematical physics, vol. 134, Amherst, MA, 1990, Contemp. Math., 1992, pp. 93–114.

[22] Gracia-Bondìa, J. M., Vàrilly, J. C., *The Moyal representation for spin*, Ann. Physics **190** (1989), 107–148.

[23] Helgason, S., *Differential geometry, Lie groups and symmetric spaces*, Grad. Stud. Math. **34** (2001).

[24] Herb, R. A., Wolf, J. A., *Wave packets for the relative discrete series I. The holomorphic case*, J. Funct. Anal. **73** (1987), 1–37.

[25] Hua, L. K., *Harmonic analysis of functions of several complex variables in the classical domains*, American Mathematical Society, Providence, R.I., 1963.

[26] Kirillov, A. A., *Lectures on the orbit method*, Grad. Stud. Math. **64** (2004).

[27] Knapp, A. W., *Representation theory of semi–simple groups. An overview based on examples*, Princeton Math. Ser. **36** (1986).

[28] Moore, C. C., *Compactifications of symmetric spaces II: The Cartan domains*, Amer. J. Math. **86** (2) (1964), 358–378.

[29] Neeb, K.-H., *Holomorphy and Convexity in Lie Theory*, de Gruyter Exp. Math. **28** (2000), xxii+778 pp.

[30] Nomura, T., *Berezin transforms and group representations*, J. Lie Theory **8** (1998), 433–440.

[31] Oliveira, M. P. De, *Some formulas for the canonical Kernel function*, Geom. Dedicata **86** (2001), 227–247.

[32] Ørsted, B., Zhang, G., *Weyl quantization and tensor products of Fock and Bergman spaces*, Indiana Univ. Math. J. **43** (2) (1994), 551–583.

[33] Peetre, J., Zhang, G., *A weighted Plancherel formula III. The case of a hyperbolic matrix ball*, Collect. Math. **43** (1992), 273–301.

[34] Satake, I., *Algebraic structures of symmetric domains*, Iwanami Sho–ten, Tokyo and Princeton Univ. Press, 1971.

[35] Stratonovich, R. L., *On distributions in representation space*, Soviet Physics JETP **4** (1957), 891–898.

[36] Unterberger, A., Upmeier, H., *Berezin transform and invariant differential operators*, Comm. Math. Phys. **164** (3) (1994), 563–597.

[37] Varadarajan, V. S., *Lie groups, Lie algebras and their representations*, Grad. Texts in Math. **102** (1984), xiii+430 pp.

[38] Wildberger, N. J., *On the Fourier transform of a compact semisimple Lie group*, J. Austral. Math. Soc. Ser. A **56** (1994), 64–116.

[39] Zhang, G., *Berezin transform on compact Hermitian symmetric spaces*, Manuscripta Math. **97** (1998), 371–388.

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