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A NEW CHARACTERIZATION OF r-STABLE HYPERSURFACES IN SPACE FORMS

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Abstract. In this paper we study the r-stability of closed hypersurfaces with constant r-th mean curvature in Riemannian manifolds of constant sectional curvature. In this setting, we obtain a characterization of the r-stable ones through of the analysis of the first eigenvalue of an operator naturally attached to the r-th mean curvature.

1. Introduction

The notion of stability concerning hypersurfaces of constant mean curvature of Riemannian ambient spaces was first studied by Barbosa and do Carmo in [3], and Barbosa, do Carmo and Eschenburg in [4], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations. In [1], Alencar, do Carmo and Colares extended to hypersurfaces with constant scalar curvature the above stability result on constant mean curvature. In order to do that, they assumed that the Riemannian ambient space had positive constant sectional curvature.

The natural generalization of mean and scalar curvatures for an n-dimensional hypersurface are the r-th mean curvatures H_r, for $r = 1, \dots, n$. In fact, H_1 is just the mean curvature and H_2 defines a geometric quantity which is related to the scalar curvature.

In [2], Barbosa and Colares studied closed hypersurfaces immersed in space forms with constant r-th mean curvature. The authors showed that such hypersurfaces are r-stable if and only if they are geodesic spheres, thus generalizing the previous results on constant mean curvature hypersurfaces. More recently, Yijun He and Haizhong Li [9] treated the case of compact hypersurfaces without boundary immersed in space forms with $\dfrac{H_{r+1}}{H_1}$ constant. They proved that such hypersurfaces are r-stable if and only if they are totally umbilical.

Motivated by these works, here we consider closed hypersurfaces with constant r-th mean curvature in a space form in order to obtain a relation between r-stability

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and the spectrum of a certain elliptic operator naturally attached to such r-th mean curvature of the hypersurfaces. Our approach is based on the use of the Newton transformations P_r and the associated second order linear differential operators L_r (cf. Section 2).

More precisely, we will prove the following characterization of r-stable hypersurfaces (cf. Theorem 3.5):

Let $\overline{M}_c^{n+1}$ be either the Euclidean space $\mathbb{R}^{n+1}(c=0)$, an open hemisphere of the sphere $\mathbb{S}^{n+1}(c=1)$, or the hyperbolic space $\mathbb{H}^{n+1}(c=-1)$. Let r be an integer satisfying the inequality $0 \le r \le n-2$, and $x\colon M^n \to \overline{M}_c^{n+1}$ be a closed hypersurface with positive constant $(r+1)$-th mean curvature H_{r+1}. Suppose that

$$\lambda = c(n-r)\binom{n}{r}H_r + nH_1\binom{n}{r+1}H_{r+1} - (r+2)\binom{n}{r+2}H_{r+2}$$

is constant. Then x is r-stable if and only if λ is the first eigenvalue of L_r on M^n.

2. Preliminaries

Let $\overline{M}_c^{n+1}$ be an orientable simply connected Riemannian manifold with constant sectional curvature c, Riemannian metric $\overline{g} = \langle\ ,\ \rangle$, volume element $d\overline{M}$ and Levi-Civita connection $\overline{\nabla}$. In this context, we consider hypersurfaces $x\colon M^n \to \overline{M}_c^{n+1}$, namely, isometric immersions from a connected, $n-$dimensional orientable Riemannian manifold M^n into $\overline{M}_c^{n+1}$. We also let ∇ denote the Levi-Civita connection of M^n.

Since M^n is orientable, one can choose a globally defined unit normal vector field N on M^n. Let A denote the shape operator with respect to N, so that, at each $p \in M^n$, A restricts to a self-adjoint linear map $A_p\colon T_pM \to T_pM$.

For $1 \le r \le n$, if we let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p, we get n smooth functions $S_r\colon M^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n}(-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in M^n$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM formed by eigenvectors of A_p, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

For $1 \le r \le n$, one defines the r-th mean curvature H_r of x by

$$\binom{n}{r}H_r = S_r(\lambda_1, \ldots, \lambda_n).$$

In particular, for $r = 1$,

$$H_1 = \frac{1}{n}\sum_{k=1}^{n}\lambda_k = H$$

is the mean curvature of M^n, which is the main extrinsic curvature of the hypersurface. When $r = 2$, H_2 defines a geometric quantity which is related to the (intrinsic) normalized scalar curvature R of the hypersurface. More precisely, it follows from the Gauss equation that

$$R = c + H_2 \,. \tag{2.1}$$

On the other hand, with a straightforward computation we verify that

$$|A|^2 = n^2H^2 - n(n-1)H_2 \,, \tag{2.2}$$

where $|A|^2$ denotes the squared norm of the shape operator of M^n.

We also define, for $0 \le r \le n$, the r-th Newton transformation P_r on M^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = S_rI - AP_{r-1} \,. \tag{2.3}$$

A trivial induction shows that

$$P_r = (S_rI - S_{r-1}A + S_{r-2}A^2 - \cdots + rA^r)\,,$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM diagonalizing A at $p \in M^n$ also diagonalize all of the P_r at p. Let $\{e_1, \ldots, e_n\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i\rangle^\perp \subset T_p\Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1}(-1)^kS_k(A_i)t^{n-1-k}\,,$$

where

$$S_k(A_i) = \sum_{\substack{1\le j_1<\ldots<j_k\le n\\ j_1,\ldots,j_k\ne i}} \lambda_{j_1}\cdots\lambda_{j_k}\,.$$

With the above notations, it is also immediate to check that $P_re_i = S_r(A_i)e_i$, and hence (cf. Lemma 2.1 of [2])

$$\begin{aligned} \operatorname{tr}(P_r) &= (-1)^r(n-r)S_r = b_rH_r\,;\\ \operatorname{tr}(AP_r) &= (-1)^r(r+1)S_{r+1} = -b_rH_{r+1}\,;\\ \operatorname{tr}(A^2P_r) &= (-1)^r(S_1S_{r+1} - (r+2)S_{r+2})\,, \end{aligned} \tag{2.4}$$

where $b_r = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r\colon C^\infty(M) \to C^\infty(M)$, given by

$$L_r(f) = \operatorname{tr}(P_r \operatorname{Hess} f)\,.$$

Note that $L_0 = \Delta$, the Laplacian operator on M.

According to [11], P_r is a divergence-free whenever $\overline{M}_c^{n+1}$ is of constant sectional curvature; consequently,

$$L_r(f) = \operatorname{div}(P_r\nabla f)\,.$$

For future use, we recall Lemma 2.6 of [6]: if (a_{ij}) denotes the matrix of A with respect to a certain orthonormal basis $\beta = \{e_1, \ldots, e_n\}$ of T_pM, then the matrix (a^r_{ij}) of P_r with respect to the same basis is given by

$$a^r_{ij} = \frac{1}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon^{j_1 \ldots j_r j}_{i_1 \ldots i_r i} a_{j_1 i_1} \ldots a_{j_r i_r}\,, \tag{2.5}$$

where

$$\epsilon^{j_1 \ldots j_r}_{i_1 \ldots i_r} = \begin{cases} \mathrm{sgn}(\sigma)\,, & \text{if the } i_k \text{ are pairwise distinct and} \\ & \sigma = (j_k) \text{ is a permutation of them;} \\ 0\,, & \text{else}\,. \end{cases}$$

For $x\colon M^n \to \overline{M}^{n+1}_c$ as above, a *variation* of it is a smooth mapping

$$X\colon M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}_c$$

satisfying the following conditions:

(1) For $t \in (-\epsilon, \epsilon)$, the map $X_t\colon M^n \to \overline{M}^{n+1}_c$ given by $X_t(p) = X(t,p)$ is an immersion such that $X_0 = x$.

(2) $X_t\big|_{\partial M} = x\big|_{\partial M}$, for all $t \in (-\epsilon, \epsilon)$.

In all that follows, we let dM_t denote the volume element of the metric induced on M by X_t and N_t the unit normal vector field along X_t.

The *variational field* associated to the variation X is the vector field $\frac{\partial X}{\partial t}\Big|_{t=0}$. Letting $f = \langle \frac{\partial X}{\partial t}, N_t \rangle$, we get

$$\frac{\partial X}{\partial t} = f N_t + \Big(\frac{\partial X}{\partial t}\Big)^{\top}, \tag{2.6}$$

where $\top$ stands for tangential components.

The *balance of volume* of the variation X is the function $\mathcal{V}\colon (-\epsilon, \epsilon) \to \mathbb{R}$ given by

$$\mathcal{V}(t) = \int_{M \times [0,t]} X^*(d\overline{M})\,,$$

and we say X is *volume-preserving* if $\mathcal{V}$ is constant.

From now on, we will consider only closed hypersurfaces $x\colon M^n \to \overline{M}^{n+1}_c$. The following lemma is enough known and can be found in (cf. [12]).

Lemma 2.1. *Let $\overline{M}^{n+1}_c$ be an orientable simply connected Riemannian manifold with constant sectional curvatures c and $x : M^n \to \overline{M}^{n+1}_c$ a closed hypersurface. If $X\colon M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}_c$ is a variation of x, then*

$$\frac{d\mathcal{V}}{dt} = \int_M f dM_t\,.$$

In particular, X is volume-preserving if and only if $\int_M f dM_t = 0$ for all t.

We remark that Lemma 2.2 of [4] shows that if $f_0\colon M \to \mathbb{R}$ is a smooth function such that $\int_M f_0 dM = 0$, then there exists a volume-preserving variation of M whose variational field is $f_0 N$.

Following [2], we define the *r-area functional* $\mathcal{A}_r\colon (-\epsilon, \epsilon) \to \mathbb{R}$ associated to the variation X be given by

$$\mathcal{A}_r(t) = \int_M F_r(S_1, S_2, \ldots, S_r) dM_t\,,$$

where $S_r = S_r(t)$ and F_r is recursively defined by setting $F_0 = 1$, $F_1 = S_1$ and, for $2 \le r \le n-1$,

$$F_r = S_r + \frac{c(n-r+1)}{r-1} F_{r-2}\,.$$

We notice that if $r = 0$, the functional $\mathcal{A}_0$ is the classical area functional.

The following result follows from Proposition 4.1 of [2]. Since it seems to us that their proof only works on a neighborhood free of umbilics, and in order to keep this work self-contained, we present an alternative one here.

Lemma 2.2. *Let $x\colon M^n \to \overline{M}_c^{n+1}$ be a closed hypersurface of the orientable simply connected Riemannian manifold $\overline{M}_c^{n+1}$ with constant curvature c, and let $X\colon M^n \times (-\epsilon, \epsilon) \to \overline{M}_c^{n+1}$ be a variation of x. Then,*

$$\frac{\partial S_{r+1}}{\partial t} = L_r f + c\,\mathrm{tr}(P_r) f + \mathrm{tr}(A^2 P_r) f + \Big\langle \Big(\frac{\partial X}{\partial t}\Big)^{\top}, \nabla S_{r+1} \Big\rangle\,. \tag{2.7}$$

Proof. Formula (2.5) gives

$$(r+1) S_{r+1} = \mathrm{tr}(A P_r) = \sum_{i,j} a_{ji} a_{ij}^r = \frac{1}{r!} \sum_{i,j,i_k,j_k} \epsilon_{i_1 \ldots i_r i}^{j_1 \ldots j_r j} a_{j_1 i_1} \ldots a_{j_r i_r} a_{ji}\,,$$

where the functions S_{r+1} are seen as functions of t. Hence, differentiation with respect to t gives

$$\begin{aligned}
(r+1) S'_{r+1} &= \frac{1}{r!} \sum_{i,j,i_k,j_k} \epsilon_{i_1 \ldots i_r i}^{j_1 \ldots j_r j} [a'_{j_1 i_1} \ldots a_{j_r i_r} a_{ji} + \cdots + a_{j_1 i_1} \ldots a_{j_r i_r} a'_{ji}] \\
&= \frac{(r+1)}{r!} \sum_{i,j,i_k,j_k} \epsilon_{i_1 \ldots i_r i}^{j_1 \ldots j_r j} a'_{ji} a_{j_1 i_1} \ldots a_{j_r i_r} \\
&= (r+1) \sum_{i,j} a'_{ji} a_{ij}^r \\
&= (r+1) \mathrm{tr}\Big(\frac{\partial A}{\partial t} P_r\Big)\,.
\end{aligned}$$

We see from the above that it is enough to compute $\mathrm{tr}\big(\frac{\partial A}{\partial t}P_r\big)$, and this is what we do next:

$$\begin{aligned}
S'_{r+1} &= \mathrm{tr}\Big(\frac{\partial A}{\partial t}P_r\Big) = \sum_k \Big\langle \frac{\partial A}{\partial t}P_r e_k, e_k\Big\rangle \\
&= \sum_k S_r(A_k)\big\langle(\nabla_{\frac{\partial X}{\partial t}}A)e_k, e_k\big\rangle \\
&= \sum_k S_r(A_k)\big[\langle\nabla_{\frac{\partial X}{\partial t}}Ae_k, e_k\rangle - \langle A\overline{\nabla}_{\frac{\partial X}{\partial t}}e_k, e_k\rangle\big] \\
&= -\sum_k S_r(A_k)\langle\overline{\nabla}_{\frac{\partial X}{\partial t}}\overline{\nabla}_{e_k}N, e_k\rangle - \sum_k S_r(A_k)\langle A\overline{\nabla}_{e_k}\partial X/\partial t, e_k\rangle\,,
\end{aligned}$$

where we used that $[\partial X/\partial t, e_k] = 0$ in the last summand. If $\overline{R}$ denotes the curvature tensor of $\overline{M}$, we have

$$\overline{R}(e_k, \partial X/\partial t)N = \overline{\nabla}_{\frac{\partial X}{\partial t}}\overline{\nabla}_{e_k}N - \overline{\nabla}_{e_k}\overline{\nabla}_{\frac{\partial X}{\partial t}}N + \overline{\nabla}_{[e_k, \frac{\partial X}{\partial t}]}N\,.$$

Thus, by using also (2.6),

$$\begin{aligned}
S'_{r+1} = &-\sum_k S_r(A_k)\big[\langle\overline{R}(e_k, \partial X/\partial t)N, e_k\rangle + \langle\overline{\nabla}_{e_k}\overline{\nabla}_{\frac{\partial X}{\partial t}}N, e_k\rangle\big] \\
&-\sum_k S_r(A_k)\langle\overline{\nabla}e_k(fN + (\partial X/\partial t)^\top), Ae_k\rangle\,.
\end{aligned}$$

Since $\overline{M}$ is of constant sectional curvature, we get

$$\langle\overline{R}(X,Y)W,Z\rangle = c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\}\,.$$

Therefore,

$$\begin{aligned}
S'_{r+1} &= -\sum_k S_r(A_k)c\,\{\langle e_k, N\rangle\langle\partial X/\partial t, e_k\rangle - \langle e_k, e_k\rangle\langle\partial X/\partial t, N\rangle\} \\
&\quad -\sum_k S_r(A_k)\langle\overline{\nabla}_{e_k}\overline{\nabla}_{\frac{\partial X}{\partial t}}N, e_k\rangle - \sum_k S_r(A_k)\langle Ae_k, \overline{\nabla}e_k fN\rangle \\
&\quad -\sum_k S_r(A_k)\langle\overline{\nabla}_{e_k}(\partial X/\partial t)^\top, Ae_k\rangle \\
&= c\sum_k S_r(A_k)f - \sum_k S_r(A_k)e_k\langle\overline{\nabla}_{\frac{\partial X}{\partial t}}N, e_k\rangle + \sum_k S_r(A_k)\langle\overline{\nabla}_{\frac{\partial X}{\partial t}}N, \overline{\nabla}_{e_k}e_k\rangle \\
&\quad -\sum_k S_r(A_k)\langle Ae_k, f\overline{\nabla}e_kN\rangle - \sum_k S_r(A_k)e_k\langle Ae_k, (\partial X/\partial t)^\top\rangle \\
&\quad +\sum_k S_r(A_k)\langle\overline{\nabla}_{e_k}Ae_k, (\partial X/\partial t)^\top\rangle\,.
\end{aligned}$$

Asking further that $\nabla_{e_i}e_j = 0$ at p (which is always possible) and writing $\frac{\partial X}{\partial t} = fN + \sum_l \alpha_l e_l$, we have

$$\begin{aligned}
\sum_k S_r(A_k)\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} N, \overline{\nabla}_{e_k} e_k\rangle &= \sum_k S_r(A_k)\langle \overline{\nabla}_{\frac{\partial X}{\partial t}} N, \lambda_k \delta_{kk} N\rangle \\
&= \sum_k S_r(A_k)\lambda_k \langle \overline{\nabla}_{fN+\sum_l \alpha_l e_l} N, N\rangle \\
&= \sum_k S_r(A_k)\lambda_k \Big\{ f\langle \overline{\nabla}_N N, N\rangle - \sum_l \alpha_l \langle Ae_l, N\rangle \Big\} \\
&= 0\,.
\end{aligned}$$

By now applying expression (2.4) for the trace of P_r, we get

$$\begin{aligned}
S'_{r+1} &= c\operatorname{tr}(P_r)f + \sum_k P_r e_k \langle N, \overline{\nabla}_{\frac{\partial X}{\partial t}} e_k\rangle + f\sum_k S_r(A_k)\langle Ae_k, Ae_k\rangle \\
&\quad - \sum_k P_r e_k \langle Ae_k, (\partial X/\partial t)^\top\rangle + \sum_k S_r(A_k)\langle \nabla_{e_k} Ae_k, (\partial X/\partial t)^\top\rangle \\
&= c\operatorname{tr}(P_r)f + \sum_k P_r e_k \langle N, \overline{\nabla}_{e_k} \partial X/\partial t\rangle + f\sum_k \langle AP_r e_k, Ae_k\rangle \\
&\quad - \sum_k P_r e_k \langle Ae_k, \partial X/\partial t\rangle + \sum_k \langle \nabla_{P_r e_k} Ae_k, (\partial X/\partial t)^\top\rangle \\
&= c\operatorname{tr}(P_r)f + \sum_k \big(P_r e_k e_k(f) - P_r e_k \langle \overline{\nabla}_{e_k} N, \partial X/\partial t\rangle \big) \\
&\quad + \operatorname{tr}(A^2 P_r)f - \sum_k P_r e_k \langle Ae_k, \partial X/\partial t\rangle + \langle \sum_k \nabla_{P_r e_k} Ae_k, (\partial X/\partial t)^\top\rangle\,.
\end{aligned}$$

Finally, by applying Codazzi's equation, we arrive at

$$\begin{aligned}
\sum_k \nabla_{P_r e_k} Ae_k &= \sum_k (\nabla_{e_k} AP_r e_k + A[P_r e_k, e_k]) \\
&= \sum_k \nabla_{e_k}(S_{r+1}I - P_{r+1})e_k + \sum_k A\,(\nabla_{P_r e_k} e_k - \nabla_{e_k} P_r e_k) \\
&= \sum_k e_k(S_{r+1})e_k + \sum_k \nabla_{e_k} P_{r+1} e_k - \sum_k A\nabla_{e_k} P_r e_k \\
&= \nabla S_{r+1} + \operatorname{div}(P_{r+1}) - A(\operatorname{div} P_r) \\
&= \nabla S_{r+1}\,,
\end{aligned}$$

since P_r is divergence-free. Hence,

$$S'_{r+1} = c\operatorname{tr}(P_r)f + L_r f + \operatorname{tr}(A^2 P_r)f + \langle \nabla S_{r+1}, (\partial X/\partial t)^\top\rangle\,. \qquad \square$$

The previous lemma allows us to compute the first variation of the r-area functional.

Proposition 2.3. *Under the hypotheses of Lemma 2.2, if X is a variation of x, then*

$$\mathcal{A}_r'(t) = -(r+1)\int_M S_{r+1} f \, dM_t \,. \tag{2.8}$$

Proof. We make an inductive argument. The case $r=0$ is well known, and to the case $r=1$ we use the classical formula

$$\frac{\partial}{\partial_t} dM_t = [-S_1 f + \mathrm{div}(\partial X/\partial t)^\top] dM_t$$

to get

$$\begin{aligned}
\mathcal{A}_1' &= \int_M F_1' dM_t + \int_M F_1 \frac{\partial}{\partial_t} dM_t \\
&= \int_M S_1' dM_t + \int_M S_1[-S_1 f + \mathrm{div}(\partial X/\partial t)^\top] dM_t \\
&= \int_M [ncf + \Delta f + (S_1^2 - 2S_2)f + \langle (\partial X/\partial t)^\top, \nabla S_1\rangle - S_1^2 f + S_1 \mathrm{div}(\partial X/\partial t)^\top] dM_t \\
&= nc\int_M f dM_t - 2\int_M S_2 f dM_t + \int_M \langle (\partial X/\partial t)^\top, \nabla S_1\rangle dM_t \\
&\quad + \int_M \mathrm{div}\left(S_1 (\partial X/\partial t)^\top\right) dM_t - \int_M \langle (\partial X/\partial t)^\top, \nabla S_1\rangle dM_t \\
&= -2\int_M S_2 f dM_t \,,
\end{aligned}$$

where in the last equality we used that M is closed and X is volume-preserving. For $r \geq 2$, the induction hypothesis and (2.7) give

$$\begin{aligned}
\mathcal{A}_r' &= \int_M F_r' dM_t + \int_M F_r \frac{\partial}{\partial_t} dM_t \\
&= \int_M \left[S_r' + \frac{c(n-r+1)}{r-1} F_{r-2}' \right] dM_t \\
&\quad + \int_M \left(S_r + \frac{c(n-r+1)}{r-1} F_{r-2} \right) \left(-S_1 f + \mathrm{div}(\partial X/\partial t)^\top \right) dM_t \\
&= \int_M S_r' dM_t - \int_M S_1 S_r f dM_t + \int_M S_r \mathrm{div}\,(\partial X/\partial t)^T dM_t \\
&\quad + \frac{c(n-r+1)}{r-1} \left\{ \int_M F_{r-2}' dM_t + \int_M F_{r-2} \frac{\partial}{\partial_t} dM_t \right\} \\
&= c\int_M \mathrm{tr}(P_{r-1}) f dM_t + \int_M L_{r-1} dM_t + \int_M \mathrm{tr}(A^2 P_{r-1}) f dM_t \\
&\quad + \int_M \langle \nabla S_r, (\partial X/\partial t)^T\rangle dM_t - \int_M S_1 S_r f dM_t + \int_M \mathrm{div}\left(S_r \, (\partial X/\partial t)^T \right) dM_t \\
&\quad - \int_M \langle \nabla S_r, (\partial X/\partial t)^T\rangle dM_t + \frac{c(n-r+1)}{r-1} A_{r-2}'
\end{aligned}$$

$$\begin{aligned}
&= c(n-r+1)\int_M S_{r-1} f dM_t + \int_M (S_1S_r - (r+1)S_{r+1})\, f dM_t \\
&\quad - \int_M S_1S_r f dM_t - \frac{c(n-r+1)}{r-1}\int_M (r-1)S_{r-1} f dM_t \\
&= -(r+1)\int_M S_{r+1} f dM_t\,. \qquad \square
\end{aligned}$$

Remark 2.4. We want to point out that Lemma 2.2 and the first variation formula (Proposition 2.3) were first proved by R. Reilly in [10].

In order to characterize hypersurfaces of constant $(r+1)-$th mean curvature, let λ be a real constant and $\mathcal{J}_r\colon(-\epsilon,\epsilon)\to\mathbb{R}$ be the *Jacobi functional* associated to the variation X, i.e.,

$$\mathcal{J}_r(t) = \mathcal{A}_r(t) + \lambda\mathcal{V}(t)\,.$$

As an immediate consequence of (2.8) we get

$$\mathcal{J}_r'(t) = \int_M [-b_rH_{r+1} + \lambda] f dM_t\,,$$

where $b_r = (r+1)\binom{n}{r+1}$. Therefore, if we choose $\lambda = b_r\overline{H}_{r+1}(0)$, where

$$\overline{H}_{r+1}(0) = \frac{1}{\mathcal{A}_0(0)}\int_M H_{r+1}(0)dM$$

is the mean of the $(r+1)$-th curvature $H_{r+1}(0)$ of M, we arrive at

$$\mathcal{J}_r'(t) = b_r\int_M [-H_{r+1} + \overline{H}_{r+1}(0)] f dM_t\,.$$

Hence, a standard argument (cf. [3]) shows that M^n is a critical point of $\mathcal{J}_r$ for all variations of x if and only if M^n has constant $(r+1)$-th mean curvature.

We wish to study spacelike immersions $x\colon M^n\to\overline{M}_c^{n+1}$ that minimize $\mathcal{A}_r$ for all volume-preserving variations X of x. The above discussion shows that M^n must have constant $(r+1)$-th mean curvature and, for such an M^n, leads us naturally to compute the second variation of $\mathcal{A}_r$. This motivates the following

Definition 2.5. Let $\overline{M}_c^{n+1}$ be an orientable simply connected Riemannian manifold with constant curvature c, and $x\colon M^n\to\overline{M}^{n+1}$ be a closed hypersurface having constant $(r+1)$-th mean curvature. We say that x is r-stable if $\mathcal{A}_r''(0)\geq 0$, for all volume-preserving variation of x.

Let $x\colon M^n\to\overline{M}_c^{n+1}$ be a closed hypersurface with constant $(r+1)$-th mean curvature and denote by $\mathcal{G}$ the set of differentiable functions $f\colon M^n\to\mathbb{R}$ with $\int_M f dM_t = 0$. Just as [2] we can establish the following criterion of stability: *x is r-stable if and only if $\mathcal{J}_r''(0)\geq 0$, for all $f\in\mathcal{G}$.*

The sought formula for the second variation of $\mathcal{J}_r$ is another straightforward consequence of Proposition 2.3.

Proposition 2.6. *Let* $x\colon M^n \to \overline{M}_c^{n+1}$ *be a closed hypersurface of orientable simply connected Riemannian manifold* $\overline{M}_c^{n+1}$, *having constant* $(r+1)$*-mean curvature* H_{r+1}. *If* $X\colon M^n \times (-\epsilon,\epsilon) \to \overline{M}_c^{n+1}$ *is a variation of* x, *then* $J_r''(0)$ *is given by*

$$\mathcal{J}_r''(0)(f) = -(r+1)\int_M \left[L_r(f) + \{c\operatorname{tr}(P_r) + \operatorname{tr}(A^2P_r)\}f\right] f dM\,. \tag{2.9}$$

3. r-Stable Hypersurfaces in Space Forms

As in the previous section, let $\overline{M}_c^{n+1}$ orientable simply connected Riemannian manifold with constant curvature c. A vector field V on $\overline{M}_c^{n+1}$ is said to be *conformal* if

$$\mathcal{L}_V\langle\ ,\ \rangle = 2\psi\langle\ ,\ \rangle \tag{3.1}$$

for some function $\psi \in C^\infty(\overline{M})$, where $\mathcal{L}$ stands for the Lie derivative of the Riemann metric of $\overline{M}$. The function ψ is called the *conformal factor* of V.

Since $\mathcal{L}_V(X) = [V,X]$ for all $X \in \mathfrak{X}(\overline{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathfrak{X}(\overline{M})$ is conformal if and only if

$$\langle\overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\psi\langle X,Y\rangle\,, \tag{3.2}$$

for all $X,Y \in \mathfrak{X}(\overline{M})$. In particular, V is a Killing vector field relatively to $\overline{g}$ if and only if $\psi \equiv 0$.

The following result was shown in [5].

Lemma 3.1. *Let* $\overline{M}_c^{n+1}$ *be an orientable simply connected Riemannian manifold with constant curvature* c *and endowed with a conformal vector field* V. *Let also* $x\colon M^n \to \overline{M}_c^{n+1}$ *be a hypersurface of* $\overline{M}_c^{n+1}$ *and* N *be a Gauss map on* M^n. *If* $\eta = \langle V, N\rangle$, *then*

$$\begin{aligned} L_r(\eta) &= \{-\operatorname{tr}(A^2P_r) - c\operatorname{tr}(P_r)\}\eta - (n-r)\binom{n}{r}H_r N(\psi) \\ &\quad - (r+1)\binom{n}{r+1}H_{r+1}\psi - \binom{n}{r+1}\langle V, \nabla H_{r+1}\rangle\,, \end{aligned} \tag{3.3}$$

where $\psi\colon \overline{M}^{n+1} \to \mathbb{R}$ *is the conformal factor of* V, H_j *is the* j*-th mean curvature of* M^n *and* ∇H_j *stands for the gradient of* H_j *on* M^n.

In particular, we obtain the following

Corollary 3.2. *Let* $\overline{M}_c^{n+1}$ *be an orientable simply connected Riemannian manifold with constant curvature* c *and endowed with a Killing vector field* W. *Let also* $x\colon M^n \to \overline{M}_c^{n+1}$ *be a hypersurface having constant* $(r+1)$*-th mean curvature* H_{r+1}, N *be a Gauss map on* M^n *and* $\eta = \langle W, N\rangle$, *then*

$$L_r(\eta) + \{c\operatorname{tr}(P_r) + \operatorname{tr}(A^2P_r)\}\eta = 0\,.$$

In particular, if $x\colon M^n \to \overline{M}_c^{n+1}$ *is a closed spacelike hypersurface with constant* $(r+1)$*-th mean curvature such that* $\lambda = c\operatorname{tr}(P_r) + \operatorname{tr}(A^2P_r)$ *is constant, then* λ *is an eigenvalue of the operator* L_r *in* M^n *with eigenfunction* η.

Although the existence of nontrivial Killing vector fields on Riemannian space forms is a standard fact, we will present an alternative proof of this result by using the ideas of [8].

Lemma 3.3. *There exist nontrivial Killing vector fields on Riemannian space forms.*

Proof. Given any fixed two linearly independent vectors u and v in the the Euclidean space $\mathbb{R}^{n+1}$ and considering the vector field $W = \langle u, \cdot\rangle v - \langle v, \cdot\rangle u$. Observe that $\langle W(p), p\rangle = 0$, that is, geometrically, $W(p)$ determines an orthogonal direction to the position vector p on the subspace spanned by u and v. Moreover, we easily verify that

$$\langle \overline{\nabla}_X W, Y\rangle + \langle X, \overline{\nabla}_Y W\rangle = 0\,,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^{n+1})$. Therefore, W is a Killing vetor field globally defined on the $\mathbb{R}^{n+1}$. Since W is orthogonal to the position vector field, then W is also a Killing vector field on the sphere

$$\mathbb{S}^n = \left\{p \in \mathbb{R}^{n+1} \,:\, \langle p, p\rangle = 1\right\}\,.$$

Similarly, fixed two linearly independent vectors u and v in the the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ and considering the vector field $W = \langle u, \cdot\rangle v - \langle v, \cdot\rangle u$ we get a Killing vector field W globally defined in $\mathbb{L}^{n+1}$ that is orthogonal to the position vector field. Then, W is also a Killing vector field on the hyperbolic space

$$\mathbb{H}^n = \left\{p \in \mathbb{L}^{n+1} \,:\, \langle p, p\rangle = -1\right\}\,. \qquad \square$$

The following result can be found in [2].

Lemma 3.4. *Let $x\colon M^n \to \overline{M}_c^{n+1}$ be a closed orientable hypersurface, where $\overline{M}_c^{n+1}$ represent the Euclidean space $\mathbb{R}^{n+1}(c = 0)$, an open hemisphere of the sphere $\mathbb{S}^{n+1}(c = 1)$ or the hyperbolic space $\mathbb{H}^{n+1}(c = -1)$. If H_r is positive then, for $1 \le j \le r$, L_j is elliptic.*

We can now state and prove our main result.

Theorem 3.5. *Let $\overline{M}_c^{n+1}$ be either the Euclidean space $\mathbb{R}^{n+1}(c = 0)$, an open hemisphere of the sphere $\mathbb{S}^{n+1}(c = 1)$, or the hyperbolic space $\mathbb{H}^{n+1}(c = -1)$. Let r be an integer satisfying the inequality $0 \le r \le n-2$, and $x\colon M^n \to \overline{M}_c^{n+1}$ be a closed hypersurface with positive constant $(r+1)$-th mean curvature H_{r+1}. Suppose that*

$$\lambda = c(n-r)\binom{n}{r}H_r + nH_1\binom{n}{r+1}H_{r+1} - (r+2)\binom{n}{r+2}H_{r+2}$$

is constant. Then x is r-stable if and only if λ is the first eigenvalue of L_r on M^n.

Proof. First we observe that Lemma 3.3 guarantees the existence of a nontrivial Killing vector field W on $\overline{M}_c^{n+1}$. On the other hand, from Lemma 3.4, the operator L_r is elliptic on M.

By using the formulas (2.4), it is easy to show that $\lambda = c\,\mathrm{tr}(P_r) + \mathrm{tr}(A^2P_r)$. Therefore, since that λ is constant and W is a Killing field on $\overline{M}_c^{n+1}$, Corollary 3.2 guarantees that λ is in the spectrum of L_r.

Let λ_1 be the first eigenvalue of L_r on M^n. If $\lambda = \lambda_1$, then

$$\lambda = \min_{f \in \mathcal{G} \setminus \{0\}} \frac{-\int_M f L_r(f) dM}{\int_M f^2 dM}\,.$$

It follows that, for any $f \in \mathcal{G}$,

$$\mathcal{J}_r''(0)(f) = (r+1)\int_M \{-fL_r(f) - \lambda f^2\} dM \geq (r+1)(\lambda - \lambda)\int_M f^2 dM = 0\,,$$

and hence x is r-stable.

Now suppose that x is $r-$stable. Then $\mathcal{J}_r''(0)(f) \geq 0$, $\forall f \in \mathcal{G}$. Let us consider f the eigenfunction associated to the first eigenvalue λ_1 of L_r. As it was already observed, there exists a volume-preserving variation of M whose variational field is fN. Consequently, by (2.9), we get

$$0 \leq \mathcal{J}_r''(0)(f) = (r+1)(\lambda_1 - \lambda)\int_M f^2 dM.$$

Therefore, since $\lambda_1 \leq \lambda$, we conclude that $\lambda_1 = \lambda$. □

Since $L_0 = \Delta$ is always elliptic and taking into account formula (2.2), we obtain the following

Corollary 3.6 ([4], Proposition 2.13)**.** *Let $\overline{M}_c^{n+1}$ be the Euclidean space $\mathbb{R}^{n+1}(c = 0)$, the sphere $\mathbb{S}^{n+1}(c = 1)$, or the hyperbolic space $\mathbb{H}^{n+1}(c = -1)$. Let $x : M^n \to \overline{M}_c^{n+1}$ be a closed hypersurface with constant mean curvature. Suppose that*

$$\lambda = cn + |A|^2$$

is constant, where $|A|^2$ denotes the squared norm of the shape operator. Then x is stable if and only if λ is the first eigenvalue of Δ on M^n.

Finally, since L_1 is just the Yau's square operator (cf. [7]), by using equation (2.1) we get the following

Corollary 3.7. *Let $\overline{M}_c^{n+1}$ be a Euclidean space $\mathbb{R}^{n+1}(c = 0)$, the sphere $\mathbb{S}^{n+1}(c = 1)$, or the hyperbolic space $\mathbb{H}^{n+1}(c = -1)$. Let $x : M^n \to \overline{M}_c^{n+1}$ be a closed hypersurface, with constant normalized scalar curvature $R > c$. If*

$$\lambda = cn(n-1)H_1 + \frac{n^2(n-1)}{2}(R - c)H_1 - \frac{n(n-1)(n-2)}{2}H_3$$

is constant, then x is 1-stable if and only if λ is the first eigenvalue of L_1 on M^n.

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