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CRITERION OF p-CRITICALITY FOR ONE TERM $2n$-ORDER DIFFERENCE OPERATORS

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Abstract. We investigate the criticality of the one term $2n$-order difference operators $l(y)_k = \Delta^n(r_k\Delta^n y_k)$. We explicitly determine the recessive and the dominant system of solutions of the equation $l(y)_k = 0$. Using their structure we prove a criticality criterion.

1. Introduction

In this paper, we deal with the $2n$-order one term difference operators and equations

$$l(y)_k := \Delta^n(r_k\Delta^n y_k) = 0\,, \quad r_k > 0,\ k \in \mathbb{Z}\,, \tag{1.1}$$

where Δ is the forward difference operator, i.e., $\Delta y_k = y_{k+1} - y_k$.

Our paper is motivated by a conjecture given in [7, Conj. 4.1] and by some results presented in [8], where the ordered system of solutions of the $2n$-order one term differential equations $\big[r(t)y^{(n)}\big]^{(n)} = 0$ is investigated. The concept of a *critical* operator was introduced in [10] for the second order Sturm-Liouville equations (via tridiagonal matrices) and in [7] for the $2n$-order Sturm-Liouville difference equations (via Hamiltonian systems). We recall these concepts in more details in the next section.

Our results are based on a structure of the solution space of Equation (1.1), which is described in [6] (we recall this structure in Lemma 2), see also [1].

The paper is organized as follows. In the next section, we recall necessary preliminaries, including the relationship between banded symmetric matrices, Sturm-Liouville difference operators, and linear Hamiltonian difference systems, and the concept of p-criticality as introduced in [7]. Section 3 is devoted to the study of the structure of the solution space of Equation (1.1) and in the last section we formulate the main results of our paper.

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2. Preliminaries

In this section we describe the relationship between Sturm-Liouville operators, banded symmetric matrices, and linear Hamiltonian systems, which is necessary for understanding the results of [7] and [10], and their connection. Let us consider the Sturm-Liouville operator

$$L(y)_k := \sum_{\nu=0}^{n} (-\Delta)^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k-\nu} \right) , \quad k \in \mathbb{Z}, \quad r_k^{[n]} \neq 0 , \tag{2.1}$$

and the equation

$$L(y)_k = 0 . \tag{2.2}$$

It was established in [12, 13], that the operator (2.1) is associated to the matrix operators

$$(\mathcal{T} y)_k = \sum_{j=k-n}^{k+n} t_{k,j} y_j , \quad k \in \mathbb{Z} . \tag{2.3}$$

defined by the infinite symmetric banded matrix

$$T = (t_{\mu,\nu}), \quad t_{\mu,\nu} = t_{\nu,\mu} , \quad \mu, \nu \in \mathbb{Z} , \quad t_{\mu,\nu} = 0 \quad \text{for} \quad |\mu - \nu| > n .$$

Expanding the differences in (2.1), we obtain the recurrence relation (2.3) with $t_{i,j}$ given by the formulas

$$\begin{aligned} t_{k,k+j} &= (-1)^j \sum_{\mu=j}^{n} \sum_{\nu=j}^{\mu} \binom{\mu}{\nu} \binom{\mu}{\nu - j} r_{k+\nu}^{[\mu]} , \\ t_{k,k-j} &= (-1)^j \sum_{\mu=j}^{n} \sum_{\nu=0}^{\mu-j} \binom{\mu}{\nu} \binom{\mu}{\nu + j} r_{k+\nu}^{[\mu]} , \end{aligned} \tag{2.4}$$

for $k \in \mathbb{Z}$ and $j \in \{0, \dots, n\}$. Therefore, one can associate the difference operator L given by (2.1) with the matrix operator $\mathcal{T}$ defined via the infinite matrix T by the formula

$$(\mathcal{T} y)_k := L(y)_k , \quad k \in \mathbb{Z} .$$

Conversely, the coefficients $r_k^{[\cdot]}$ can be expressed in terms of the elements of the matrix T. Having any symmetric banded matrix $T = (t_{\mu,\nu})$ with the bandwidth $2n + 1$, we can associate this matrix with the Sturm-Liouville operator (2.1) with $r^{[\mu]}$, $\mu = 0, \dots, n$, given by the formula

$$r_{k+\mu}^{[\mu]} = (-1)^\mu \sum_{s=\mu}^{n} \left[\binom{s}{\mu} t_{k,k+s} + \sum_{l=1}^{s-\mu} \frac{s}{l} \binom{\mu+l-1}{l-1} \binom{s-l-1}{s-\mu-l} t_{k-l,k-l+s} \right] , \tag{2.5}$$

where $k \in \mathbb{Z}$, $0 \leq \mu \leq n$.

For Equation (1.1) we get formulas (2.4) and (2.5) in the form

$$t_{k,k+j} = (-1)^j \sum_{\nu=j}^{n} \binom{n}{\nu}\binom{n}{\nu-j} r_{k+\nu}\,,$$

$$t_{k,k-j} = (-1)^j \sum_{\nu=0}^{n-j} \binom{n}{\nu}\binom{n}{\nu+j} r_{k+\nu}\,,$$

and

$$r_{k+n} = (-1)^n t_{k,k+n}\,.$$

Now, we recall some basic facts concerning linear Hamiltonian systems (see papers [3, 5, 9] and books [2, 11])

$$\Delta x_k = A_k x_{k+1} + B_k u_k\,, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k\,, \tag{2.6}$$

where A_k, B_k, and C_k are $n \times n$ matrices, B_k and C_k are symmetric, and $I - A_k$ is invertible (where I stands for the identity matrix of the appropriate dimension). Let y be a solution of Equation (2.2) and let

$$x_k = \begin{pmatrix} y_{k-1} \\ \Delta y_{k-2} \\ \vdots \\ \Delta^{n-1} y_{k-n} \end{pmatrix}, \quad u_k = \begin{pmatrix} \sum_{\mu=1}^{n} (-1)^{\mu-1} \Delta^{\mu-1}(r_k^{[\mu]} \Delta^\mu y_{k-\mu}) \\ \vdots \\ -\Delta(r_k^{[n]} \Delta^n y_{k-n}) + r_k^{[n-1]} \Delta^{n-1} y_{k-n+1} \\ r_k^{[n]} \Delta^n y_{k-n} \end{pmatrix}.$$

Then $\binom{x}{u}$ solves the linear Hamiltonian difference system (2.6) with a constant matrix

$$A_k = A := a_{ij} = \begin{cases} 1 & \text{if } j = i+1,\ i = 1, \dots, n-1\,, \\ 0 & \text{elsewhere,} \end{cases} \tag{2.7}$$

and matrices

$$B_k = \operatorname{diag}\left\{0, \dots, 0, \frac{1}{r_k^{[n]}}\right\}, \quad C_k = \operatorname{diag}\left\{r_k^{[0]}, \dots, r_k^{[n-1]}\right\}. \tag{2.8}$$

We say that the solution $\binom{x}{u}$ of (2.6) is generated by the solution y_k of (2.2). For Equation (1.1) we obtain this system with $C_k = 0$.

Let us consider the matrix linear Hamiltonian system

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^T U_k, \tag{2.9}$$

where the matrices A_k, B_k, and C_k are given by (2.7) and (2.8). We say that a solution (X, U) of (2.9) is generated by the solutions $y^{[1]}, \dots, y^{[n]}$ of (2.2) if and only if its columns $\binom{x^{[1]}}{u^{[1]}}, \dots, \binom{x^{[n]}}{u^{[n]}}$ (the solutions of (2.6)) are generated by $y^{[1]}, \dots, y^{[n]}$, respectively. On the other hand, if we have the solution (X, U) of (2.9), the elements from the first line of the matrix X are exactly the solutions $y^{[1]}, \dots, y^{[n]}$ of (2.2).

Let (X, U) and $(\tilde{X}, \tilde{U})$ be two solutions of (2.9). Then

$$X_k^T \tilde{U}_k - U_k^T \tilde{X}_k \equiv W \tag{2.10}$$

holds with a constant matrix W. (This is an analog of the continuous *Wronskian identity.*) We say that the solution (X,U) of (2.9) is a *conjoined basis* if

$$X_k^T U_k \equiv U_k^T X_k \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = n\,.$$

Two conjoined bases (X,U), $(\tilde{X},\tilde{U})$ of (2.9) are called *normalized* conjoined bases of (2.9) if $W = I$ in (2.10).

System (2.6) is said to be *right disconjugate* in a discrete interval $[l,m]$, $l,m \in \mathbb{Z}$, if the solution $\binom{X}{U}$ of (2.9) given by the initial condition $X_l = 0$, $U_l = I$ satisfies

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^{\dagger}(I-A)^{-1}B_k \geq 0$$

for $k = l,\dots,m-1$, see [3]. Here Ker, † and $\geq$ stand for the kernel, Moore-Penrose generalized inverse, and non-negative definiteness of a matrix indicated, respectively. Similarly, (2.6) is said to be *left disconjugate* on $[l,m]$ if the solution given by the initial condition $X_m = 0$, $U_m = -I$ satisfies

$$\operatorname{Ker} X_k \subseteq \operatorname{Ker} X_{k+1} \quad \text{and} \quad X_{k+1}X_k^{\dagger}B_k(I-A)^{T-1} \geq 0\,, \quad k = l,\dots,m-1\,,$$

see [4]. System (2.6) is disconjugate on $\mathbb{Z}$ if it is right disconjugate (which is the same as left disconjugate, see e.g. [4, Th. 1]) on $[l,m]$ for every $l,m \in \mathbb{Z}$, $l < m$. System (2.6) is said to be *non-oscillatory* at ∞ (*non-oscillatory* at $-\infty$) if there exists $l \in \mathbb{Z}$ ($m \in \mathbb{Z}$) such that it is right disconjugate on $[l,m]$ for every $m > l$ (left disconjugate on $[l,m]$ for every $l < m$).

System (2.6) is said to be *eventually controllable* if there exist $N,\kappa \in \mathbb{N}$ such that for any $m \geq N$ the trivial solution $\binom{x}{u} = \binom{0}{0}$ is the only solution for which $x_m = x_{m+1} = \cdots = x_{m+\kappa} = 0$. Note that Hamiltonian system (2.6) corresponding to Sturm-Liouville Equation (2.2) is controllable with the constant $\kappa = n$, see [3, Rem. 9].

We call a conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (2.9) the *recessive solution* at ∞ if the matrices $\tilde{X}_k$ are nonsingular, $\tilde{X}_k\tilde{X}_{k+1}^{-1}(I-A_k)^{-1}B_k \geq 0$, both for large k, and for any other conjoined basis $\binom{X}{U}$ for which the (constant) matrix $X^T\tilde{U} - U^T\tilde{X}$ is nonsingular we have

$$\lim_{k\to\infty} X_k^{-1}\tilde{X}_k = 0\,.$$

The solution (X,U) is usually called *dominant* at ∞. The recessive solution at ∞ is determined uniquely up to a right multiple by a nonsingular constant matrix and exists whenever (2.6) is non-oscillatory and eventually controllable. The recessive solution at $-\infty$ is defined analogously.

We say that a pair $\binom{x}{u}$ is *admissible* for system (2.6) if and only if the first equation in (2.6) holds.

Finally, we can define the oscillatory properties of (2.2) via the corresponding properties of the associated Hamiltonian system (2.6) with matrices A_k, B_k, and C_k given by (2.7) and (2.8). E.g., Equation (2.2) is disconjugate if and only if the associated system (2.6) is disconjugate, the system of solutions $y^{[1]}\dots,y^{[n]}$ is said to be recessive if and only if it generates the recessive solution X of (2.9), etc.

Now, let us recall the concept of p-critical operators as it is introduced in [7]. Let $\hat{y}^{[i]}$ and $\tilde{y}^{[i]}$, $i = 1, \dots, n$, be the recessive systems of solutions of (2.2) at $-\infty$ and ∞, respectively. We introduce the linear spaces

$$\mathcal{V}^- = \operatorname{Lin}\{\hat{y}^{[1]}, \dots, \hat{y}^{[n]}\}, \quad \mathcal{V}^+ = \operatorname{Lin}\{\tilde{y}^{[1]}, \dots, \tilde{y}^{[n]}\}, \quad \mathcal{H} = \mathcal{V}^- \cap \mathcal{V}^+ .$$

Definition 1. Let (2.2) be disconjugate on $\mathbb{Z}$ and let $\dim \mathcal{H} = p \in \{1, \dots, n\}$. Then we say that the operator L given by (2.1) (or Equation (2.2)) is *p-critical* on $\mathbb{Z}$. If $\dim \mathcal{H} = 0$, we say that L is *subcritical* on $\mathbb{Z}$. If (2.2) is not disconjugate on $\mathbb{Z}$, we say that L is *supercritical* on $\mathbb{Z}$.

The following theorem describes a very important property of the p-critical operators – their resistance to negative perturbations of their coefficients. We use a notation $|J|$ for a number of elements of a set J.

Theorem 1 ([7, Th. 4.1])**.** *Let the operator L be p-critical on $\mathbb{Z}$, and let $m \in \mathbb{Z}$ and $\varepsilon > 0$ be arbitrary. Further, let $J \subseteq \{0, \dots, n-1\}$ with $|J| = n - p + 1$ and let us consider the sequences*

$$\hat{r}_m^{[\mu]} = \begin{cases} r_m^{[\mu]} - \varepsilon , & \text{for } \mu \in J , \\ r_m^{[\mu]} , & \text{otherwise}, \end{cases}$$
$$\hat{r}_k^{[\mu]} = r_k^{[\mu]} , \quad \text{for } \ k \neq m , \quad (\mu = 0, \dots, n) .$$

Then the operator

$$\widehat{L}(y) := \sum_{\nu=0}^{n} (-\Delta)^\nu \left(\hat{r}_k^{[\nu]} \Delta^\nu y_{k-\nu} \right)$$

is supercritical on $\mathbb{Z}$, i.e., it is not disconjugate.

Remark 1. If we consider the operator l from (1.1) as a special case of the operator L with $r^{[i]} \equiv 0$, $i = 0, \dots, n-1$, then Theorem 1 is applicable.

3. Recessive and dominant system of solutions

In this section we describe the recessive and the dominant system of solutions of Equation (1.1). Let us recall, that $\mathcal{H} = \mathcal{V}^+ \cap \mathcal{V}^-$, where $\mathcal{V}^+$ and $\mathcal{V}^-$ denote the subspaces of the solution space of Equation (1.1) generated by the recessive system of solutions at ∞ and $-\infty$, respectively. To prove the results in this section, we need the following statements, where we use the generalized power function

$$k^{(0)} = 1 , \quad k^{(i)} = k(k-1) \dots (k-i+1) , \quad i \in \mathbb{N} .$$

Lemma 1 ([7])**.** (i) *Let z_k be any sequence and*

$$y_k := \frac{1}{(n-1)!} \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} z_j ,$$

then $\Delta^n y_k = z_k$.

(ii) *The generalized power function has the binomial expansion*

$$(k-j)^{(n)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} k^{(n-i)} (j+i-1)^{(i)} .$$

We distinguish two types of solutions of (1.1). The *polynomial* solutions $k^{(i)}$, $i = 0, \dots, n-1$, for which $\Delta^n y_k = 0$, and *non-polynomial* solutions

$$\sum_{j=0}^{k-1} (k-j-1)^{(n-1)} j^{(i)} r_k^{-1} , \quad i = 0, \dots, n-1 ,$$

for which $\Delta^n y_k \neq 0$. (Using Lemma 1 we obtain that $\Delta^n y_k = (n-1)! k^{(i)} r_k^{-1}$.)
The following Lemma describes the structure of the solution space of (1.1).

Lemma 2 ([6, Sec. 2])**.** *Equation* (1.1) *is disconjugate on* $\mathbb{Z}$ *and possesses a system of solutions* $y^{[j]}, \tilde{y}^{[j]}$, $j = 1, \dots, n$, *such that*

$$y^{[1]} \prec \dots \prec y^{[n]} \prec \tilde{y}^{[1]} \prec \dots \prec \tilde{y}^{[n]} \tag{3.1}$$

as $k \to \infty$*, where* $f \prec g$ *as* $k \to \infty$ *for a pair of sequences* f, g *means that* $\lim_{k\to\infty}(f_k/g_k) = 0$*. If* (3.1) *holds, the solutions* $y^{[j]}$ *form the recessive system of solutions at* ∞*, while* $\tilde{y}^{[j]}$ *form the dominant system,* $j = 1, \dots, n$*. The analogous statement holds for the ordered system of solutions as* $k \to -\infty$*.*

Using Lemma 2 we can explicitly describe the recessive and the dominant system of solutions of Equation (1.1). We split this problem into two partially different cases.

Theorem 2. *Suppose that* $m \in \{0, \dots, n-1\}$, $p := n - m - 1, p \leq m+1$, *and*

$$\sum_{k=0}^{\infty} \Big[k^{(p)}\Big]^2 r_k^{-1} = \infty , \qquad \sum_{k=0}^{\infty} k^{(p)} k^{(p-1)} r_k^{-1} < \infty . \tag{3.2}$$

Then

$$\{1, k, \dots, k^{(m)}\} \subseteq \mathcal{V}^+ , \qquad \{k^{(m+1)}, \dots, k^{(n-1)}\} \nsubseteq \mathcal{V}^+ .$$

Proof. Let us consider the following non-polynomial solutions of Equation (1.1)

$$\begin{aligned} y_k^{[\ell]} &= \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} j^{(p+\ell-1)} r_j^{-1} \\ &\quad - \sum_{i=0}^{p-\ell} \Big[(-1)^i \binom{n-1}{i} (k-1)^{(n-1-i)} \sum_{j=0}^{\infty} j^{(p+\ell-1)} (j+i-1)^{(i)} r_j^{-1} \Big] , \end{aligned}$$

for $\ell = 1-p, \dots, p$, and

$$y_k^{[\ell]} = \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} j^{(p+\ell-1)} r_j^{-1} ,$$

for $\ell = p+1,\dots,m+1$. It is clear that these solutions are ordered, i.e., $y^{[i]} \prec y^{[i+1]}$, $i = 1-p,\dots,m$, as well as the polynomial solutions, i.e., $k^{(i)} \prec k^{(i+1)}$, $i = 0,\dots,n-2$.

Now, we prove that

$$\{1,\dots,k^{(m)},y_k^{[1-p]},\dots,y_k^{[0]}\} \prec \{y_k^{[1]},\dots,y_k^{[m+1]},k^{(m+1)},\dots,k^{(n-1)}\} \tag{3.3}$$

which is sufficient for the statement of Theorem 2.

At first, we show that for $\ell = 1,\dots,p$ it holds that $y_k^{[\ell]} \prec k^{(m+\ell)}$, which means that $y_k^{[1]}$ is the smallest solution in the set on the right-hand side of (3.3) (the recessive system of solutions). We have

$$\begin{aligned}
\Delta^{m+\ell} y_k^{[\ell]} &= \frac{(n-1)!}{(n-m-1-\ell)!}\sum_{j=0}^{k-1}(k-j-1)^{(n-m-1-\ell)} j^{(p+\ell-1)} r_j^{-1} \\
&\quad - \sum_{i=0}^{p-\ell}\Big[(-1)^i\binom{n-1}{i}\frac{(n-1-i)!}{(n-m-1-\ell-i)!}(k-1)^{(n-m-1-\ell-i)} \\
&\quad \times \sum_{j=0}^{\infty} j^{(p+\ell-1)}(j+i-1)^{(i)} r_j^{-1}\Big] \\
&= \frac{(n-1)!}{(p-\ell)!}\sum_{j=0}^{k-1}(k-j-1)^{(p-\ell)} j^{(p+\ell-1)} r_j^{-1} \\
&\quad - \sum_{i=0}^{p-\ell}\Big[(-1)^i\frac{(n-1)!(n-1-i)!}{(n-1-i)!i!(p-\ell-i)!}(k-1)^{(p-\ell-i)} \\
&\quad \times \sum_{j=0}^{\infty} j^{(p+\ell-1)}(j+i-1)^{(i)} r_j^{-1}\Big] \\
&= \frac{(n-1)!}{(p-\ell)!}\Big\{\sum_{j=0}^{k-1}(k-j-1)^{(p-\ell)} j^{(p+\ell-1)} r_j^{-1} \\
&\quad - \sum_{i=0}^{p-\ell}\Big[(-1)^i\binom{p-\ell}{i}(k-1)^{(p-\ell-i)}\sum_{j=0}^{\infty} j^{(p+\ell-1)}(j+i-1)^{(i)} r_j^{-1}\Big]\Big\} \\
&= \frac{(n-1)!}{(p-\ell)!}\Big[\sum_{j=0}^{k-1}(k-j-1)^{(p-\ell)} j^{(p+\ell-1)} r_j^{-1} - \sum_{j=0}^{\infty}(k-j-1)^{(p-\ell)} j^{(p+\ell-1)} r_j^{-1}\Big] \\
&= -\frac{(n-1)!}{(p-\ell)!}\sum_{j=k}^{\infty}(k-j-1)^{(p-\ell)} j^{(p+\ell-1)} r_j^{-1} \\
&= (-1)^{p-\ell+1}\frac{(n-1)!}{(p-\ell)!}\sum_{j=k}^{\infty}(j+1-k)^{(p-\ell)} j^{(p+\ell-1)} r_j^{-1}
\end{aligned}$$

and

$$\sum_{j=k}^{\infty}(j+1-k)^{(p-\ell)}j^{(p+\ell-1)}r_j^{-1} \le \sum_{j=k}^{\infty} j^{(p-\ell)}j^{(p+\ell-1)}r_j^{-1} \le \sum_{j=k}^{\infty} j^{(p)}j^{(p-1)}r_j^{-1}\,,$$

hence,

$$\lim_{k\to\infty}\frac{y_k^{[\ell]}}{k^{(m+\ell)}} = \lim_{k\to\infty}\Delta^{m+\ell}y_k^{[\ell]} = 0,$$

thus $y_k^{[\ell]} \prec k^{(m+\ell)}, \ell = 1,\dots,p$, holds.

Now, we show that $k^{(m)} \prec y_k^{[1]}$. We have

$$\begin{aligned}
\frac{(p-1)!}{(n-1)!}\Big|\sum_{i=0}^{k-1}\Delta^{m+1}y_i^{[1]}\Big| &= \sum_{i=0}^{k-1}\sum_{j=i}^{\infty}(j+1-i)^{(p-1)}j^{(p)}r_j^{-1}\\
&= \sum_{j=0}^{k-1}\sum_{i=0}^{j}(j+1-i)^{(p-1)}j^{(p)}r_j^{-1} + \sum_{j=k}^{\infty}\sum_{i=0}^{k-1}(j+1-i)^{(p-1)}j^{(p)}r_j^{-1}\\
&= \sum_{j=0}^{k-1}j^{(p)}r_j^{-1}\left[-\frac{(j+1-i)^{(p)}}{p}\right]_0^{j+1} + \sum_{j=k}^{\infty}j^{(p)}r_j^{-1}\left[-\frac{(j+1-i)^{(p)}}{p}\right]_0^{k}\\
&= \frac{1}{p}\sum_{j=0}^{k-1}j^{(p)}r_j^{-1}(j+1)^{(p)} + \frac{1}{p}\sum_{j=k}^{\infty}j^{(p)}r_j^{-1}\left[(j+1)^{(p)}-(j+1-k)^{(p)}\right],
\end{aligned}$$

where for $k \ge p$ the first sum tends to infinity as $k\to\infty$ (using the assumption $\sum^{\infty} j^{(p)}j^{(p)}r_j^{-1} = \infty$) and the second sum is positive. Therefore, we have

$$\lim_{k\to\infty}\frac{y_k^{[1]}}{k^{(m)}} = \frac{1}{m!}\lim_{k\to\infty}\sum_{i=0}^{k-1}\Delta^{m+1}y_i^{[1]} = \infty,$$

which means that $k^{(m)} \prec y_k^{[1]}$.

Altogether, we have obtained that $k^{(m)} \prec y_k^{[1]}$ and $y_k^{[\ell]} \prec k^{(m+\ell)}, \ell = 1,\dots,p$, where $m+p = n-1$. Thus (3.3) (and therefore the statement of Theorem 2) holds. $\square$

Theorem 3. *Suppose that* $m \in \{0,\dots,n-1\}$, $p := n-m-1, p \ge m+1$, *and* (3.2) *holds. Then*

$$\{1,k,\dots,k^{(m)}\} \subseteq \mathcal{V}^+\,, \qquad \{k^{(m+1)},\dots,k^{(n-1)}\} \not\subseteq \mathcal{V}^+\,.$$

Proof. Here, we use the non-polynomial solutions

$$y_k^{[\ell]} = \sum_{j=k}^{\infty}(k-j-1)^{(n-1)}j^{(p+\ell-1)}r_j^{-1}\,,$$

for $\ell = 1-p, \dots, p-n+1$, and

$$y_k^{[\ell]} = \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} j^{(p+\ell-1)} r_j^{-1}$$
$$- \sum_{i=0}^{p-\ell} \left[(-1)^i \binom{n-1}{i} (k-1)^{(n-1-i)} \sum_{j=0}^{\infty} j^{(p+\ell-1)} (j+i-1)^{(i)} r_j^{-1} \right],$$

for $\ell = p-n+2, \dots, m+1$, and we can proceed as in the proof of Theorem 2. $\square$

The following Corollary follows directly from the proofs of Theorems 2 and 3 and from Lemma 2.

Corollary 1. *Let $m \in \{0, \dots, n-1\}$ and $p := n-m-1$ and suppose that* (3.2) *holds. Then the recessive and dominant systems of solutions of Equation* (1.1) *are*

$$\left\{1, \dots, k^{(m)}, y_k^{[1-p]}, \dots, y_k^{[0]}\right\} \quad \text{and} \quad \left\{k^{(m+1)}, \dots, k^{(n-1)}, y_k^{[1]}, \dots, y_k^{[m+1]}\right\},$$

respectively, where the solutions $y^{[1-p]}, \dots, y^{[m+1]}$ are given in the proof of Theorem 2 for $p \le m+1$ and (*or*) *in the proof of Theorem 3 for $p \ge m+1$.*

Remark 2. To find a counterpart of Theorems 2 and 3 and Corollary 1 at $-\infty$, it suffices to replace $\sum^{\infty}$ by $\sum_{-\infty}$.

Remark 3. The previous analysis shows that only polynomial solutions can be simultaneously contained both in the recessive systems of solutions at ∞ and $-\infty$.

4. Criticality of one term operator

Now, we can formulate the main results of this paper. The first one follows directly from Theorems 2 and 3 and from Remarks 2 and 3.

Theorem 4. *Let $\mathcal{V}^+$ and $\mathcal{V}^-$ denote the subspaces of the solution space of Equation* (1.1) *generated by the recessive system of solutions at ∞ and $-\infty$, respectively. If for some $m \in \{0, \dots, n-1\}$*

$$\sum_{k=-\infty}^{0} \left[k^{(n-m-1)}\right]^2 r_k^{-1} = \infty = \sum_{k=0}^{\infty} \left[k^{(n-m-1)}\right]^2 r_k^{-1},$$

then

$$\operatorname{Lin}\{1, \dots, k^{(m)}\} \subseteq \mathcal{V}^+ \cap \mathcal{V}^-.$$

If in addition

$$\sum_{k=-\infty}^{0} k^{(n-m-1)} k^{(n-m-2)} r_k^{-1} < \infty \quad \text{or} \quad \sum_{k=0}^{\infty} k^{(n-m-1)} k^{(n-m-2)} r_k^{-1} < \infty,$$

then

$$\operatorname{Lin}\{1, \dots, k^{(m)}\} = \mathcal{V}^+ \cap \mathcal{V}^-,$$

i.e., (1.1) *is* $(m+1)$*-critical on* $\mathbb{Z}$.

In the last theorem we formulate a criterion of subcriticality.

Theorem 5. *Let us consider Equation* (1.1) *and let at least one of the sums*

$$\sum_{k=-\infty}^{0}\left[k^{(n-1)}\right]^2 r_k^{-1}, \qquad \sum_{k=0}^{\infty}\left[k^{(n-1)}\right]^2 r_k^{-1} \tag{4.1}$$

be convergent. Then Equation (1.1) *is subcritical, i.e.,* $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$.

Proof. Let $\sum_{k=0}^{\infty}\left[k^{(n)}\right]^2 r_k^{-1} < \infty$. The case where only the first sum in (4.1) is convergent can be treated analogically. We consider the following non-polynomial solutions of Equation (1.1)

$$y_k^{[\ell]} = \sum_{j=k}^{\infty}(k-j-1)^{(n-1)} j^{(n-1+\ell)} r_j^{-1},$$

where $\ell = 1-n,\dots,0$. For $k > 1$ we have

$$|y_k^{[0]}| = \Big|\sum_{j=k}^{\infty}(k-j-1)^{(n-1)} j^{(n-1)} r_j^{-1}\Big| \le \sum_{j=k}^{\infty} j^{(n-1)} j^{(n-1)} r_j^{-1}.$$

Therefore by (4.1)

$$\lim_{k\to\infty} y_k^{[0]} = 0.$$

Thus $y_k^{[0]} \prec 1$ and we have obtained the ordered system of solutions

$$y_k^{[1-n]} \prec \dots \prec y_k^{[0]} \prec 1 \prec \dots \prec k^{(n-1)}.$$

Therefore, by Lemma 2, there is no polynomial solution in the recessive system of solutions of (1.1) at ∞ and therefore $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$. □

Remark 4. Theorem 1 deals with perturbations of $n-p+1$ coefficients at one point $m \in \mathbb{Z}$. If we consider the matrix operator (2.3) we can see, using (2.4), that these perturbations affect the matrix T in rows (and columns) from $m+1-n$ to $m+1$. Hence a natural question arises, whether a perturbation of only one coefficient at more points will cause the same effect. Theorem 4, the Sections IV. and V. in [8], together with the proof of Lemma 4.1 in [7] have lead us to the following conjecture, in which we sufficiently (in the sense of (4.2)) perturb some of the diagonal elements of the matrix T. This conjecture is a subject of the present investigation.

Conjecture 1. *Let there exists an integer* $m \in \{0,\dots,n-1\}$ *and real constants* $c_0,\dots,c_m$ *such that*

$$\sum_{k=-\infty}^{0}\left[k^{(n-m-1)}\right]^2 r_k^{-1} = \infty = \sum_{k=0}^{\infty}\left[k^{(n-m-1)}\right]^2 r_k^{-1},$$

and the sequence $z_k := c_0 + c_1 k + \dots + c_m k^{(m)}$ *satisfies*

$$\limsup_{K\downarrow-\infty, L\uparrow\infty} \sum_{k=K}^{L} q_k z_k^2 \le 0. \tag{4.2}$$

If $q \not\equiv 0$, then the equation

$$(-\Delta)^n \left(r_k \Delta^n y_{k-n}\right) + q_k y_k = 0$$

is not disconjugate.

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