ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 83–89

A NOTE ON EXISTENCE THEOREM OF PEANO

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Abstract. An ODE with non-Lipschitz right hand side has been considered. A family of solutions with L^p-dependence of the initial data has been obtained. A special set of initial data has been constructed. In this set the family is continuous. The measure of this set has been estimated.

1. Introduction

Consider a system of ordinary differential equations of the following form

$$\dot{x} = v(t,x)\,, \quad x \in \mathbb{R}^m\,. \tag{1.1}$$

The vector-function v is defined in the cross product of some interval $[-T,T]$ and a domain $D \subseteq \mathbb{R}^m$.

The simplest and often occurred situation is when the vector field v is continuous and fulfills the Lipschitz condition in the second variable:

$$\|v(t,x') - v(t,x'')\| \le c\|x' - x''\|\,. \tag{1.2}$$

In such a case problem (1.1) has a unique solution $x(t)$ that satisfies the initial condition $x(0) = x_0 \in D$. This result is known as Cauchy-Picard existence theorem. (All the classical facts we mention without reference are contained in [7].)

In general, the solution $x(t)$ is defined not in the whole interval $[-T,T]$ but in its smaller subinterval. In the described above conditions the solution $x(t)$ depends continuously on the initial data x_0.

It should be noted that the Cauchy-Picard existence theorem as well as its proof transmit literally from the case $x \in \mathbb{R}^m$ to the case when x belongs to an infinite dimensional Banach space.

If we refuse Lipschitz hypothesis (1.2) then our problem becomes widely complicated. Particularly, it is known that in an infinite dimensional Banach space problem (1.1) may have no solutions [16], [6]. In the finite dimensional case the existence is guaranteed by Peano's theorem.

So, when the function v is only continuous in $[-T,T] \times D$ then for the same initial datum x_0 there may be several solutions. (An example is contained in the

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2010 *Mathematics Subject Classification*: primary 34A12.
Key words and phrases: Peano existence theorem, non-Lipschitz nonlinearity, non-uniqueness, IVP, ODE, Cauchy problem.
Partially supported by grants RFBR 08-01-00681, Science Sch.-8784.2010.1.
Received October 10, 2010. Editor O. Došlý.

next section.) Nevertheless suppose that by some reason for any initial condition x_0 the solution is unique then it depends continuously on the initial data.

There are a lot of works devoted to investigating of different types of the uniqueness conditions. As far as the author knows this activity has been started from Kamke [8] and Levy [11]. Their results have been generalized in different directions. See for example [12], [1] and references therein. Another approach is contained in [10], [2].

In this paper we are not concern with Carathéodory theory which is devoted to ODE with non-continuous right-hand side. Introduction to this topic see in [3].

When problem (1.1) admits non-uniqueness then for some initial data x_0 there are many ways to pick up a solution $x(t)$ such that $x(0) = x_0$. Actually we even do not know how many ways to do this we have and how many such points x_0 are there. An attempt to clarify the last question has been done in [14] (see also references therein). The main result of that article is as follows: the initial data with non-unique solution form a Borel set of the class $F_{\sigma\delta}$.

Anyway for each x_0 we can choose one of the solutions $x(t)$ such that $x(0) = x_0$ and write

$$x(t) = x(t, x_0)\,, \quad x(0, x_0) = x_0\,.$$

At this moment our argument is heavily rested on the Axiom of Choice.

From analysis we know that the Axiom of Choice is the best device to produce very irregular functions. It is sufficient to recall that all the examples of non-measurable functions are based on the Choice Axiom.

Thus a priori we should not expect anything good from the function $x(t, x_0)$.

The aim of this article is to show that under suitable choice of the correspondence $x(t, x_0)$ the function $x_0 \mapsto x(t, x_0)$ is continuous in a closed set of big measure, moreover $x(t, x_0)$ is measurable in the second argument in D.

Comparing this result with [14] it is important to note that if a function $f\colon D \to \mathbb{R}$ is continuous in the set $D \backslash Q$ where Q is a Borel set, then it does not imply that f is a measurable function. Indeed, it is sufficient to take $D = [0, 1]$, $Q = [1/2, 1]$ and $f\mid_{[0,1/2)} \equiv 0$, and for the $f\mid_Q$ to take any non-measurable function.

2. Main Theorems

Equip the space $\mathbb{R}^m = \{x = (x^1, \dots, x^m)\}$ with a norm

$$\|x\| = \max_{k=1,\dots,m} |x^k|.$$

Let B_R stands for the open ball of $\mathbb{R}^m$ with radius R and the center at the origin. By I_T denote an interval $I_T = (-T, T)$.

Introduce a vector-function $f(t, x) = (f^1, \dots, f^m)(t, x) \in C(\mathbb{R}_t \times \mathbb{R}^m_x, \mathbb{R}^m)$. Suppose that

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^m} \|f(t, x)\| = M < \infty.$$

This assumption is made only for simplicity, actually it is sufficient to have f defined in a ball. The reader may consider f to be extendable to the whole space $\mathbb{R}^m$.

Our aim is to study the set of solutions to the following initial value problem:

$$\dot{y} = f(t, y)\,, \quad y(0) = x\,.$$

From Peano's existence theorem we know that for all x this IVP has a solution, $y(t) \in C^1(\mathbb{R})$. It is also well known that for the same initial condition there may be several solutions.

We are going to study whether an initial condition x can be put in correspondence with a solution $u(t, x)$ such that the function $u(t, x)$ possesses reasonable properties.

We will look for solutions to the following IVP.

$$u_t(t, x) = f\big(t, u(t, x)\big)\,, \quad u(0, x) = x\,. \tag{2.1}$$

In such a setup problem (2.1) is no longer a Cauchy problem for finite-dimensional ODE, it is an infinite dimensional Cauchy problem. Indeed, for any fixed t the function $u(t, x)$ is a function of variable x i.e. $t \mapsto u(t, x)$ is a curve of an infinite dimensional functional space. For the functional space it is useful to take $L^p(B_R)$, $p \in [1, \infty)$.

In the Introduction it has already been noted that such an infinite dimensional Cauchy problems may have no solutions. All the existence results concerning this type of IVP use some compactness argument. For example in [15] it is imposed that f is weakly continuous mapping of a reflexive Banach space. Another approach see for example in [9].

In our situation there are no reasonable compactness argument but the space $L^p(B_R)$ has an additional structure: it has a partial order. This fact appears to be decisive.

To enable this structure let us make the following hypothesis.

For each $t \in \overline{I}_T$ and for all $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$ and $y = (y^1, \ldots, y^m) \in \mathbb{R}^m$ such that

$$x^j \le y^j\,, \qquad j = 1, \ldots, m$$

one has

$$f^j(t, x) \le f^j(t, y)\,, \qquad j = 1, \ldots, m\,. \tag{2.2}$$

This hypothesis does not prevent the effect of non-uniqueness. To see this it is sufficient to consider our IVP with $f(t, y) = \sqrt{y}$ provided $y \ge 0$ and $f(t, y) = 0$ otherwise and $y(0) = 0$.

Theorem 1. *For any positive constants T, R and $p \in [1, \infty)$ problem* (2.1) *has a solution $w(t, x) \in C(\overline{I}_T, L^p(B_R)) \bigcap C^1(I_T, L^p(B_R))$.*

Let μ stands for the standard Lebesgue measure in B_R.

Theorem 2. *For any $\varepsilon > 0$ there is a closed set $M_\varepsilon \subset B_R$ such that $\mu(B_R \backslash M_\varepsilon) < \varepsilon$ and $w(t, x) \in C(M_\varepsilon, C(\overline{I}_T))$.*

Proof of Theorem 2. Arrange a countable set $Z = \overline{I}_T \bigcap \mathbb{Q}$ as follows: $Z = \{t_i\}_{i \in \mathbb{N}}$.

Then by Luzin's theorem [13] we choose closed sets

$$M_i \subseteq B_R\,, \quad \mu(B_R\backslash M_i) < \frac{\varepsilon}{2^i}$$

such that $w(t_i, x) \in C(M_i)$.

Let us put $M_\varepsilon = \bigcap_i M_i$ then

$$\mu(B_R\backslash M_\varepsilon) = \mu\Big(\bigcup_i B_R\backslash M_i\Big) \le \sum_i \mu(B_R\backslash M_i) < \varepsilon\,.$$

Take a sequence $x_k \to x$, $\{x_k\} \subseteq M_\varepsilon$. For all $t_i \in Z$ we have

$$\|w(t_i, x_k) - w(t_i, x)\| \to 0\,.$$

Observe that the sequence $\{w(t, x_k)\}$ is uniformly continuous in $\overline{I}_T$:

$$\|w(t', x_k) - w(t'', x_k)\| = \Big\| \int_{t''}^{t'} f\big(s, w(s, x_k)\big)\, ds \Big\| \le M|t' - t''|\,, \quad t', t'' \in \overline{I}_T\,.$$

Thus the sequence $\{w(t, x_k)\}$ converges uniformly in Z [13]. And since the set Z is dense in $\overline{I}_T$ this sequence converges uniformly in $\overline{I}_T$.

The theorem is proved. □

3. Proof of Theorem 1

For convenience of the reader we recall several propositions which are used in the sequel.

The following proposition is a corollary from the Vitali convergence theorem [5].

Proposition 1. *Let $(X, \mathfrak{S}, \mu)$ be a measure space, $\mu(X) < \infty$. And a sequence of measurable functions $\{f_n\}$ is such that for all $n \in \mathbb{N}$ and for almost all $x \in X$ we have $|f_n(x)| \le const$. Assume that $\{f_n\}$ is a Cauchy sequence in measure. Then it converges in measure to a measurable function f and $\int_X (f_n - f)d\mu \to 0$.*

Formulate another fact.

Proposition 2 ([5])**.** *Let $D \subset \mathbb{R}^m$ be a measurable set with respect to the standard Lebesgue measure. Consider a function $\psi \in C(\overline{B}_R, \mathbb{R}^k)$. If $f_n \to f$ in measure in D and $\|f_n(x)\| \le R$ almost everywhere in D then $\psi \circ f_n \to \psi \circ f$ in measure.*

As usual we formulate our IVP in terms of the integral equation

$$u(t,x) = F(u)(t,x)\,, \quad F(u)(t,x) = x + \int_0^t f\big(s, u(s,x)\big)\, ds\,. \tag{3.1}$$

Definition 1. *We shall say that the function $u(t,x)$ belongs to a set X if*

1. *$u(t,x) \in C(\overline{I}_T, L^p(B_R))$,*
2. *for every $t \in \overline{I}_T$ the inequality $\|u(t,x)\| \le R + TM$ holds almost everywhere in B_R;*
3. *for every $t', t'' \in \overline{I}_T$ the estimate*

$$\|u(t',x) - u(t'',x)\| \le M|t' - t''|$$

holds almost everywhere in B_R.

Lemma 1. *The mapping F takes the set X to itself.*

The proof of this Lemma is straightforward.
We must only check that

$$f\big(t, u(t,x)\big) \in C\big(\overline{I}_T, L^p(B_R)\big)\,, \quad u(t,x) \in X\,.$$

Take a sequence $t_k \to t$. Then $u(t_k, x) \to u(t,x)$ in $L^p(B_R)$ and in measure. By Propositions 2, 1

$$f\big(t_k, u(t_k, x)\big) \to f\big(t, u(t,x)\big)$$

in measure and in $L^p(B_R)$. This implies the strong measurability of the mapping $t \mapsto f(t, u(t,x))$. Lemma is proved.

Now let us endow the space X with a partial order $\preceq$. We shall say that $u(t,x) = (u^1, \dots, u^m)(t,x) \in X$ and $v(t,x) = (v^1, \dots, v^m)(t,x) \in X$ satisfy the relation $u \preceq v$ iff for every $t \in \overline{I}_T$ the inequality $u^k(t,x) \le v^k(t,x)$, $k = 1, \dots, m$ holds almost everywhere in B_R.

Lemma 2. *A set $E = \{u \in X \mid u \preceq F(u)\}$ possesses a maximal element:*

$$w = \max E\,.$$

Observe that by Lemma 1 the space E is non void: $-(R{+}TM, \dots, R{+}TM) \in E$.

Proof of Lemma 2. The assertion of the lemma is surely based on the Zorn Lemma. So it is sufficient to prove that any chain $C \subseteq E$ has an upper bound.

The space $L^p(B_R)$ is separable and the interval $\overline{I}_T$ is compact. So the space $C(\overline{I}_T, L^p(B_R))$ is separable [4].

Since the set C belongs to $C(\overline{I}_T, L^p(B_R))$, it is also separable. This implies that there is a countable set $Q \subseteq C$ such that for any element $p \in C$ there exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq Q$ and $\max_{t\in\overline{I_T}} \|p_n(t,\cdot) - p(t,\cdot)\|_{L^p(B_R)} \to 0$ as $n \to \infty$.

Arrange the set Q as a sequence: $Q = \{g_j\}_{j\in\mathbb{N}}$ and consider a sequence $h_l = \max\{g_1, \dots, g_l\}$, $\{h_l\} \subseteq Q$. Here max stands in regard to the relation $\preceq$.

We claim that for each $t \in \overline{I}_T$ this sequence converges almost everywhere to a function h and this function is the desired upper bound of C.

Since for all $t \in \overline{I}_T$ and for almost all $x \in B_R$ the inequalities

$$\|h_l(t,x)\| \le R + TM\,, \quad h_l^n(t,x) \le h_{l+1}^n(t,x)\,, \quad n = 1, \dots, m$$

fulfill for all $l \in \mathbb{N}$, then for every $t \in \overline{I}_T$ the sequence h_l converges to a function h almost everywhere in $x \in B_R$. And for every t, t', $t'' \in \overline{I}_T$ and almost everywhere in B_R we also get

$$\|h(t,x)\| \le R + TM\,, \quad \|h(t',x) - h(t'',x)\| \le M|t' - t''|\,. \tag{3.2}$$

By the Dominated convergence theorem for every $t \in \overline{I}_T$ the function $h(t,x) \in L^\infty(B_R)$ and $h_l(t,\cdot) \to h(t,\cdot)$ in $L^p(B_R)$.

Since the functions $h_l(t,x)$ satisfy item (3) of Definition 1 we write

$$\|h_l(t',\cdot) - h_l(t'',\cdot)\|_{L^p(B_R)} \le M\big(\mu(B_R)\big)^{1/p}|t' - t''|\,.$$

Thus the sequence $\{h_l\} \subset C(\overline{I}_T, L^p(B_R))$ is uniformly continuous in t and it converges to h in $C(\overline{I}_T, L^p(B_R))$ [13]. Particularly we have $h \in C(\overline{I}_T, L^p(B_R))$ and from formulas (3.2) it follows that $h \in X$.

Owing to the continuity of the function f for every $t \in \overline{I}_T$ we obtain

$$f\big(t, h_l(t,x)\big) \to f\big(t, h(t,x)\big)$$

almost everywhere in B_R.

By the Dominated convergence theorem we have

$$\|f\big(t, h_l(t,x)\big) \to f\big(t, h(t,x)\big)\|_{L^p(B_R)} \to 0\,, \quad t \in \overline{I}_T\,.$$

Now we apply the Dominated convergence theorem again, but this time we use its Bochner integral version to yield:

$$\Big\| \int_0^t f\big(s, h_l(s,x)\big)\,ds - \int_0^t f\big(s, h(s,x)\big)\,ds \Big\|_{L^p(B_R)} \to 0\,.$$

From this formula it follows that there exists a subsequence $\{h_{l_i}\}$ such that

$$\int_0^t f\big(s, h_{l_i}(s,x)\big)\,ds \to \int_0^t f\big(s, h(s,x)\big)\,ds$$

for almost all $x \in B_R$. This states that $h \in E$.

Obviously the function h is an upper bound for Q. Check that h is an upper bound for C.

Assume the converse: there exists an element $b \in C$ such that the relation $b \preceq h$ does not hold. This implies that for some $t' \in \overline{I}_T$ and for some index k a set

$$D' = \{x \in B_R \mid b^k(t',x) - h^k(t',x) > 0\}$$

has non zero measure: $\mu(D') > 0$.

Actually there exists a set $D \subseteq D'$, $\mu(D) > 0$ such that for some constant $c > 0$ one has $b^k(t',x) - h^k(t',x) \geq c$, $x \in D$. Indeed, if it is not true then we can take a sequence

$$\{c_l\}_{l\in\mathbb{N}}\,, \qquad c_l > 0\,, \quad c_l \to 0$$

and consider sets $D_l = \{x \in D' \mid b^k(t',x) - h^k(t',x) \geq c_l\}$. By the assumption for all l we have $\mu(D_l) = 0$ but on the other hand $D' = \bigcup_l D_l$ and $\mu(D') \leq \sum_l \mu(D_l) = 0$.

Take a sequence $\{b_j\}_{j\in\mathbb{N}} \subseteq Q$ such that $b_j \to b$ in $C(\overline{I}_T, L^p(B_R))$. We obtain

$$c + h^k(t',x) - b_j^k(t',x) \leq b^k(t',x) - b_j^k(t',x) \tag{3.3}$$

almost everywhere in D. It is obvious $h^k(t',x) - b_j^k(t',x) \geq 0$ almost everywhere in B_R and from formula (3.3) we get

$$b^k(t',x) - b_j^k(t',x) \geq c \tag{3.4}$$

almost everywhere in D.

The L^p-convergence implies the convergence in measure [5] thus for every q, $\sigma > 0$ there is an index J such that if $j > J$ then

$$\mu\big(\{x \in B_R \mid \; |b^k(t',x) - b_j^k(t',x)| \geq q\}\big) < \sigma\,.$$

Putting in this formula $q = c$ and $\sigma = \mu(D)/2$ we obtain a contradiction with inequality (3.4).

Lemma is proved. □

Now we are ready to prove the Theorem. By Lemma 1 and inequality (2.2) it follows that $F(E) \subseteq E$. Particularly $F(w) \in E$, where $w = \max E$ is a maximal element given by Lemma 2. Consequently the relation $w \preceq F(w)$ implies that $w = F(w)$.

Theorem 2 is proved.

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