ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 133–138

NATURAL EXTENSION OF A CONGRUENCE OF A LATTICE TO ITS LATTICE OF CONVEX SUBLATTICES

S. Parameshwara Bhatta and H. S. Ramananda*

Abstract. Let L be a lattice. In this paper, corresponding to a given congruence relation Θ of L, a congruence relation Ψ_Θ on $CS(L)$ is defined and it is proved that

1. $CS(L/\Theta)$ is isomorphic to $CS(L)/\Psi_\Theta$;
2. L/Θ and $CS(L)/\Psi_\Theta$ are in the same equational class;
3. if Θ is representable in L, then so is Ψ_Θ in $CS(L)$.

1. Introduction

Let L be a lattice and $CS(L)$ be the set of all convex sublattices of L. It is proved in [3] that, there exists a partial order on $CS(L)$ with respect to which $CS(L)$ is a lattice such that both L and $CS(L)$ are in the same equational class. A natural question that arises is the following:

If Θ is a congruence relation of L, does there exists a natural extension Ψ_Θ of Θ to $CS(L)$ such that L/Θ and $CS(L)/\Psi_\Theta$ are in the same equational class?

This paper gives an affirmative answer to this question. Further, it is proved that, if Θ is representable in L, then so is Ψ_Θ in $CS(L)$.

2. Notation and definitions

Let L be a lattice and $CS(L)$ be the set of all convex sublattices of L. Define an ordering $\leq$ on $CS(L)$ by, for $A, B \in CS(L)$, $A \leq B$ if and only if for each $a \in A$ there exists $b \in B$ such that $a \leq b$ and for each $b \in B$ there exists $a \in A$ such that $b \geq a$. Then $(CS(L); \leq)$ is a lattice called the *lattice of convex sublattices* of L (see [3]), denoted by $CS(L)$ in this paper.

Let L be a lattice and A and B be convex sublattices of L. Then in $CS(L)$,

$$A \wedge B := \{z \in L \mid a_1 \wedge b_1 \leq z \leq a_2 \wedge b_2 \text{ for some } a_1,\ a_2 \in A,\ b_1,\ b_2 \in B\}\,;$$
$$A \vee B := \{z \in L \mid a_1 \vee b_1 \leq z \leq a_2 \vee b_2 \text{ for some } a_1,\ a_2 \in A,\ b_1,\ b_2 \in B\}$$

(see [3]).

2010 *Mathematics Subject Classification*: primary 06B20; secondary 06B10.
Key words and phrases: lattice of convex sublattices of a lattice, congruence relation, representable congruence relation.
*Corresponding author.
Received October 21, 2010, revised January 11, 2011. Editor J. Rosický.

Let L be a lattice and X be a sublattice of L. Then the convex sublattice generated by X in L, denoted by $\langle X\rangle$, is given by

$$\langle X\rangle = \{z \in L|\ a_1 \leq z \leq a_2 \text{ for some } a_1,\ a_2 \in X\}$$

(see [1]).

Let L be a lattice and Θ be a congruence relation of L. Then L/Θ denotes the quotient lattice of L modulo Θ and for $a \in L$, a/Θ denotes the congruence class containing a (see [2]).

A congruence relation Θ of a lattice L is said to be *representable* if there is a sublattice L_1 of L such that the map $f\colon L_1 \to L/\Theta$ defined by $f(a) = a/\Theta$ is an isomorphism (see [1]).

3. Extending a congruence relation of L to $CS(L)$

The following lemma is often used in the paper.

Lemma 3.1. *Let L be a lattice, Θ be a congruence relation of L and A be a convex sublattice of L. Suppose that the elements x_1, x, x_2 of L satisfy the following conditions:*
(1) $x_1 \leq x \leq x_2$;
(2) $x_1 \equiv a_1(\Theta)$ *for some* $a_1 \in A$;
(3) $x_2 \equiv a_2(\Theta)$ *for some* $a_2 \in A$.
Then there exists $y \in A$ such that $x \equiv y(\Theta)$.

Proof. From (1) and (2), we get

$$x = x \vee x_1 \equiv x \vee a_1(\Theta) \tag{3.1}$$

and from (1) and (3), we get

$$x = x \wedge x_2 \equiv x \wedge a_2(\Theta). \tag{3.2}$$

Take $y = (a_1 \wedge a_2) \vee (a_2 \wedge x)$. Then

$$a_1 \wedge a_2 \leq y \leq a_2 \tag{3.3}$$

and

$$a_2 \wedge x \leq y \leq a_1 \vee x\,. \tag{3.4}$$

Now from (3.1), (3.2) and (3.4), $x \equiv y(\Theta)$ and from (3.3), $y \in A$. □

In the following lemma a congruence relation on $CS(L)$ corresponding to a congruence relation of a lattice L is constructed. Note that, in [4], a similar congruence relation is defined on $I(L)$ of a trellis L, and it is used for proving some results.

Lemma 3.2. *Let L be a lattice and Θ be a congruence relation of L. Then the binary relation Ψ on $CS(L)$ defined by "$X \equiv Y(\Psi)$ if and only if for each $x \in X$ there exists $y \in Y$ such that $x \equiv y(\Theta)$ and for each $y \in Y$ there exists $x \in X$ such that $x \equiv y(\Theta)$", is a congruence relation on $CS(L)$.*

Proof. Clearly Ψ is an equivalence relation on $CS(L)$. To show that Ψ satisfies the substitution property, consider A, B, $C \in CS(L)$ with $A \equiv C(\Psi)$. It is enough to prove that

$$A \wedge B \equiv C \wedge B(\Psi);$$

$$A \vee B \equiv C \vee B(\Psi).$$

Let $x \in A \wedge B$. Then, by the definition of $A \wedge B$ in $CS(L)$, there exist a_1, $a_2 \in A$ and b_1, $b_2 \in B$ such that $a_1 \wedge b_1 \leq x \leq a_2 \wedge b_2$. Since $a_1 \in A$ and $A \equiv C(\Psi)$, there exists $c_1 \in C$ such that $a_1 \equiv c_1(\Theta)$. But then $a_1 \wedge b_1 \equiv c_1 \wedge b_1(\Theta)$. Similarly, $a_2 \wedge b_2 \equiv c_2 \wedge b_2(\Theta)$ for some $c_2 \in C$. Note that $c_1 \wedge b_1$ and $c_2 \wedge b_2 \in C \wedge B$. Applying Lemma 3.1 for $a_1 \wedge b_1$, x, $a_2 \wedge b_2$ in L, noting that $C \wedge B \in CS(L)$, there exists $y \in C \wedge B$ such that $x \equiv y(\Theta)$.

Similarly, for each $x \in C \wedge B$ there exists $y \in A \wedge B$ such that $x \equiv y(\Theta)$. Hence $A \wedge B \equiv C \wedge B(\Psi)$.

By the dual argument it follows that $A \vee B \equiv C \vee B(\Psi)$. □

Definition 3.3. For a given congruence relation Θ on L, the congruence relation on $CS(L)$ defined in Lemma 3.2 is denoted by Ψ_Θ.

One can easily verify the following lemma.

Lemma 3.4 ([3])**.** *L/Θ is a suborder of $CS(L)$ for any $\Theta \in \operatorname{Con} L$.*

Theorem 3.5. *Let L be a lattice and Θ be a congruence relation of L. Then $CS(L/\Theta)$ is isomorphic to $CS(L)/\Psi_\Theta$.*

Proof. Define a map $f\colon CS(L/\Theta) \to CS(L)/\Psi_\Theta$ by

$$f(X) = (\cup X)/\Psi_\Theta\,.$$

It is easy to see that $\cup X$ is a convex sublattice of L and hence the map f is well-defined.

To prove f is one to one, suppose that $(\cup X)/\Psi_\Theta = (\cup Y)/\Psi_\Theta$. We assert that $\cup X = \cup Y$ which eventually proves $X = Y$. Let $x \in \cup X$. Since $(\cup X) \equiv (\cup Y)(\Psi_\Theta)$, there is a $y \in \cup Y$ such that $x \equiv y(\Theta)$. Now $x/\Theta = y/\Theta \in Y$ so that $x \in \cup Y$. Hence $\cup X \subseteq \cup Y$. Similarly it follows that $\cup Y \subseteq \cup X$. Thus f is one to one.

To prove f is onto, we need some preliminary considerations.

Let $A \in CS(L)$ and $S = \bigcup\{B \in CS(L) \mid B \equiv A(\Psi_\Theta)\}$.

Claim 1: S is a convex sublattice of L.

Let x, $y \in S$. Then $x \in A_1 \equiv A(\Psi_\Theta)$ and $y \in A_2 \equiv A(\Psi_\Theta)$ for some A_1, $A_2 \in CS(L)$. Now $A_1 \underset{CS(L)}{\wedge} A_2 \equiv A_1 \underset{CS(L)}{\vee} A_2 \equiv A(\Psi_\Theta)$. Note that $x \wedge y \in A_1 \underset{CS(L)}{\wedge} A_2$ and $x \vee y \in A_1 \underset{CS(L)}{\vee} A_2$. Hence $x \wedge y$ and $x \vee y \in S$.

Let $a \leq x \leq b$ in L and $a, b \in S$. Then $a \in A_1 \equiv A(\Psi_\Theta)$ and $b \in A_2 \equiv A(\Psi_\Theta)$ for some A_1, $A_2 \in CS(L)$. We can assume w.l.g that $A_1 \underset{CS(L)}{\leq} A_2$. Let $C = [A_1) \cap (A_2]$, where $[A_1)$ is the filter of L generated by A_1 and $(A_2]$ is the ideal of L generated by A_2. Then C is a convex sublattice of L. Also $A_1 \underset{CS(L)}{\leq} C \underset{CS(L)}{\leq} A_2$ so that $A_1 \equiv C \equiv A_2(\Psi_\Theta)$. Thus $x \in C \subseteq S$. Claim 1 holds.

Claim 2: $S \equiv A(\Psi_\Theta)$.
Let $x \in A$. Since $A \subseteq S$, clearly $x \in S$ and $x \equiv x(\Theta)$. On the other hand, let $y \in S$. Then $y \in B \equiv A(\Psi_\Theta)$ for some B in $CS(L)$, i.e. there exists $x \in A$ such that $y \equiv x(\Theta)$. Claim 2 holds.

Now set

$$X := \{x/\Theta \in L/\Theta |\ x \in S\}.$$

We shall prove that X is a convex sublattice of L/Θ. Let a/Θ, $b/\Theta \in X$. Then $a/\Theta = x/\Theta$ and $b/\Theta = y/\Theta$ for some x, $y \in S$. Now, since S is a sublattice of L, $x \wedge y$ and $x \vee y \in S$. Therefore $x \wedge y/\Theta = x/\Theta \wedge y/\Theta = a/\Theta \wedge b/\Theta \in X$ and $x \vee y/\Theta = x/\Theta \vee y/\Theta = a/\Theta \vee b/\Theta \in X$.

Let $a/\Theta \underset{L/\Theta}{\leq} c/\Theta \underset{L/\Theta}{\leq} b/\Theta$ and a/Θ, $b/\Theta \in X$. We can assume w.l.g that a, $b \in S$. Using Lemma 3.4, there exist $x \in c/\Theta$ and $b_1 \in b/\Theta$ such that $a \leq x \leq b_1$. Applying Lemma 3.1 for $a \leq x \leq b_1$ in L and $S \in CS(L)$, there exists $y \in S$ such that $x \equiv y(\Theta)$, i.e., $y/\Theta = x/\Theta = c/\Theta \in X$. Hence X is a convex sublattice of L/Θ.

It is easy to see that $\cup X \equiv S(\Psi_\Theta)$. Now $X \in CS(L/\Theta)$ and from claim 2, $\cup X \equiv S \equiv A(\Psi_\Theta)$, so that f is onto.

To prove that f is order preserving, let $X \underset{CS(L/\Theta)}{\leq} Y$. Consider any $x \in \cup X$. Then $x/\Theta \in X \underset{CS(L/\Theta)}{\leq} Y$ and hence there exists $y/\Theta \in Y$ such that $x/\Theta \underset{L/\Theta}{\leq} y/\Theta$. Now $x/\Theta \vee y/\Theta = (x \vee y)/\Theta = y/\Theta \in Y$. Hence $x \vee y \in \cup Y$ and also $x \leq x \vee y$. Similarly for each $y \in \cup Y$ we can find $x \in \cup X$ such that $x \leq y$. Thus $\cup X \underset{CS(L)}{\leq} \cup Y$. Therefore $(\cup X)/\Psi_\Theta \underset{CS(L)/\Psi_\Theta}{\leq} (\cup Y)/\Psi_\Theta$, proving f is order preserving.

It remains to prove that f^{-1} is order preserving. First we observe the following fact.

Claim 3: Let $X \in CS(L/\Theta)$ and $S = \cup\{A \in CS(L) | A \equiv \cup X(\Psi_\Theta)\}$. Then $S = \cup X$. Since $\cup X \in CS(L)$ and $\cup X \equiv \cup X(\Psi_\Theta)$, $\cup X \subseteq S$. On the other hand, if $x \in S$, then $x \in A \equiv \cup X(\Psi_\Theta)$, for some $A \in CS(L)$. Now there exists $y \in \cup X$ such that $x \equiv y(\Theta)$. But then, $x/\Theta = y/\Theta \in X$. Hence $x \in \cup X$. Claim 3 holds.

Let $(\cup X)/\Psi_\Theta \underset{CS(L)/\Psi_\Theta}{\leq} (\cup Y)/\Psi_\Theta$. We prove that $\cup X \underset{CS(L)}{\leq} \cup Y$ which leads to $X \underset{CS(L/\Theta)}{\leq} Y$. Using Claim 3, it can be assumed that $\cup X = S_1$ and $\cup Y = S_2$ where S_1 and S_2 are as defined in Claim 3. It remains to show that $S_1 \underset{CS(L)}{\leq} S_2$.

Let $x \in S_1$. Then $x \in A \equiv \cup X(\Psi_\Theta)$, for some $A \in CS(L)$.
Since $S_1/\Psi_\Theta \underset{CS(L)/\Psi_\Theta}{\leq} S_2/\Psi_\Theta$ and $A \in S_1/\Psi_\Theta$; by Lemma 3.4, there exists $B \in S_2/\Psi_\Theta$ such that $A \underset{CS(L)}{\leq} B$. Since $x \in A \underset{CS(L)}{\leq} B$, there exists $y \in B$ such that $x \leq y$. Clearly $B \subseteq S_2$, so that $y \in S_2$. Similarly one can prove that for each $x \in S_2$ there exists $y \in S_1$ such that $y \leq x$. Thus $S_1 \underset{CS(L)}{\leq} S_2$. $\square$

With the aid of Theorem 3.5, we obtain the following result.

Corollary 3.6. *Let L be a lattice and Θ be a congruence relation of L. Then L/Θ and $CS(L)/\Psi_\Theta$ are in the same equational class.*

Proof. It is known that for a lattice L, L/Θ and $CS(L/\Theta)$ are in the same equational class ([3]). Now by Theorem 3.5, $CS(L)/\Psi_\Theta$ is also in the same equational class. □

Next theorem shows that, the map $\Theta \to \Psi_\Theta$, preserves representability. But it requires a lemma.

In the following lemma a sublattice of $CS(L)$ corresponding to a sublattice of L is constructed.

Lemma 3.7. *Let L_1 be a sublattice of L. Let*

$$Cvx(L_1) := \{\langle X\rangle \in CS(L) |\, X \in CS(L_1)\}.$$

Then $Cvx(L_1)$ is a sublattice of $CS(L)$.

Proof. The result follows by noting that, for $\langle X\rangle$, $\langle Y\rangle \in Cvx(L_1)$,

$$\langle X\rangle \underset{CS(L)}{\wedge} \langle Y\rangle = \Big\langle X \underset{CS(L_1)}{\wedge} Y\Big\rangle$$

and

$$\langle X\rangle \underset{CS(L)}{\vee} \langle Y\rangle = \Big\langle X \underset{CS(L_1)}{\vee} Y\Big\rangle\,.$$

□

Theorem 3.8. *If Θ is a representable congruence relation of L, then so is Ψ_Θ of $CS(L)$.*

Proof. Let Θ be a representable congruence relation of L. Then there exists a sublattice L_1 of L such that the map $L_1 \to L/\Theta$, $a \mapsto a/\Theta$, defines an isomorphism. Let $Cvx(L_1)$ be the sublattice of $CS(L)$ as defined in Lemma 3.7.

Define a map $f\colon Cvx(L_1) \to CS(L)/\Psi_\Theta$ by

$$f(\langle X\rangle) = \langle X\rangle\, /\Psi_\Theta\,,$$

where $X \in CS(L_1)$. We shall prove that f is an isomorphism.

Clearly f is well defined and a homomorphism.

Let $\langle X\rangle \equiv \langle Y\rangle\, (\Psi_\Theta)$. We claim that $X = Y$, which proves that f is one to one. Let $x \in X$. Then there exists $y \in \langle Y\rangle$ such that $x \equiv y(\Theta)$. Since $y \in \langle Y\rangle$, there exist y_1, $y_2 \in Y$ such that $y_1 \le y \le y_2$. Then

$$y_1 = y \wedge y_1 \equiv x \wedge y_1(\Theta) \tag{3.5}$$

and

$$y_2 = y \vee y_2 \equiv x \vee y_2(\Theta)\,. \tag{3.6}$$

Since x, y_1, $y_2 \in L_1$ and L_1 has only one element in each congruence class, (3.5) and (3.6) give $y_1 \le x \le y_2$. Now $x \in Y$ by the convexity of Y in L_1. Therefore $X \subseteq Y$. Similarly, by interchanging X and Y, we get $Y \subseteq X$.

To prove that f is onto, let $A \in CS(L)$. Set

$$X := \{x \in L_1 |\, A \cap (x/\Theta) \neq \emptyset\}\,.$$

Then X is nonempty. In fact, A is nonempty therefore there exists an element $a \in A$ and

$$A = A \cap L = A \cap (\bigcup_{x \in L_1} x/\Theta) = \bigcup_{x \in L_1} \big(A \cap (x/\Theta)\big)$$

so that $a \in A \cap (x/\Theta)$ for some $x \in L_1$. But then $x \in X$.

We prove that X is a convex sublattice of L_1. Let a, $b \in X$. Since L_1 is a sublattice of L, $a \wedge b$ and $a \vee b \in L_1$. Further, since $A \cap (a/\Theta) \neq \emptyset$ and $A \cap (b/\Theta) \neq \emptyset$, take $x \in A \cap (a/\Theta)$ and $y \in A \cap (b/\Theta)$. Then $x \wedge y \in A \cap ((a \wedge b)/\Theta)$ and $x \vee y \in A \cap ((a \vee b)/\Theta)$, proving $A \cap ((a \wedge b)/\Theta) \neq \emptyset$ and $A \cap ((a \vee b)/\Theta) \neq \emptyset$. Thus $a \wedge b$ and $a \vee b \in X$.

Let x_1, $x_2 \in X$ and $x_1 \underset{L_1}{\leq} x \underset{L_1}{\leq} x_2$. Since $A \cap (x_1/\Theta) \neq \emptyset$ and $A \cap (x_2/\Theta) \neq \emptyset$, take $a \in A \cap (x_1/\Theta)$ and $b \in A \cap (x_2/\Theta)$. By Lemma 3.1, there exists $y \in A$ such that $x \equiv y(\Theta)$. Therefore $y \in A \cap (x/\Theta)$, so that $A \cap (x/\Theta) \neq \emptyset$. Thus $x \in X$. Hence X is a convex sublattice of L_1.

Now we prove that $\langle X \rangle \equiv A(\Psi_\Theta)$.

Let $x \in \langle X \rangle$. Then there exist x_1, $x_2 \in X$ such that $x_1 \leq x \leq x_2$. Since $A \cap (x_1/\Theta) \neq \emptyset$ and $A \cap (x_2/\Theta) \neq \emptyset$, take $b_1 \in A \cap (x_1/\Theta)$ and $b_2 \in A \cap (x_2/\Theta)$. Then again by Lemma 3.1, there is a $y \in A$ such that $x \equiv y(\Theta)$.

On the other hand, if $x \in A$, then $x \in A \cap (y/\Theta)$ for some $y \in L_1$. Clearly $y \in X$ and $y \equiv x(\Theta)$ holds. $\square$

References

[1] Grätzer, G., *General Lattice Theory*, 2nd ed., Birkhäuser Verlag, 1998.

[2] Grätzer, G., *The Congruence of a Finite Lattice, A Proof by Picture Aproach*, Birkhäuser Boston, 2006.

[3] Lavanya, S., Parameshwara Bhatta, S., *A new approach to the lattice of convex sublattice of a lattice*, Algebra Universalis **35** (1996), 63–71.

[4] Parameshwara Bhatta, S., Ramananda, H. S., *On ideals and congruence relations in trellises*, Acta Math. Univ. Comenian. **2** (2010), 209–216.

Department of Mathematics, Mangalore University,
Mangalagangothri, 574 199, Karnataka State, INDIA
E-mail: `s_p_bhatta@yahoo.co.in`

Department of Mathematics, Mangalore University,
Mangalagangothri, 574 199, Karnataka State, INDIA
E-mail: `ramanandahs@gmail.com`