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THE FIRST EIGENVALUE OF SPACELIKE SUBMANIFOLDS IN INDEFINITE SPACE FORM R_p^{n+p}

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Abstract. In this paper, we prove that the first eigenvalue of a complete spacelike submanifold in R_p^{n+p} with the bounded Gauss map must be zero.

1. Introduction

Let M^n be a complete noncompact Riemannian manifold and $\Omega \subset M^n$ be a domain with compact closure and nonempty boundary $\partial\Omega$. The Dirichlet eigenvalue $\lambda_1(\Omega)$ of Ω is defined by

$$\lambda_1(\Omega) = \inf \Big(\frac{\int_\Omega |\nabla f|^2 dM}{\int_M f^2 dM} : f \in L^2_{1,0}(\Omega)\ \{0\} \Big) ,$$

where dM is the volume element on M^n and $L^2_{1,0}(\Omega)$ the completion of C_0^∞ with respect to the norm

$$\|\varphi\|^2_\Omega = \int_M \varphi^2 dM + \int_M |\nabla \varphi|^2 \, dM \, .$$

If $\Omega_1 \subset \Omega_2$ are bounded domains, then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \geq 0$. Thus one may define the first Dirichlet eigenvalue of M^n as the following limit

$$\lambda_1(M) = \lim_{r\to\infty} \lambda_1\big(B(p,r)\big) \geq 0 \, ,$$

where $B(p,r)$ is the geodesic ball of M^n with radius r centered at p. It is clear that the definition of $\lambda_1(M)$ does not depend on the center point p. It is interesting to ask that for what geometries a noncompact manifold M^n has zero first eigenvalue. Cheng and Yau [1] showed that $\lambda_1(M) = 0$ if M^n has polynomial volume growth.

In [5], B. Wu proved the following result.

Theorem A. *Let M^n be a complete spacelike hypersurface in R_1^{n+1} whose Gauss map is bounded, then $\lambda_1(M) = 0$.*

In this note, we discover that Wu's result still holds for higher codimensional complete spacelike submanifolds in R_p^{n+p}. In fact, we prove

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Theorem 3.1. *Let M^n be a complete spacelike submanifold in R_p^{n+p} whose Gauss map is bounded, then $\lambda_1(M) = 0$.*

2. The geometry of pseudo-Grassmannian

In this section we review some basic properties about the geometry of pseudo--Grassmannian. For details one referred to see [6, 3].

Let R_p^{n+p} be the $(n+p)$-dimensional pseudo-Euclidean space with index p, where, for simplicity, we assume that $n \geq p$. The case $n < p$ can be treated similarly. We choose a pseudo-Euclidean frame field $\{e_1, \ldots, e_{n+p}\}$ such that the pseudo-Euclidean metric of R_p^{n+p} is given by $ds^2 = \sum_i (\omega_i)^2 - \sum_\alpha \omega_\alpha = \sum_A \varepsilon_A (\omega_A)^2$, where $\{\omega_1, \ldots, \omega_{n=p}\}$ is the dual frame field of $\{e_1, \ldots, e_{n+p}\}$, $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Here and in the following we shall use the following convention on the ranges of indices:

$$1 \leq i, j, \dots \leq n; \quad n+1 \leq \alpha, \beta, \dots \leq n+p\,; \quad 1 \leq A, B, \dots \leq n+p\,.$$

The structure equations of R_p^{n+p} are given by

$$\begin{aligned} de_A &= -\sum_B \varepsilon_A \omega_{AB} e_B\,, \\ d\omega_A &= -\sum_B \omega_{AB} \wedge \omega_B\,, \quad \omega_{AB} + \omega_{BA} = 0\,, \\ d\omega_{AB} &= -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB}\,. \end{aligned}$$

Let $G_{n,p}^p$ be the pseudo-Grassmannian of all spacelike n-subspace in R_p^{n+p}, and $\widetilde{G_{n,p}^p}$ be the pseudo-Grassmannian of all timelike p-subspace in R_p^{n+p}. They are specific Cartan-Hadamard manifolds, and the canonical Riemannian metric on $G_{n,p}^p$ and $\widetilde{G_{n,p}^p}$ is

$$ds_G = ds_{\widetilde{G}} = \sum_{i\alpha} (\omega_{\alpha i})^2\,.$$

Let 0 be the origin of R_p^{n+p}. Let $SO^0(n+p,p)$ denote the identity component of the Lorentzian group $O(n+p,p)$. $SO^0(n+p,p)$ can be viewed as the manifold consisting of all pseudo-Euclidean frames $(0; e_i, e_\alpha)$, and $SO^0(n+p,p)/SO(n) \times SO(p)$ can be viewed as $G_{n,p}^p$ or $\widetilde{G_{n,p}^p}$. Any element in $G_{n,p}^p$ can be represented by a unit simple n-vector $e_1 \wedge \cdots \wedge e_n$, while any element in $\widetilde{G_{n,p}^p}$ can be represented by a unit simple p-vector $e_{n+1} \wedge \cdots \wedge e_{n+p}$. They are unique up to an action of $SO(n) \times SO(p)$. The Hodge star $*$ provides an one to one correspondence between $G_{n,p}^p$ and $\widetilde{G_{n,p}^p}$. The product $\langle , \rangle$ on $G_{n,p}^p$ for $e_1 \wedge \cdots \wedge e_n$, $v_1 \wedge \cdots \wedge v_n$ is defined by

$$\langle e_1 \wedge \cdots \wedge e_n, v_1 \wedge \cdots \wedge v_n \rangle = \det\big(\langle e_i, v_j \rangle\big)\,.$$

The product on $\widetilde{G_{n,p}^p}$ can be defined similarly.

Now we fix a standard pseudo-Euclidean frame e_i, e_α for R_p^{n+p}, and take $g_0 = e_1 \wedge \cdots \wedge e_n \in G_{n,p}^p$, $\widetilde{g_0} = *g_0 = e_{n+1} \wedge \cdots \wedge e_{n+p} \in \widetilde{G_{n,p}^p}$. Then we can span the spacelike n-subspace g in a neighborhood of g_0 by n spacelike vectors f_i:

$$f_i = e_i + \sum_\alpha z_{i\alpha} e_\alpha \,,$$

where $(z_{i\alpha})$ are the local coordinates of g. By an action of $SO(n) \times SO(p)$ we can assume that

$$(z_{i\alpha}) = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_p \\ & 0 & \end{pmatrix} .$$

From [3] we know that the normal geodesic $g(t)$ between g_0 and g has local coordinates

$$(z_{i\alpha}) = \begin{pmatrix} \tanh(\lambda_1 t) & & \\ & \ddots & \\ & & \tanh(\lambda_p t) \\ & 0 & \end{pmatrix} ,$$

for real numbers $\lambda_1 \dots \lambda_p$ such that $\sum_{i=1}^p \lambda_i^2 = 1$. This means that $g(t)$ is spanned by $f_1(t) = e_1 + \tanh(\lambda_1 t)e_{n+1}, \dots, f_p(t) = e_p + \tanh(\lambda_p t)e_{n+p}, f_{p+1} = e_{p+1}, \dots, f_n = e_n$. Consequently, $g(t)$ can also be represented by a unit simple n-vector as following:

$$\begin{aligned} g(t) = \big(\cosh(\lambda_1 t)e_1 + \sinh(\lambda_1 t)e_{n+1}\big) \wedge \cdots \wedge \big(\cosh(\lambda_p t)e_1 \\ + \sinh(\lambda_p t)e_{n+p}) \wedge e_{p+1} \wedge \cdots \wedge e_n \,. \end{aligned}$$

Set $\lambda_\alpha = \lambda_{\alpha-n}$, then it is clear that

$$\begin{aligned} &\cosh(\lambda_1 t)e_1 + \sinh(\lambda_1 t)e_{n+1}, \dots, \cosh(\lambda_p t)e_1 + \sinh(\lambda_p t)e_{n+p}, e_{p+1}, \dots, e_n \,, \\ &\sinh(\lambda_{n+1} t)e_1 + \cosh(\lambda_{n+1} t)e_{n+1}, \dots, \sinh(\lambda_{n+p} t)e_p + \cosh(\lambda_{n+p} t)e_{n+p} \end{aligned}$$

is again a pseudo-Euclidean frame for R_p^{n+p}, so we have

$$\begin{aligned} \widetilde{g(t)} = *g(t) = \big(\sinh(\lambda_{n+1} t)e_1 + \cosh(\lambda_{n+1} t)e_{n+1}\big) \wedge \cdots \wedge \big(\sinh(\lambda_{n+p} t)e_p \\ + \cosh(\lambda_{n+p} t)e_{n+p}\big) \in \widetilde{G^p_{n,p}} \,. \end{aligned}$$

Thus we have

$$\langle g_0, g\rangle = (-1)^p \langle *g_0, *g\rangle = (-1)^p \langle \widetilde{g_0}, \widetilde{g}\rangle = \prod_\alpha \cosh(\lambda_\alpha t) \,.$$

In this note, we also need the following lemma,

Lemma 2.1 ([4])**.** *Let $\mu_1 \ge 1, \dots, \mu_p \ge 1$ and $\prod_\alpha \mu_\alpha = C$. Then $\sum_\alpha \cosh^2(\lambda_\alpha) \le C^2 + p - 1$, and the equality holds if and only if $\mu_{i_0} = C$ for some $1 \le i_0 \le p$ and $\mu_i = 1$ for any $i \ne i_0$.*

3. Main results for space-like submanifolds

In this note, we get the following result:

Theorem 3.1. *Let M^n be a complete space-like submanifold in R_p^{n+p} whose Gauss map is bounded, then we have $\lambda_1(M) = 0$.*

Proof. We choose a local frames $e_1 \ldots, e_{n+p}$ in R_p^{n+p} such that restricted to M^n, $e_1, \ldots, e_n$ are tangent to M^n, $e_{n+1}, \ldots, e_{n+p}$ are normal to M^n, the Gauss map is defined by $e_{n+1} \wedge \cdots \wedge e_{n+p} \colon M^n \to \widetilde{G^p_{n,p}}$. Let us fix p-vector and n-vector $a_{n+1} \wedge \cdots \wedge a_{n+p} \in \widetilde{G^p_{n,p}}$, $a_1 \wedge \cdots \wedge a_n \in G^p_{n,p}$, where $\langle a_\alpha, a_\beta \rangle = -\delta_{\alpha\beta}$ and $\langle a_i, a_j \rangle = \delta_{ij}$. We defined the projection $\Pi \colon M^n \to R^n_a$ by

$$\Pi(x) = x + \sum_{\alpha=n+1}^{n+p} \langle x, a_\alpha \rangle a_\alpha \,, \tag{1}$$

where $\langle , \rangle$ is the standard indefinite inner product on R_p^{n+p} and R^n_a the totally geodesic Euclidean n-space determined by $a = a_{n+1} \wedge \cdots \wedge a_{n+p}$ which is defined by

$$R^n_a = \{x \in R_p^{n+p} : \langle x, a_{n+1} \rangle = \cdots = \langle x, a_{n+p} \rangle = 0\} \,. \tag{2}$$

It is clear from (1) that

$$\mathrm{d}\Pi(X) = X + \sum_{\alpha=n+1}^{n+p} \langle X, a_\alpha \rangle a_\alpha \tag{3}$$

for any tangent vector field on M^n and consequently,

$$|\mathrm{d}\Pi(X)|^2 = |X|^2 + \sum_{\alpha=n+1}^{n+p} \langle X, a_\alpha \rangle^2 \,. \tag{4}$$

From the equation (4), we know that the map $\Pi \colon M^n \to R^n_a$ increases the distance. If a map, from a complete Riemannian manifold M_1 into another Riemannian manifold M_2 of same dimension, increases the distance, then it is a covering map and M_2 is complete (in [2, VIII, Lemma 8.1]). Hence Π is a covering map, but R^n_a being simply connected this means that Π is in face a diffeomorphism between M^n and R^n_a, and thus M^n is noncompact. Now assume that the Gauss map $e_{n+1} \wedge \cdots \wedge e_{n+p} \colon M^n \to \widetilde{G^p_{n,p}}$ is bounded, then there exists $\rho > 0$ such that

$$1 \le (-1)^p \langle e_{n+1} \wedge \cdots \wedge e_{n+p}, a_{n+1} \wedge \cdots \wedge a_{n+p} \rangle \le \rho \,. \tag{5}$$

From Section 2 we know that by an action of $SO(n) \times SO(p)$ we can assume that

$$\begin{aligned} e_{n+1} &= \sinh(\lambda_{n+1} t) a_1 + \cosh(\lambda_{n+1} t) a_{n+1}, \ldots, e_{n+p} \\ &= \sinh(\lambda_{n+p} t) a_1 + \cosh(\lambda_{n+p} t) a_{n+p} \,, \end{aligned}$$

where $\sum_\alpha \lambda_\alpha^2 = 1$ and $t \in R$.

Write

$$a_\alpha = a^\top - \sum_{\beta=n+1}^{n+p} \langle a_\alpha, e_\beta\rangle e_\beta\,, \tag{6}$$

where $a_\alpha^\top$ denote the component of a_α which is tangent to M^n, and $\alpha = n+1, \dots, n+p$. Since $\langle a_\alpha, a_\beta\rangle = -\delta_{\alpha\beta}$, we have

$$-1 = |a_\alpha^\top|^2 - \sum_{\beta=n+1}^{n+p} \langle a_\alpha, e_\beta\rangle^2 = |a_\alpha^\top|^2 - \cosh^2(\lambda_\alpha t)\,, \tag{7}$$

where $\alpha = n+1, \dots, n+p$. It follows from Lemma 2.1 and Eq. (5), (7), we have

$$1 + \sum_{\alpha=n+1}^{n+p} |a_\alpha^\top|^2 = \sum_{\alpha=n+1}^{n+p} \cosh^2(\lambda_\alpha t) - p + 1 \le \prod \cosh^2(\lambda_\alpha t) \le \rho^2\,. \tag{8}$$

From Eq.(4) and (8), we have

$$|\mathrm{d}\Pi(X)|^2 = |X|^2 + \sum_{\alpha=n+1}^{n+p} \langle X, a_\alpha^\top\rangle^2 \le |X|^2 (1 + \sum_{\alpha=n+1}^{n+p} |a_\alpha^\top|^2) \le \rho^2 |X|^2\,. \tag{9}$$

for any tangent vector field on M^n. Let $B(p,r)$ is the geodesic ball of M^n with radius r centered at $p \in M^n$. We claim that $\Pi(B(p,r)) \subset \widetilde{B}(\widetilde{p}, \rho r)$, where $\widetilde{B}(\widetilde{p}, \rho r)$ denotes the geodesic ball of R_a^n with radius ρr centered at $\widetilde{p} = \Pi(p)$. In fact, for any $\widetilde{q} \in \Pi(B(p,r))$ let $q \in B(p,r)$ be the unique point such that $\Pi(q) = \widetilde{q}$, and $\gamma\colon [a,b] \to M^n$ is the minimal geodesic joining p and q, then from (9) we have

$$\widetilde{d}(\widetilde{p}, \widetilde{q}) \le L(\Pi \circ r) = \int_a^b \left|\mathrm{d}\Pi\big(\gamma'(t)\big)\right| dt \le \rho \int_a^b |\gamma'(t)|\, dt = \rho L(\gamma) = \rho d(p,q) \le \rho r\,,$$

where $\widetilde{d}$ and d denote the distance in R_a^n and M^n, respectively. This prove our claim.

Let dV denotes the n-dimensional volume element on R_a^n. Using (3) and (6) it follows that

$$\begin{aligned}
\Pi^*(dV)(X_1, \dots, X_n) &= \det\big(\mathrm{d}\Pi(X_1), \dots, \mathrm{d}\Pi(X_n), a_{n+1}, \dots, a_{n+p}\big) \\
&= \det(X_1, \dots, X_n, a_{n+1}, \dots, a_{n+p}) \\
&= (-1)^p \langle e_{n+1} \wedge \cdots \wedge e_{n+p}, a_{n+1} \wedge \cdots \wedge a_{n+p}\rangle \\
&\quad\ \det(X_1, \dots, X_n, e_{n+1}, \dots, e_{n+p}) \\
&= (-1)^p \langle e_{n+1} \wedge \cdots \wedge e_{n+p}, a_{n+1} \wedge \cdots \wedge a_{n+p}\rangle \\
&\quad\ dM(X_1, \dots, X_n)
\end{aligned}$$

for any tangent vector fields $X_1, \dots, X_n$ of M^n. In other words,

$$\Pi^*(dV) = (-1)^p \langle e_{n+1} \wedge \cdots \wedge e_{n+p}, a_{n+1} \wedge \cdots \wedge a_{n+p}\rangle dM \ge dM\,. \tag{10}$$

Since $\Pi(B(p,r)) \subset \widetilde{B}(\widetilde{p},\rho r)$ and $\Pi\colon M^n \to R^n_a$ is diffeomorphism, it follows from Eq. (10) that

$$\begin{aligned} \rho^n r^n \omega_n &= \operatorname{Vol}\left(\widetilde{B}(\widetilde{p},\rho r)\right) \geq \operatorname{Vol}\left(\Pi(B(p,r))\right) = \int_{\Pi(B(p,r))} dV \\ &= \int_{B(p,r)} \Pi^* dV \geq \int_{B(p,r)} dM = \operatorname{Vol}\left(B(p,r)\right), \end{aligned} \tag{11}$$

where ω_n denotes the volume of unit ball in Euclidean n-space. (11) means that the order of the volume growth of M^n is not larger than n, thus by [1] we see that $\lambda_1(M) = 0$. □

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