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ON COMPLETE SPACELIKE HYPERSURFACES WITH $R = aH + b$ IN LOCALLY SYMMETRIC LORENTZ SPACES

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Abstract. In this note, we investigate n-dimensional spacelike hypersurfaces M^n with $R = aH+b$ in locally symmetric Lorentz space. Two rigidity theorems are obtained for these spacelike hypersurfaces.

1. Introduction

Let M_1^{n+1} be an $(n+1)$-dimensional Lorentz space, i.e. a pseudo-Riemannian manifold of index 1. When the Lorentz space M_1^{n+1} is of constant curvature c, we call it a Lorentz space form, denoted by $M_1^{n+1}(c)$. A hypersurface M^n of a Lorentz space is said to be spacelike if the induced metric on M^n from that of the Lorentz space is positive definite. Since Goddard's conjecture (see [7]), several papers about spacelike hypersurfaces with constant mean curvature in de Sitter space $S_1^{n+1}(1)$ have been published. For a more complete study of spacelike hypersurfaces in general Lorentzian space with constant mean curvature, we refer to [2]. For the study of spacelike hypersurface with constant scalar curvature in de Sitter space $S_1^{n+1}(1)$, there are also many results such as [4, 9, 14, 15]. There are some results about spacelike hypersurfaces with constant scalar curvature in general Lorentzian space, such as [8] and [13].

It is natural to study complete spacelike hypersurfaces in the more general Lorentz spaces, satisfying the assumptions $R = aH + b$, where R is the normalized scalar curvature at a point of space-like hypersurface, H is the mean curvature and $a, b \in \mathbb{R}$ are constants. First of all, we recall that Choi et al. [6, 12] introduced the class of $(n+1)$-dimensional Lorentz spaces M_1^{n+1} of index 1 which satisfy the following two conditions for some fixed constants c_1 and c_2:

(i) for any spacelike vector u and any timelike vector v,

$$K(u,v) = -\frac{c_1}{n}\,,$$

(ii) for any spacelike vectors u and v,

$$K(u,v) \geq c_2\,.$$

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(Here, and in the sequel, K denotes the sectional curvature of M_1^{n+1}.)

Convention. When M_1^{n+1} satisfies conditions (i) and (ii), we shall say that M_1^{n+1} satisfies condition $(*)$.

We compute the scalar curvature at a point of Lorentz space M_1^{n+1},

$$\bar{R} = \sum_A \epsilon_A \bar{R}_{AA} = -2\sum_{i=1}^{n} \bar{R}_{n+1iin+1} + \sum_{ij} \bar{R}_{ijji} = -2c_1 + \sum_{ij} \bar{R}_{ijji}, \tag{1}$$

where $\bar{R}_{n+1iin+1} = -K(e_i, e_{n+1}) = \frac{c_1}{n}$, for $i = 1, \ldots, n$.

It is known that $\bar{R}$ is constant when the Lorentz space M_1^{n+1} is locally symmetric, so $\sum_{ij} \bar{R}_{ijji}$ is constant. In this note, we shall prove the following main results:

Theorem 1.1. *Let M^n be a complete spacelike hypersurface with bounded mean curvature in locally symmetric Lorentz space M_1^{n+1} satisfying the condition $(*)$. If $R = aH + b$, $(n-1)^2a^2 + 4\sum_{ij} \bar{R}_{ijji} - 4n(n-1)b \geq 0$, and $a \geq 0$, then the following properties hold.*

(1) *If $\sup H^2 < \frac{4(n-1)}{n^2}c$, where $c = \frac{c_1}{n} + 2c_2$, then $c > 0$, $S = nH^2$ and M^n is totally umbilical.*

(2) *If $\sup H^2 = \frac{4(n-1)}{n^2}c$, then $c \geq 0$ and either $S = nH^2$ and M^n is totally umbilical, or $\sup S = nc$.*

(3-a) *If $c < 0$, then either $S = nH^2$ and M^n is totally umbilical, or $n \sup H^2 < \sup S \leq S^+$.*

(3-b) *If $c \geq 0$ and $\sup H^2 \geq c > \frac{4(n-1)}{n^2}c$, then either $S = nH^2$ and M^n is totally umbilical, or $n \sup H^2 < \sup S \leq S^+$.*

(3-c) *If $c \geq 0$ and $c > \sup H^2 > \frac{4(n-1)}{n^2}c$, then either $S = nH^2$ and M^n is totally umbilical, or $S^- \leq \sup S \leq S^+$.*

(4)

$$S \equiv \frac{n}{2(n-1)}[n^2 \sup H^2 + (n-2)\sup|H|\sqrt{n^2 \sup H^2 - 4(n-1)c}] - nc\,, \tag{2}$$

if and only if M is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Here $S^+ = \frac{n}{2(n-1)}[n^2 \sup H^2 + (n-2)\sup|H|\sqrt{n^2 \sup H^2 - 4(n-1)c}] - nc$, and $S^- = \frac{n}{2(n-1)}[n^2 \sup H^2 - (n-2)\sup|H|\sqrt{n^2 \sup H^2 - 4(n-1)c}] - nc$.

Theorem 1.2. *Let M^n $(n > 1)$ be a complete spacelike hypersurface in locally symmetric Lorentz space M_1^{n+1} satisfying the condition $(*)$. If $c = \frac{c_1}{n} + c_2 > 0$, $c_2 > 0$ and*

$$W^2 = \operatorname{tr}(W)W\,, \tag{3}$$

where W is the shape operator with respect to e_{n+1}, then M^n must be totally geodesic.

Remark 1.3. The Lorentz space form $M_1^{n+1}(c)$ satisfies the condition $(*)$, where $-\frac{c_1}{n} = c_2 = \text{const}$.

2. Preliminaries

Let M^n be a spacelike hypersurface of Lorentz space M_1^{n+1}. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \dots, e_n, e_{n+1}\}$ in M_1^{n+1} such that, restricted to M^n, $e_1, \dots, e_n$ are tangent to M^n and e_{n+1} is the unit timelike normal vector. Denote by $\{\omega_A\}$ the corresponding dual coframe and by $\{\omega_{AB}\}$ the connection forms of M_1^{n+1}. Then the structure equations of M_1^{n+1} are given by

$$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B\,, \quad \omega_{AB} + \omega_{BA} = 0\,, \quad \epsilon_i = 1\,, \quad \epsilon_{n+1} = -1\,, \tag{4}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{CD} \epsilon_C \epsilon_D \bar{R}_{ABCD} \omega_C \wedge \omega_D\,, \tag{5}$$

where $A, B, C, \dots = 1, \dots, n+1$ and $i, j, l, \dots = 1, \dots, n$. The components $\bar{R}_{CD}$ of the Ricci tensor and the scalar curvature $\bar{R}$ of M_1^{n+1} are given by

$$\bar{R}_{CD} = \sum_B \epsilon_B \bar{R}_{BCDB}\,, \quad \bar{R} = \sum_A \epsilon_A \bar{R}_{AA}\,. \tag{6}$$

The components $\bar{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor $\bar{R}$ are defined by

$$\begin{aligned} \sum_E \epsilon_E \bar{R}_{ABCD;E} \omega_E = d\bar{R}_{ABCD} - \sum_E \epsilon_E (&\bar{R}_{EBCD}\omega_{EA} \\ &+ \bar{R}_{AECD}\omega_{EB} + \bar{R}_{ABED}\omega_{EC} + \bar{R}_{ABCE}\omega_{ED})\,. \end{aligned} \tag{7}$$

We restrict these forms to M^n, then $\omega_{n+1} = 0$ and the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since

$$0 = d\omega_{n+1} = -\sum_i \omega_{n+1,i} \wedge \omega_i\,, \tag{8}$$

by Cartan's lemma we may write

$$\omega_{n+1,i} = \sum_j h_{ij}\omega_j\,, \qquad h_{ij} = h_{ji}\,. \tag{9}$$

From these formulas, we obtain the structure equations of M^n:

$$\begin{aligned} d\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,, \\ d\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,, \\ R_{ijkl} &= \bar{R}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl})\,, \end{aligned} \tag{10}$$

where R_{ijkl} are the components of curvature tensor of M^n. Components R_{ij} of Ricci tensor and scalar curvature R of M^n are given by

$$R_{ij} = \sum_k \bar{R}_{kijk} - \Big(\sum_k h_{kk}\Big) h_{ij} + \sum_k h_{ik} h_{jk}\,, \tag{11}$$

$$n(n-1)R = \sum_{ij} \bar{R}_{ijji} + S - n^2 H^2\,. \tag{12}$$

We call

$$B = \sum_{i,j,\alpha} h_{ij} \omega_i \otimes \omega_j \otimes e_{n+1} \tag{13}$$

the second fundamental form of M^n. The mean curvature vector is $h = \frac{1}{n}\sum_i h_{ii} e_{n+1}$. We denote $S = \sum_{i,j}(h_{ij})^2$, $H^2 = |h|^2$ and $W = (h_{ij})_{i,j=1}^n$. We call that M^n is maximal if its mean curvature vector vanishes, i.e. $h = 0$.

Let h_{ijk} and h_{ijkl} denote the covariant derivative and the second covariant derivative of h_{ij}^α. Then we have $h_{ijk} = h_{ikj} + \bar{R}_{(n+1)ijk}$ and

$$h_{ijkl} - h_{ijlk} = -\sum_m h_{im} R_{mjkl} - \sum_m h_{mj} R_{mikl}\,. \tag{14}$$

Restricting the covariant derivative $\bar{R}_{ABCD;E}$ on M^n, then $\bar{R}_{(n+1)ijk;l}$ is given by

$$\begin{aligned} \bar{R}_{(n+1)ijk;l} &= \bar{R}_{(n+1)ijkl} + \bar{R}_{(n+1)i(n+1)k} h_{jl} \\ &\quad + \bar{R}_{(n+1)ij(n+1)} h_{kl} + \sum_m \bar{R}_{mijk} h_{ml}\,, \end{aligned} \tag{15}$$

where $\bar{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\bar{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\begin{aligned} \bar{R}_{(n+1)ijkl} &= g\bar{R}_{(n+1)ijk} - \sum_l \bar{R}_{(n+1)ljk}\omega_{li} - \sum_l \bar{R}_{(n+1)ilk}\omega_{lj} \\ &\quad - \sum_l \bar{R}_{(n+1)ijl}\omega_{lk}\,. \end{aligned} \tag{16}$$

The Laplacian $\triangle h_{ij}$ is defined by $\triangle h_{ij} = \sum_k h_{ijkk}$. Using Gauss equation, Codazzi equation Ricci identity and (2), a straightforward calculation will give

$$\begin{aligned} \frac{1}{2}\triangle S &= \sum_{ijk} h_{ijk}^2 + \sum_{ij} h_{ij}\triangle h_{ij} \\ &= \sum_{ijk} h_{ijk}^2 + \sum_{ij} (nH)_{ij} h_{ij} + \sum_{ijk} (\bar{R}_{(n+1)ijk;k} + \bar{R}_{(n+1)kik;j}) h_{ij} \\ &\quad - (\sum_{ij} nH h_{ij} \bar{R}_{(n+1)ij(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}) \\ &\quad - 2\sum_{ijkl} (h_{kl} h_{ij} \bar{R}_{lijk} + h_{il} h_{ij} \bar{R}_{lkjk}) - nH \sum_{ijl} h_{il} h_{lj} h_{ij} + S^2\,. \end{aligned} \tag{17}$$

Set $\Phi_{ij} = h_{ij} - H\delta_{ij}$, it is easy to check that Φ is traceless and $|\Phi|^2 = S - nH^2$. In this note we consider the spacelike hypersurface with $R = aH + b$ in locally symmetric Lorentz space M_1^{n+1}, where a, b are real constants. Following Cheng-Yau [5], we introduce a modified operator acting on any C^2-function f by

$$L(f) = \sum_{ij}(nH\delta_{ij} - h_{ij})f_{ij} + \frac{n-1}{2}a\Delta f\,. \tag{18}$$

We need the following algebraic Lemmas.

Lemma 2.1 ([11]). *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $F\colon M^n \to \mathbb{R}$ be a smooth function which is bounded above on M^n. Then there exists a sequence of points $x_k \in M^n$ such that*

$$\begin{aligned}
&\lim_{k\to\infty} F(x_k) = \sup(F)\,,\\
&\lim_{k\to\infty} |\nabla F(x_k)| = 0\,,\\
&\lim_{k\to\infty} \sup\max\{(\nabla^2(F)(x_k))(X,X) : |X| = 1\} \le 0\,.
\end{aligned}$$

Lemma 2.2 ([1, 10]). *Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta \ge 0$ is constant. Then*

$$\Big|\sum_i \mu_i^3\Big| \le \frac{n-2}{\sqrt{n(n-1)}}\beta^3\,, \tag{19}$$

and equality holds if and only if at least $n-1$ of μ_i's are equal.

3. Proof of the theorems

First, we give the following lemma.

Lemma 3.1. *Let M^n be a complete spacelike hypersurface in locally symmetric Lorentz space M_1^{n+1} satisfying the condition $(*)$. If $R = aH + b$, $a, b \in \mathbb{R}$ and $(n-1)^2a^2 + 4\sum_{ij}\bar{R}_{ijji} - 4n(n-1)b \ge 0$.*

(1) *We have the following inequality,*

$$L(nH) \ge |\Phi|^2\Big(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi| + nc - nH^2\Big)\,. \tag{20}$$

where $c = 2c_2 + \frac{c_1}{n}$.

(2) *If the mean curvature H is bounded, then there is a sequence of points $\{x_k\} \in M$ such that*

$$\begin{aligned}
&\lim_{k\to\infty} nH(x_k) = \sup(nH)\,, \quad \lim_{k\to\infty} |\nabla nH(x_k)| = 0\,,\\
&\lim_{k\to\infty} \sup\big(L(nH)(x_k)\big) \le 0\,.
\end{aligned} \tag{21}$$

Proof. (1) Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h_{ij} = \lambda_i\delta_{ij}$ and $\Phi_{ij} = \lambda_i\delta_{ij} - H\delta_{ij}$. Let $\mu_i = \lambda_i - H$ and denote $\Phi^2 = \sum_i \mu_i^2$. From (12),

(18) and the relation $R = aH + b$, we have

$$\begin{aligned}
L(nH) &= \sum_{ij}(nH\delta_{ij} - h_{ij})(nH)_{ij} + \frac{(n-1)a}{2}\triangle(nH) \\
&= nH\triangle(nH) - \sum_{ij} h_{ij}(nH)_{ij} + \frac{1}{2}\triangle(n(n-1)R - n(n-1)b) \\
&= \frac{1}{2}\triangle[(nH)^2 + n(n-1)R] - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij} \\
&= \frac{1}{2}\triangle\Big[\sum_{ij}\bar{R}_{ijji} + S\Big] - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij} \\
&= \frac{1}{2}\triangle S - n^2|\nabla H|^2 - \sum_{ij} h_{ij}(nH)_{ij}\,.
\end{aligned}$$

From (17) and M_1^n is locally symmetric, we have

$$L(nH) = \underbrace{\sum_{ijk} h_{ijk}^2 - n^2|\nabla H|^2}_{\text{I}} \underbrace{-nH\sum_i \lambda_i^3 + S^2}_{\text{II}}$$
$$\underbrace{-\Big(\sum_{ij} nH\lambda_i\bar{R}_{(n+1)ii(n+1)} + S\sum_k \bar{R}_{(n+1)k(n+1)k}\Big) - 2\sum_{ijkl}(\lambda_k\lambda_i\bar{R}_{kiik} + \lambda_i^2\bar{R}_{ikik})}_{\text{III}}\,.$$

Firstly, we estimate (I):

From Gauss equation, we have

$$\sum_{ijji}\bar{R}_{ijji} + S - n^2H^2 = n(n-1)R = n(n-1)(aH+b)\,, \tag{22}$$

Taking the covariant derivative of the above equation, we have

$$2\sum_{ijk} h_{ij}h_{ijk} = 2n^2HH_k + n(n-1)aH_k\,. \tag{23}$$

Therefore

$$4S\sum_{ijk} h_{ijk}^2 \geq 4\sum_k\Big(\sum_{ij} h_{ij}h_{ijk}\Big)^2 = [2n^2H + n(n-1)a]^2|\nabla H|^2\,. \tag{24}$$

Since we know

$$\begin{aligned}
[2n^2H + n(n-1)a]^2 - 4n^2S &= 4n^4H^2 + n^2(n-1)^2a^2 + 4n^3(n-1)aH \\
&\quad - 4n^2\Big[n^2H^2 + n(n-1)R - \sum_{ij}\bar{R}_{ijji}\Big] \\
&= n^2\Big[(n-1)^2a^2 + 4\sum_{ijji}\bar{R}_{ijji} - 4n(n-1)b\Big] \geq 0\,.
\end{aligned}$$

if follows that

$$\sum_{ijk} h_{ijk}^2 \geq n^2 |\nabla H|^2 \,. \tag{25}$$

Secondly, we estimate (II):

It is easy to know that

$$\sum_i \lambda_i^3 = nH^3 + 3H \sum_i \mu_i^2 + \sum_i \mu_i^3 \,. \tag{26}$$

By applying Lemma 2.2 to real numbers $\mu_1, \dots, \mu_n$, we get

$$\begin{aligned} S^2 - nH \sum_i \lambda_i^3 &= (|\Phi|^2 + nH^2)^2 - n^2 H^4 - 3nH^2 |\Phi|^2 - nH \sum_i \mu_i^3 \\ &\geq |\Phi|^4 - nH^2 |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| |\Phi|^3 \,. \end{aligned} \tag{27}$$

Finally, we estimate (III):

Using curvature condition $(*)$, we get

$$-\Big(\sum_{ij} nH \lambda_i \bar{R}_{(n+1)ii(n+1)} + S \sum_k \bar{R}_{(n+1)k(n+1)k} \Big) = c_1 (S - nH^2) \,. \tag{28}$$

Notice that $S - nH^2 = \frac{1}{2n} \sum_{ij} (\lambda_i - \lambda_j)^2$, we also have

$$\begin{aligned} -2 \sum_{ik} (\lambda_k \lambda_i \bar{R}_{kiik} + \lambda_i^2 \bar{R}_{ikik}) &= -2 \sum_{ik} (\lambda_i \lambda_k - \lambda_i^2) R_{ikki} \\ &\geq c_2 \sum_{ik} (\lambda_i - \lambda_k)^2 = 2nc_2 (S - nH^2) \,. \end{aligned} \tag{29}$$

From (25), **??**, (28), (29) and set $c = 2c_2 + \frac{c_1}{n}$, we have

$$L(nH) \geq |\Phi|^2 \Big(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} |H| \, |\Phi| + nc - nH^2 \Big) \,.$$

(2) Choose a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. By definition, $L(nH) = \sum_i (nH - \lambda_i)(nH)_{ii} + \frac{(n-1)a}{2} \sum_i (nH)_{ii}$. If $H \equiv 0$ the result is obvious. Let suppose that H is not identically zero. By changing the orientation of M^n if necessary, we may assume that $\sup H > 0$. From

$$\begin{aligned} (\lambda_i)^2 \leq S &= n^2 H^2 + n(n-1)R - \sum_{ij} \bar{R}_{ijji} \\ &= n^2 H^2 + n(n-1)(aH + b) - \sum_{ij} \bar{R}_{ijji} \\ &= (nH + \frac{(n-1)a}{2})^2 - \frac{1}{4}(n-1)^2 a^2 - \sum_{ij} \bar{R}_{ijji} + n(n-1)b \\ &\leq (nH + \frac{(n-1)a}{2})^2 \,, \end{aligned} \tag{30}$$

we have

$$|\lambda_i| \leq |nH + \frac{(n-1)a}{2}|\,. \tag{31}$$

Since H is bounded and Eq. (30), we know that S is also bounded. From the Eq. (10),

$$\begin{aligned} R_{ijji} &= \bar{R}_{ijji} - h_{ii}h_{jj} + (h_{ij})^2 \geq c_2 - h_{ii}h_{jj} \\ &= c_2 - \lambda_i\lambda_j \geq c_2 - S\,. \end{aligned} \tag{32}$$

This shows that the sectional curvatures of M^n are bounded from below because S is bounded. Therefore we may apply Lemma 2.1 to the function nH, and obtain a sequence of points $\{x_k\} \in M^n$ such that

$$\begin{aligned} &\lim_{k\to\infty} nH(x_k) = \sup(nH)\,, \quad \lim_{k\to\infty} |\nabla(nH)(x_k)| = 0\,, \\ &\lim_{k\to\infty} \sup\big(nH_{ii}(x_k)\big) \leq 0\,. \end{aligned} \tag{33}$$

Since H is bounded, taking subsequences if necessary, we can arrive to a sequence $\{x_k\} \in M^n$ which satisfies (33) and such that $H(x_k) \geq 0$ (by changing the orientation of M^n if necessary). Thus from (31) we get

$$\begin{aligned} 0 &\leq nH(x_k) + \frac{(n-1)a}{2} - |\lambda_i(x_k)| \leq nH(x_k) + \frac{(n-1)a}{2} - \lambda_i(x_k) \\ &\leq nH(x_k) + \frac{(n-1)a}{2} + |\lambda_i(x_k)| \leq 2(nH(x_k) + \frac{(n-1)a}{2})\,. \end{aligned} \tag{34}$$

Using once the fact that H is bounded, from (34) we infer that $\{nH(x_k) - \lambda_i^{n+1}(x_k)\}$ is non-negative and bounded. By applying $L(nH)$ at x_k, taking the limit and using (33) and (34) we have

$$\begin{aligned} &\lim_{k\to\infty} \sup(L(nH))(x_k) \\ &\qquad \leq \sum_i \lim_{k\to\infty} \sup(nH + \frac{(n-1)a}{2} - \lambda_i)(x_k)nH_{ii}(x_k) \leq 0\,. \end{aligned} \tag{35}$$

□

Remark 3.2. When $a = 0$, then $R = b$ is constant, the inequality (20) appeared in [3, 8, 13].

Proof of Theorem 1.1. According to Lemma 3.1 (2), there exists a sequence of points $\{x_k\}$ in M^n such that

$$\lim_{k\to\infty} nH(x_k) = \sup(nH)\,, \quad \lim_{k\to\infty} \sup\big(L(nH)(x_k)\big) \leq 0\,. \tag{36}$$

From Gauss equation, we have that

$$|\Phi|^2 = S - nH^2 = n(n-1)H^2 + n(n-1)(aH+b) - \sum_{ij} \bar{R}_{ijji}\,. \tag{37}$$

Notice that $\lim_{k\to\infty}(nH)(x_k) = \sup(nH)$, $a \geq 0$ and $\sum_{ij} \bar{R}_{ijji}$ is constant, we have

$$\lim_{k\to\infty} |\Phi|^2(x_k) = \sup|\Phi|^2 . \tag{38}$$

Evaluating (20) at the points x_k of the sequence, taking the limit and using (36), we obtain that

$$\begin{aligned} 0 &\geq \lim_{k\to\infty} \sup \big(L(nH)(x_k)\big) \\ &\geq \sup|\Phi|^2 \Big(\sup|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup|H| \sup|\Phi| + nc - n\sup H^2 \Big) . \end{aligned} \tag{39}$$

Consider the following polynomial given by

$$P_{\sup H}(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} \sup|H| x + nc - n\sup H^2 . \tag{40}$$

(1) If $\sup H^2 < \frac{4(n-1)}{n^2}c$ holds, then we have $c > 0$ and $P(\sup|\Phi|) > 0$. From (39), we know that $\sup|\Phi| = 0$, that is $|\Phi| = 0$. Thus, we infer that $S = nH^2$ and M^n is totally umbilical.

(2) If $\sup H^2 = \frac{4(n-1)}{n^2}c$ holds, then we have $c \geq 0$ and $P(|\Phi|) = \big(|\Phi| - \frac{n-2}{\sqrt{n}}\sqrt{c}\big)^2 \geq 0$. If $\big(|\Phi| - \frac{n-2}{\sqrt{n}}\sqrt{c}\big)^2 > 0$, from (39) we have, $\sup|\Phi| = 0$, that is $|\Phi| = 0$. Thus, we infer that $S = nH^2$ and M^n is totally umbilical. If $\sup|\Phi| = \frac{n-2}{\sqrt{n}}\sqrt{c}$, we have that $\sup S = nc$.

(3) If $\sup H^2 > \frac{4(n-1)}{n^2}c$, we know that $P(x)$ has two real roots $x^-_{\sup H}$ and $x^+_{\sup H}$ given by

$$\begin{aligned} x^-_{\sup H} &= \sqrt{\frac{n}{4(n-1)}} \Big\{ (n-2)\sup|H| - \sqrt{n^2 \sup H^2 - 4(n-1)c} \Big\} \\ x^+_{\sup H} &= \sqrt{\frac{n}{4(n-1)}} \Big\{ (n-2)\sup|H| + \sqrt{n^2 \sup H^2 - 4(n-1)c} \Big\} \end{aligned}$$

It is easy to know that $x^+_{\sup H}$ is always positive. In this case, we also have that

$$P_{\sup H}(x) = (\sup|\Phi| - x^-_{\sup H})(\sup|\Phi| - x^+_{\sup H}) . \tag{41}$$

From (39) and (41), we have that

$$0 \geq \sup|\Phi|^2 (\sup|\Phi| - x^-_{\sup H})(\sup|\Phi| - x^+_{\sup H}) . \tag{42}$$

(3-a) If $c < 0$, we know that $x^-_{\sup H} < 0$. Therefore, from (42), we have, $\sup|\Phi| = 0$, in this case M^n is totally umbilical, or $0 < \sup|\Phi| \leq x^+_{\sup H}$, i.e.

$$n\sup H^2 < \sup S \leq S^+ .$$

(3-b) If $c \geq 0$ and $\sup(H)^2 \geq c > \frac{4(n-1)}{n^2}c$, we know that $x^-_{\sup H} < 0$. Therefore, from (42), we have, $\sup|\Phi| = 0$, in this case M^n is totally umbilical, or $0 < \sup|\Phi| \leq x^+_{\sup H}$, i.e.

$$n\sup H^2 < \sup S \leq S^+ .$$

(3-c) If $c \geq 0$ and $c > \sup(H)^2 > \frac{4(n-1)}{n^2}c$, then we have $x^-_{\sup H} > 0$. Therefore, from (39), we have that $\sup|\Phi| = 0$, in this case M^n is totally umbilical or $x^-_{\sup H} \leq \sup|\Phi| \leq x^+_{\sup H}$, i.e.

$$S^- \leq \sup S \leq S^+ .$$

(4) If $S \equiv \frac{n}{2(n-1)}[n^2 \sup H^2 + (n-2)\sup|H|\sqrt{n^2 \sup H^2 - 4(n-1)c}] - nc$ holds, from Gauss equation, we have $S = nH^2 + n(n-1)(aH+b) - \sum_{ij} \bar{R}_{ijji}$. Since S is constant, then H is also constant. We know that these inequalities in the proof of Lemma 2.2, and (27) are equalities and $S > nH^2$. Hence, we have $H^2 \geq \frac{4(n-1)}{n^2}c$ from (1) in Theorem 1.1. Thus, we can infer that $n-1$ of the principal curvatures λ_i are equal. Since S and H is constant, we know that principal curvatures are constant on M^n. Thus, M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple. This proves Theorem 1.1. □

Proof of Theorem 1.2. From (3), we have that

$$\sum_k h_{ik}h_{jk} = nHh_{ij}\,, \quad \text{for} \quad i,j \in \{1,\dots,n\}\,, \tag{43}$$

and

$$\sum_{ij} h_{ij}^2 = n^2H^2\,, \qquad \text{i.e.} \quad S = n^2H^2\,. \tag{44}$$

Choose a local orthonormal frame field $\{e_1,\dots,e_n\}$ such that $R_{ij} = \upsilon_i\delta_{ij}$. From (11) and (43), we have $R_{ii} = \sum_k \bar{R}_{kiik} \geq (n-1)c_2 > 0$, that is, $\upsilon_i \geq (n-1)c_2 > 0$, so we know that $\mathrm{Ric} = (R_{ij}) \geq (n-1)c_2 I$, we see by the Bonnet-Myers theorem that M^n is bounded and hence compact.

From (12) and (44), we have that $n(n-1)R = \sum_{ij}\bar{R}_{ijji}$ is constant, then from Lemma 3.1 for $a = 0$, we have the following inequality

$$L(nH) \geq |\Phi|^2\Big(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nc - nH^2\Big)\,. \tag{45}$$

Since L is self-adjoint and M^n is compact, we have

$$0 \geq \int_{M^n} |\Phi|^2\Big(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi| + nc - nH^2\Big)\,. \tag{46}$$

Since $n^2|H|^2 = S$ and $|\Phi|^2 = S - nH^2 = n(n-1)H^2$, we have

$$\begin{aligned} &nc - nH^2 + |\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}\,|H|\,|\Phi| \\ &\quad = nc - nH^2 + n(n-1)H^2 - n(n-2)H^2 = nc > 0\,. \end{aligned}$$

so we know that $|\Phi|^2 = 0$, that is, $S = nH^2$. From Eq. (44), we know that $n^2H^2 = nH^2$, so we have $H = 0$, i.e. $S = nH^2 = 0$, so M^n is totally geodesic. This proves Theorem 1.2. □

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