ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 217–228

ON THE OSCILLATION OF SOME IMPULSIVE PARABOLIC EQUATIONS WITH SEVERAL DELAYS

R. Atmania and S. Mazouzi

Abstract. In this paper, several oscillation criteria are established for some nonlinear impulsive functional parabolic equations with several delays subject to boundary conditions. We shall mainly use the divergence theorem and some corresponding impulsive delayed differential inequalities.

1. Introduction

In fact, several real world phenomena, especially in biological or medical domain, population dynamics, ecology, industrial robotics and other domains are characterized by short-term perturbations in the form of impulses because the duration of the perturbation is very short compared with the evolution duration for the phenomenon itself. A suitable mathematical simulation of some phenomena characterized by contiguous time intervals is the impulsive partial differential equations setting and to know more about this kind of partial differential equations, we refer the reader to [1]. In the last few years the theory of impulsive partial differential equations has been investigated by many authors. We notice that some of those studies were devoted to the oscillation character of the solutions of problems with or without delay, see for instance [2, 3, 4, 5, 6]. Regarding the delayed partial differential equations the solution depends not only on its present state but on its history as well.

Here, we are concerned by impulsive functional boundary value parabolic problems with a finite number of delays. We consider the following delayed functional parabolic equation

$$
\begin{aligned}
(1)\qquad & \frac{\partial}{\partial t}\Big(u(t,x)-\sum_{j=1}^{l} b_j(t)u(\rho_j(t),x)\Big) \\
&\quad -\sum_{i=1}^{n} a_i(t)\frac{\partial^2 u}{\partial x_i^2}(\tau(t),x)+\sum_{r=1}^{m} g_r(t,x)h_r\big(u(t-\eta_r,x)\big) \\
&= f(t,x)\,,\quad (t,x)\in\mathbb{R}_+\times\Omega\,,\qquad t\neq t_k\,,\quad k=1,2,\dots,
\end{aligned}
$$

2010 *Mathematics Subject Classification*: primary 35B05; secondary 35K61, 35R12.
Key words and phrases: impulsive condition, delayed parabolic equation, oscillation, divergence theorem, impulsive differential inequality.
Received July 11, 2010, revised March 2011. Editor M. Feistauer.

subject to the impulsive conditions

$$u(t_k^+,x)-u(t_k^-,x)=I\big(t_k,x,u(t_k,x)\big)\,,\qquad k=1,2\dots,\quad x\in\overline{\Omega}\,, \tag{2}$$

with the boundary condition

$$u(t,x)=\psi(t,x),\quad (t,x)\in\mathbb{R}_+\times\partial\Omega\,,\quad t\neq t_k\,,\quad k=1,2,\dots\,. \tag{3}$$

Inspired by the results of Fu et al. [4] where they considered an impulsive parabolic system with delay where $b_j(t)=0$, $j=1,\dots,l$, $\tau(t)=t$, $m=1$, $f(t,x)=0$, we intend to extend some of those results and obtain some practical oscillation criteria for the problem (1)–(3) subject to one of the two kinds of boundary conditions used in [4] by using the divergence theorem and some appropriate impulsive delayed differential inequalities based on the property of the first positive eigenvalue and the corresponding positive eigenfunction of a Dirichlet problem. Actually, the techniques that we are going to use in the sequel are applied by some authors to obtain other oscillation criteria. For example, Bainov and Minchev [1] treated such a problem under the assumptions $b_j(t)=0$, $j=1,\dots,l$, $\tau(t)=t$, $a_i(t)=a(t)$, $i=1,\dots,n$ and $m=1$, with two kinds of boundary conditions while Cui et al. [3] investigated a similar problem under two kinds of boundary conditions with $b_j(t)=0$, $j=1,\dots,l$, the diffusion term has somehow a different form and $f(t,x,u)=-p(t,x)u(t,x)$. The results are obtained under different conditions on the diffusion term, the delay arguments as well as on the impulses effect.

2. Preliminaries

In this section, we shall introduce the notations and the basic definitions which will be used throughout the paper. We set the following assumptions:

(H1) Ω is a bounded domain with smooth boundary $\partial\Omega$, $t\in\mathbb{R}_+=[0,+\infty)$ and the impulses times are such that $0<t_1<t_2\cdots<t_k\dots$; $\lim_{k\to\infty}t_k=+\infty$,

(H2) $a_i\in C(\mathbb{R}_+;\mathbb{R}_+)$ with $a_i(t)\geq a_0$, $i=1,2,\dots,n$, for some positive constant a_0; $\tau\in\mathcal{PC}(\mathbb{R}_+;\mathbb{R})$ with $\tau(t)\leq t$ and $\lim_{t\to\infty}\tau(t)=+\infty$; $b_j\in\mathcal{PC}^1(\mathbb{R}_+;\mathbb{R}_+)$, $\rho_j\in\mathcal{PC}^1(\mathbb{R}_+;\mathbb{R})$ with $\rho_j(t)\leq t$ and $\lim_{t\to\infty}\rho_j(t)=+\infty$, $j=1,\dots,l$;

(H3) $g_r\in\mathcal{PC}(\mathbb{R}_+\times\Omega;\mathbb{R}_+)$; $h_r\in\mathcal{PC}(\mathbb{R},\mathbb{R})$, η_r are positive constants $r=1,\dots,m$ with $\max\limits_{1\leq r\leq m}\eta_r=\eta$ and $I\colon\mathbb{R}_+\times\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$.

We add

(A1) $f\in\mathcal{PC}(\mathbb{R}_+\times\Omega;\mathbb{R})$; $\psi\in\mathcal{PC}(\mathbb{R}_+\times\partial\Omega;\mathbb{R})$.

We recall that $\mathcal{PC}$ is the space of piecewise continuous functions in t with first kind discontinuities at $t=t_k$, for $k=1,2,\dots$ and left continuous at $t=t_k$, for $k=1,2,\dots$.

$u(t_k^+,x)$ and $u(t_k^-,x)$ are respectively the right and left limits at $t=t_k$, for each $x\in\overline{\Omega}$.

To accommodate the delays $\rho_j(t)$, $j=1,\dots,l$; $\tau(t)$, η_r, $r=1,\dots,m$; the function $u(t,x)$ is defined and given for $(t,x)\in[\mu,0]\times\overline{\Omega}$ where $\mu=\min_{1\leq j\leq l}\big(\inf_{t\geq0}\rho_j(t)\big)$, $\inf_{t\geq0}\tau(t),-\eta)$ and $u(t,x)$ is continuous differentiable with respect to $t\in[\mu,0]$ for $x\in\Omega$ and twice continuously differentiable with respect to $x\in\Omega$ for $t\in[\mu,0]$ i.e. $u(t,x)\in C^{1,2}\left([\mu,0]\times\Omega,\mathbb{R}\right)$.

Definition 1. A solution to the problem (1)–(3) is a function $u(t,x) : [\mu,+\infty) \times \overline{\Omega} \to \mathbb{R}$ such that

1) $u(t,x)$ is given for $(t,x) \in [\mu,0] \times \overline{\Omega}$ and $u(t,x) \in C^{1,2}([\mu,0] \times \Omega, \mathbb{R})$;
2) $\dfrac{\partial u}{\partial t}, \dfrac{\partial^2 u}{\partial x_i \partial x_j}$; $i,j = 1,\dots,n$ exist, are continuous and $u(t,x)$ satisfies (1) for $(t,x) \in \left(\mathbb{R}_+ \setminus \{t_k\}_{k\geq 1}\right) \times \Omega$;
3) $u(t_k^+,x)$ and $u(t_k^-,x)$ exist such that $u(t_k^-,x) = u(t_k,x)$ and $u(t,x)$ satisfies (2) for $(t,x) \in \{t_k\}_{k\geq 1} \times \overline{\Omega}$ and (3)for $(t,x) \in (\mathbb{R}_+ \setminus \{t_k\}_{k\geq 1}) \times \partial\Omega$.

Such a solution is said to be nonoscillatory on $\mathbb{R}_+ \times \Omega$ if there exists a number $\sigma \geq 0$ for which $u(t,x)$ has a constant sign for $(t,x) \in [\sigma,+\infty[\times \Omega$; otherwise, it is said to be oscillatory.

(Note that the two notations $[a,+\infty[$ and $[a,+\infty)$ are used in this paper to give the same meaning.)

Definition 2. We mean by a positive (resp. negative) solution to the problem (1)–(3) in some domain $[\sigma,+\infty[\times\Omega$, $\sigma > 0$ that $u(t,x) > 0$ (resp. < 0), $u(\tau(t),x) > 0$ (resp. < 0), $u(\rho_j(t),x) > 0$ (resp. < 0), $j = 1,\dots,l$ and $u(t-\eta_r,x) > 0$ (resp. < 0), $r = 1,\dots,m$ for $(t,x) \in [\sigma,+\infty[\times \Omega$.

Remark 1. If there exists some $\sigma_1 \geq 0$ such that $u(t,x) > 0$ (resp. < 0), $t \geq \sigma_1$, then, there exist some positive constants $\sigma_2, \sigma_3 = \max\limits_{1\leq j\leq l} \sigma_3^j$ such that

$$\begin{aligned}
u\big(\tau(t),x\big) > 0 \quad &(\text{resp. } < 0) \quad \text{for } t \geq \sigma_2 \text{ such that } \tau(t) \geq \sigma_1 \\
u\big(\rho_j(t),x\big) > 0 \quad &(\text{resp. } < 0) \quad \text{for } t \geq \sigma_3^j \text{ such that } \rho_j(t) \geq \sigma_1\,,\ j = 1,\dots,l\,; \\
u(t-\eta_r,x) > 0 \quad &(\text{resp. } < 0) \quad \text{for } t \geq \sigma_1 + \eta \text{ such that } t-\eta_r \geq \sigma_1\,,\ r = 1,\dots,m\,.
\end{aligned}$$

So, $u(t,x)$ is positive (resp. negative) solution to the problem (1)–(3) in $[\sigma,+\infty[\times\Omega$ for $\sigma = \max(\sigma_1+\eta, \sigma_2, \sigma_3)$.

We have to use the following Lemma.

Lemma 1. *If there is a constant $a_0 > 0$ such that $a_i(t) \geq a_0$, $i = 1,2,\dots,n$, then there is a first positive eigenvalue λ_1 with corresponding positive eigenfunction $\Phi(x)$ for the problem*

$$\text{(4)} \qquad \sum_{i=1}^{n} a_i(t) \frac{\partial^2 \Phi(x)}{\partial x_i^2} + \lambda \Phi(x) = 0\,, \quad \textit{in} \quad \Omega\,,$$
$$\Phi(x) = 0\,, \quad \textit{on} \quad \partial\Omega\,.$$

Remark 2. λ_1 satisfies the inequality $\lambda_1 \geq a_0 \lambda_0$, where λ_0 is the first positive eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta\omega(x) = \lambda\omega(x)\,, & x \in \Omega \\ \omega(x) = 0\,, & x \in \partial\Omega\,. \end{cases}$$

We shall use the following notations

$$
\begin{aligned}
(5) \qquad U(t) &= K_\Phi^{-1} \int_\Omega u(t,x)\, \Phi(x)\, dx\,, \\
K_\Phi &= \int_\Omega \Phi(x)\, dx\,, \quad G(t) = \min_{1\le r\le m} \big(\inf_{x\in\Omega} g_r(t,x) \big)\,, \\
F(t) &= K_\Phi^{-1} \int_\Omega f(t,x)\Phi(x)\, dx\,, \\
\Psi\big(\tau(t)\big) &= K_\Phi^{-1} \int_{\partial\Omega} \psi\big(\tau(t),x\big) \nabla\Phi(x)\cdot A(t)\pi(x)\, dS\,,
\end{aligned}
$$

where $A(t) = (\alpha_{ij}(t))_{1\le i,j\le n}\colon \alpha_{ii}(t) = a_i(t)$ and $\alpha_{ij}(t) = 0$, if $i \neq j$, dS is a surface measure on $\partial\Omega$, ∇ is the divergence operator, $\pi(x) = (\pi_i(x))_{1\le i\le n} = (\cos(N,x_i))_{1\le i\le n}$, N is the unit outer normal vector to $\partial\Omega$, $u(t,x)$ is the solution to problem (1)–(3) and $\Phi(x)$ is the eigenfunction defined in problem (4).

3. Main results

First we shall establish some correspondence results between the impulsive parabolic boundary value problem (1)–(3) and some impulsive differential inequalities.

Theorem 1. *Besides assumptions* (H1)–(H3) *and* (A1) *assume that*

(H4) *h_r, $r = 1,\dots,m$ are positive and convex functions on $\mathbb{R}_+$.*

(H5) *There are positive constants α_k, $k = 1,2,\dots$ such that for any function $v \in \mathcal{PC}(\mathbb{R}_+ \times \overline{\Omega}; \mathbb{R}_+)$, we have*

$$
\int_\Omega I(t_k, x, v(t_k,x)t)\, dx \le \alpha_k \int_\Omega v(t_k,x)\, dx\,; \quad k = 1,2,\dots
$$

If $u(t,x)$ is a positive solution to problem (1)–(3) *in some domain $[\sigma,+\infty[\, \times \Omega$; $\sigma > 0$, then $U(t)$ defined by* (5) *is a positive solution in $[\sigma,+\infty[$ to the corresponding impulsive delayed differential inequality*

$$
(6) \qquad \begin{cases}
\dfrac{d}{dt}\Big(U(t) - \sum\limits_{j=1}^{l} b_j(t) U\big(\rho_j(t)\big)\Big) + \lambda_1 U\left(\tau(t)\right) + G(t) \sum\limits_{r=1}^{m} h_r\big(U(t-\eta_r)\big) \\
\qquad\qquad \le F(t) - \Psi\big(\tau(t)\big)\,, \quad t \neq t_k;\ t \ge \sigma \\
U(t_k^+) \le (1+\alpha_k) U(t_k)\,, \qquad k = 1,2,\dots
\end{cases}
$$

Proof. Let $u(t,x)$ be a positive solution satisfying problem (1)–(3) in $[\sigma,+\infty[\, \times \Omega$. For every $t \neq t_k$, we obtain from (1), after multiplication by $\Phi(x)$ and K_Φ^{-1}, and integration over Ω,

$$\frac{\partial}{\partial t}\Big(K_\Phi^{-1}\int_\Omega\Big(u(t,x)-\sum_{j=1}^{l}b_j(t)u\big(\rho_j(t),x\big)\Big)\Phi(x)\,dx\Big)$$
$$-K_\Phi^{-1}\sum_{i=1}^{n}a_i(t)\int_\Omega\frac{\partial^2 u}{\partial x_i^2}\big(\tau(t),x\big)\Phi(x)\,dx$$
$$+K_\Phi^{-1}\sum_{r=1}^{m}\int_\Omega g_r(t,x)h_r\big(u(t-\eta_r,x)\big)\Phi(x)\,dx$$
$$=K_\Phi^{-1}\int_\Omega f(t,x)\Phi(x)\,dx\,,\qquad t\geq\sigma\,. \tag{7}$$

We infer from the given assumptions and Jensen's inequality the following

$$K_\Phi^{-1}\sum_{r=1}^{m}\int_\Omega g_r(t,x)h_r\big(u(t-\eta_r,x)\big)\Phi(x)\,dx$$
$$\geq G(t)\sum_{r=1}^{m}h_r\Big(K_\Phi^{-1}\int_\Omega u(t-\eta_r,x)\Phi(x)\,dx\Big)$$
$$\geq G(t)\sum_{r=1}^{m}h_r\big(U(t-\eta_r)\big)\,;\qquad t\geq\sigma\,. \tag{8}$$

Next, by Lemma 1 and divergence theorem we obtain

$$K_\Phi^{-1}\sum_{i=1}^{n}a_i(t)\int_\Omega\frac{\partial^2 u}{\partial x_i^2}\big(\tau(t),x\big)\Phi(x)\,dx=\;K_\Phi^{-1}\sum_{i=1}^{n}a_i(t)\Big(\int_\Omega u\big(\tau(t),x\big)\frac{\partial^2\Phi(x)}{\partial x_i^2}\,dx$$
$$-\int_{\partial\Omega}u\big(\tau(t),x\big)\frac{\partial\Phi(x)}{\partial x_i}\pi_i(x)\,dS\Big)=-\lambda_1K_\Phi^{-1}\int_\Omega u\big(\tau(t),x\big)\Phi(x)\,dx$$
$$-K_\Phi^{-1}\int_{\partial\Omega}\psi\big(\tau(t),x\big)\nabla\Phi(x)\cdot A(t)\pi(x)\,dS=-\lambda_1U\big(\tau(t)\big)-\Psi\big(\tau(t)\big)\,;\quad t\geq\sigma\,. \tag{9}$$

So, by using (8) and (9), in (7)we get for $t\neq t_k$

$$\frac{\partial}{\partial t}\Big(U(t)-\sum_{j=1}^{l}b_j(t)U\big(\rho_j(t)\big)\Big)+\lambda_1U\big(\tau(t)\big)+G(t)\sum_{i=r}^{m}h_r\big(U(t-\eta_r)\big)$$
$$\leq F(t)-\Psi\big(\tau(t)\big)\,,\qquad t\geq\sigma\,. \tag{10}$$

For every $t=t_k$, thanks to assumption (H5), we have

$$K_\Phi^{-1}\int_\Omega\big(u(t_k^+,x)-u(t_k^-,x)\big)\Phi(x)\,dx$$
$$=K_\Phi^{-1}\int_\Omega I(t_k,x,u)\Phi(x)\,dx\leq K_\Phi^{-1}\alpha_k\int_\Omega u(t_k,x)\Phi(x)\,dx\,,$$

so that

$$K_\Phi^{-1}\int_\Omega u(t_k^+,x)\Phi(x)\,dx \le (\alpha_k+1)K_\Phi^{-1}\int_\Omega u(t_k,x)\Phi(x)\,dx\,.$$

Thus

$$U(t_k^+) \le (\alpha_k+1)U(t_k)\,, \qquad k=1,2,\dots\,. \tag{11}$$

We deduce immediately from (10), (11) and Definition 2 that $U(t)$ is a positive solution to the differential inequality (6) in $[\sigma,+\infty[$, which completes the proof. □

Theorem 2. *Suppose that hypotheses* (H1)–(H5), (A1) *hold and*

(H6) $h_r(-v)=-h_r(v)$, $r=1,\dots,m$ *for* $v\in\mathbb{R}_+$*;*

(H7) *for each* $v\in\mathcal{PC}(\mathbb{R}_+\times\overline{\Omega};\mathbb{R}_+)$ *and* $k=1,2,\dots,$ *we have*

$$I\,(t_k,x,-v)=-I(t_k,x,v)\,.$$

If $u\,(t,x)$ *is a negative solution to problem* (1)–(3) *in some domain* $[\sigma,+\infty[\times\Omega$*;* $\sigma\ge 0$*, then* $U(t)$ *defined by* (4) *is a negative solution to the corresponding impulsive delayed differential inequality* (6) *in* $[\sigma,+\infty[$.

Proof. Assume that $u(t,x)<0$ in $[\sigma,+\infty[\times\Omega$, then, by virtue of hypotheses (H6) and (H7), the function $v(t,x)=-u(t,x)$ is a positive solution to the following impulsive parabolic boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(v(t,x)-\sum\limits_{j=1}^{l} b_j(t)v\big(\rho_j(t),x\big)\Big)-\sum\limits_{i=1}^{n} a_i(t)\dfrac{\partial^2 v}{\partial x_i^2}\big(\tau(t),x\big) \\ \qquad +\sum\limits_{r=1}^{m} g_r(t,x)h_r\big(v(t-\eta_r,x)\big)=-f(t,x)\,, \qquad t\ne t_k\,;\quad x\in\Omega \\ v(t_k^+,x)-v(t_k^-,x)=I(t_k,x,v)\,, \qquad k=1,2,\dots\,;\quad x\in\overline{\Omega} \\ v(t,x)=-\psi(t,x)\,, \qquad (t,x)\in\mathbb{R}_+\times\partial\Omega;\quad t\ne t_k\,,\quad k=1,2,\dots \end{cases} \tag{12}$$

According to Theorem 1, we see that the function

$$V(t)=K_\Phi^{-1}\int_\Omega v(t,x)\Phi(x)\,dx$$

is a positive solution to the following impulsive delayed differential inequality

$$\begin{cases} \dfrac{d}{dt}\Big(V(t)-\sum\limits_{j=1}^{l} b_j(t)V\big(\rho_j(t)\big)\Big)+\lambda_1 V\big(\tau(t)\big)+G(t)\sum\limits_{r=1}^{m} h_r\big(V(t-\eta_r)\big) \\ \qquad \le -\big(F(t)-\Psi\big(\tau(t)\big)\big)\,,\quad t\ne t_k \\ V(t_k^+)\le(1+\alpha_k)V(t_k)\,,\quad k=1,2,\dots\,, \end{cases} \tag{13}$$

which implies that $U(t)=-V(t)$ is a negative solution of inequality (6) in $[\sigma,+\infty[$ and the proof is complete. □

It is obvious that $U(t)$ is piecewise continuous in t with discontinuities of first kind at $t = t_k$, for $k = 1, 2, \ldots$ and left continuous at $t = t_k$, for $k = 1, 2, \ldots$ i.e. $U\left(t_k^+\right)$ and $U\left(t_k^-\right)$ exist and $U(t_k) = U\left(t_k^-\right)$.

Now we are in position to state and prove our first oscillation criterion.

Theorem 3. *Under hypotheses* (H1)–(H7) *and* (A1)*, if* $U(t)$ *defined by* (4) *is an oscillatory solution to the impulsive delayed differential inequality* (6) *for* $t \in \mathbb{R}_+$*, then* $u(t,x)$ *is an oscillatory solution to problem* (1)–(3) *in* $\mathbb{R}_+ \times \Omega$*.*

Proof. It is easy to see that from Theorems 1 and 2 if $u(t,x)$ is nonoscillatory solution to (1)–(3) in some domain $[\sigma, +\infty[\times \Omega$; $\sigma > 0$ then $U(t)$ defined by (4) is a nonoscillatory solution to (6) in $[\sigma, +\infty[$. The proof is complete. □

In the following, we investigate a special case of the problem (1)–(3) which implies that under assumptions (H1)–(H7) Theorems 1 and 2 remain true. To do so we replace hypothesis (A1) with the following:

(A2) $f(t,x) = 0$, for $(t,x) \in \mathbb{R}_+ \times \Omega$; and $\psi(t,x) = 0$, for $(t,x) \in \mathbb{R}_+ \times \partial\Omega$, $t \neq t_k$, $k = 1, 2, \ldots$.

So, consider the delayed functional parabolic equation of the form

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(u(t,x) - \sum_{j=1}^{l} b_j(t) u\big(\rho_j(t), x\big)\Big) - \sum_{i=1}^{n} a_i(t) \frac{\partial^2 u}{\partial x_i^2}\big(\tau(t), x\big) \\ &\qquad + \sum_{r=1}^{m} g_r(t,x) h_r\big(u(t-\eta_r, x)\big) = 0\,, \\ &\qquad t \neq t_k\,, \quad k = 1, 2, \ldots\,, \quad (t,x) \in \mathbb{R}_+ \times \Omega \end{aligned} \tag{14}$$

subject to the impulsive condition (2), and the boundary condition

$$u(t,x) = 0\,, \quad t \neq t_k\,,\ k = 1, 2, \ldots\,,\ (t,x) \in \mathbb{R}_+ \times \partial\Omega\,. \tag{15}$$

The corresponding impulsive delayed differential inequality is

$$\begin{cases} \dfrac{d}{dt}\Big(U(t) - \sum\limits_{j=1}^{l} b_j(t) U\big(\rho_j(t)\big)\Big) + \lambda_1 U\big(\tau(t)\big) \\ \qquad + G(t) \sum\limits_{r=1}^{m} h_r\big(U(t-\eta_r)\big) \leq 0\,, \quad t \neq t_k\,,\ t > 0\,, \\ U(t_k^+) \leq (1+\alpha_k) U(t_k)\,, \quad k = 1, 2, \ldots \end{cases} \tag{16}$$

Next, we shall use the following lemma which gives approximately the number of impulses in some interval to obtain two oscillation criteria in Theorems 4 and 5 depending on the impulses effect being in the considered interval.

Lemma 2. *Let* ξ *be a positive constant. If there exists a positive constant* $\delta < \xi$ *such that* $t_{k+1} - t_k \geq \delta$, $k = 1, 2, \ldots$*, then there exists an integer number* $p \geq 1$ *such that the number of impulse moments in intervals of the form* $[t, t+\xi]$, $t > 0$*, is not greater than* p*.*

Remark 3. We may take $p \geq 1 + [\xi/\delta]$.

Theorem 4. *Assume that hypotheses* (H1)–(H7) *and* (A2) *hold. Assume further that*

(H8) $b_j(t_k) = 0$; $j = 1, \dots, l$; $k = 1, 2, \dots,$

(H9) *there exists a nondecreasing function* $\overline{h} \in \mathcal{PC}(\mathbb{R}, \mathbb{R}_+)$ *such that* $h_r(u) \geq \overline{h}(u)$ *and there exists a positive constant* K *such that* $\dfrac{\overline{h}(u)}{u} > K$, *for* $u > 0$.

If there exist two positive constants α *and* δ *such that* $0 < \alpha_k < \alpha$; $0 < \delta < \eta$ *and* $t_{k+1} - t_k \geq \delta$, $k = 1, 2, \dots,$ *for which we have*

$$\limsup_{k\to\infty} \int_{t_k}^{t_k+\eta} G(s)\, ds > \frac{(1+\alpha)^p}{mK},$$

then each non trivial solution to the problem (14)–(2)–(15) *is oscillatory in* $\mathbb{R}_+ \times \Omega$.

Proof. Suppose the contrary that $u(t, x)$ were a nonoscillatory solution to problem (14)–(2)–(15). If $u(t, x)$ is positive solution in $[\sigma, +\infty[\times\Omega$; $\sigma = \max(\sigma_1 + \eta, \sigma_2, \sigma_3) > 0$, then $U(t)$ is a positive solution to the differential inequality (16) for $t \geq \sigma$ such that $U(t) > 0$ for $t \geq \sigma_1$ therefore we have $U(\tau(t)) > 0$ for $t \geq \sigma_2$ such that $\tau(t) \geq \sigma_1$ and we have $U(\rho_j(t)) > 0$, for $t \geq \sigma_3$ such that $\rho_j(t) \geq \sigma_1$, $j = 1, \dots, l$. Moreover, we have $h(U(t - \eta_r)) > 0$, $r = 1, \dots, m$ for $t \geq \sigma_1 + \eta$.

Next, for every $t \neq t_k$, $t \geq \sigma$, we put

$$W(t) = U(t) - \sum_{j=1}^{l} b_j(t) U\big(\rho_j(t)\big). \tag{17}$$

It follows from the fact that $U(\rho_j(t))$ and $b_j(t)$, $j = 1, \dots, l$ are positive, then

$$U(t) \geq W(t), \quad \text{for} \quad t \geq \sigma. \tag{18}$$

We infer from (16) that

$$\frac{d}{dt} W(t) + \lambda_1 U\big(\tau(t)\big) + G(t) \sum_{r=1}^{m} h_r\big(U(t - \eta_r)\big) \leq 0, \quad \text{for} \quad t \geq \sigma, \tag{19}$$

implying that

$$\frac{d}{dt} W(t) \leq -G(t) \sum_{r=1}^{m} h_r\big(U(t - \eta_r)\big) - \lambda_1 U\big(\tau(t)\big) < 0, \quad \text{for} \quad t \geq \sigma. \tag{20}$$

We conclude that $W(t)$ is a nonincreasing function for $t \geq \sigma$, $t \neq t_k$, and thus

$$0 < G(t) \sum_{r=1}^{m} h_r\big(U(t - \eta_r)\big) < -\frac{d}{dt} W(t), \quad \text{for} \quad t \geq \sigma,\, t \neq t_k. \tag{21}$$

Integrating (21) over $[t, t_{k+1}[\subset]t_k, t_{k+1}[$, we obtain

$$0 < \int_t^{t_{k+1}} G(s) \sum_{r=1}^{m} h_r\big(U(s - \eta_r)\big)\, ds < -\int_t^{t_{k+1}} W'(s)\, ds = W(t) - W(t_{k+1})$$

giving $W(t) > W(t_{k+1})$. It follows from (H8) that $W(t_k) = U(t_k) \geq 0$, $k = 1, 2, \ldots$. Hence $W(t) \geq 0$ for $t \geq \sigma$ and accordingly, $\liminf_{t\to\infty} W(t) \geq 0$ which shows that $W(t)$ is positive and nonincreasing.

Next, using (H9) and the fact that $W(t)$ is positive and nonincreasing, we get from (19), for $t \neq t_k$, $t > \sigma$, the following

$$\begin{aligned}-\frac{d}{dt}W(t) &\geq \lambda_1 W\big(\tau(t)\big) + G(t)\sum_{r=1}^{m} \overline{h}\big(W(t-\eta_r)\big)\\ &\geq \lambda_1 W(t) + G(t)\sum_{r=1}^{m} KW(t-\eta_r) \geq \lambda_1 W(t) + mG(t)KW(t)\,.\end{aligned}$$

Therefore,

$$\frac{d}{dt}W(t) + \lambda_1 W(t) + mKG(t)W(t) \leq 0. \tag{22}$$

Multiplying (22) by $\exp(\lambda_1(t-T))$, $t > T > \sigma$ and setting $Z(t) = W(t)\exp(\lambda_1(t-T))$, $t > T$, we obtain

$$\frac{d}{dt}Z(t) + mKG(t)Z(t) \leq 0\,, \quad t \neq t_k\,. \tag{23}$$

It is easy to see that $Z(t)$ is a nonincreasing function.

For $t = t_k$, $W(t_k) = U(t_k)$; so, we have

$$\begin{aligned}Z\left(t_k^+\right) - Z\left(t_k^-\right) &= \left(W\left(t_k^+\right) - W\left(t_k\right)\right)\exp\left(\lambda_1(t_k - T)\right)\\ &\leq \alpha_k W\left(t_k\right)\exp\left(\lambda_1(t_k-T)\right) \leq \alpha_k Z\left(t_k\right)\,.\end{aligned} \tag{24}$$

Integrating (23) from t_k to $t_k + \eta$ we get

$$Z(t_k+\eta) - Z(t_k^+) - \sum_{i=k}^{k+p-1} \alpha_i Z(t_i) + mK\int_{t_k}^{t_k+\eta} Z(s)G(s)\,ds \leq 0\,.$$

Thus

$$mK\cdot\int_{t_k}^{t_k+\eta} Z(s)G(s)\,ds \leq Z(t_k^+) - Z(t_k+\eta) + \sum_{i=k}^{k+p-1}\alpha_i Z(t_i)$$

from which we get

$$\begin{aligned}mK\cdot Z(t_k)\int_{t_k}^{t_k+\eta} G(s)\,ds &\leq (1+\alpha_k)Z(t_k) + \sum_{i=k+1}^{k+p-1}\alpha_i Z(t_i)\\ &\leq (1+\alpha)Z(t_k) + \alpha\sum_{i=k+1}^{k+p-1} Z(t_i)\,.\end{aligned} \tag{25}$$

Now, since $Z(t)$ is nonincreasing and $0 < \alpha_k \leq \alpha$, $k = 1, 2, \ldots$, we have

$$Z(t_{k+1}) \leq Z(t_k^+) \leq (1+\alpha)Z(t_k);$$

$$Z(t_{k+2}) \leq Z(t_{k+1}^+) \leq (1+\alpha)Z(t_{k+1}) \leq (1+\alpha)^2 Z(t_k)$$

by induction we obtain that

$$Z(t_{k+i}) \le (1+\alpha)^i Z(t_k) \quad \text{for} \quad i = 1, \dots, p-1\,,$$

then

$$\sum_{i=k+1}^{k+p-1} Z(t_i) \le Z(t_k) \sum_{i=1}^{p-1} (1+\alpha)^i = Z(t_k)(1+\alpha)\frac{(1+\alpha)^{p-1}-1}{\alpha}\,.$$

Substituting in (25) we get

$$\begin{aligned} &mK \cdot Z(t_k) \int_{t_k}^{t_k+\eta} G(s)\,ds \\ &\qquad \le (1+\alpha)Z(t_k) + \alpha Z(t_k)(1+\alpha)\frac{(1+\alpha)^{p-1}-1}{\alpha} = Z(t_k)(1+\alpha)^p\,, \end{aligned}$$

implying that

$$\int_{t_k}^{t_k+\eta} G(s)\,ds \le \frac{(1+\alpha)^p}{mK}\,,$$

this is a contradiction. On the other hand, if $u(t,x) < 0$, then $v(t,x) = -u(t,x)$ is a positive solution of (14)–(2)–(15) and $V(t)$ is a positive solution to the inequality (16); so by analogous arguments we arrive at the same conclusion which completes the proof . □

Theorem 5. *We assume that* (H1)–(H9) *and* (A2) *are fulfilled.*

If there exists a positive constant $\delta > \eta$ such that $t_{k+1} - t_k \ge \delta$, $k = 1, 2, \dots$, and

$$\limsup_{k\to\infty} \frac{1}{(1+\alpha_k)} \int_{t_k}^{t_k+\eta} G(s)\,ds > \frac{1}{mK}\,,$$

for $\alpha_k > -1$, $k = 1, 2, \dots$, then each non trivial solution of the problem (14)–(2)–(15) *is oscillatory in $\mathbb{R}_+ \times \Omega$.*

Proof. Reasoning by contradiction as in the proof of Theorem 4 and setting $Z(t) = W(t)\exp(\lambda_1(t-T))$, $t > T > \sigma$, we obtain at once (23), for $t \ne t_k$, and (24), for $t = t_k$.

Integrating (23) from t_k to $t_k + \eta$, observing there is no impulses effect, we obtain

$$Z(t_k+\eta) - Z(t_k^+) + \int_{t_k}^{t_k+\eta} mG(s)KZ(s)\,ds \le 0\,.$$

From (H9) and the nonincreasing character of the function $Z(t)$ we have

$$mK \cdot Z(t_k) \int_{t_k}^{t_k+\eta} G(s)\,ds \le Z(t_k^+) - Z(t_k+\eta) \le (1+\alpha_k)Z(t_k)\,.$$

and since $\alpha_k > -1$, then

$$\frac{1}{(1+\alpha_k)} \int_{t_k}^{t_k+\eta} G(s)\,ds \le \frac{1}{mK}$$

which is a contradiction. For the negative case we obtain a contradiction by a similar reasoning. The proof is complete. □

We illustrate the obtained results by the following concrete example.

Example 1. Consider the following impulsive delayed parabolic problem

$$
(26)\qquad \begin{cases} \dfrac{\partial}{\partial t}\Big(u(t,x) - \sum\limits_{j=1}^{l} \dfrac{|\sin t|}{j}\, u\big(\tfrac{t}{j}, x\big)\Big) - 2(1+t)\dfrac{\partial^2 u}{\partial x^2}(\dfrac{t}{2}, x) \\ \qquad + \sum\limits_{r=1}^{m} \dfrac{4\pi |\sin t| (x^2+1) |u\left(t - \frac{11}{2r}\pi, x\right)|}{r} = 0\,, \\ \qquad t \neq k\pi\,;\ \ t \geq 0\,,\ \ x \in \Omega = (-1,1)\,, \\ u(t_k^+, x) - u(t_k^-, x) = \dfrac{u(t_k,x)}{t_k}(1-\cos x)\,, \quad t_k = k\pi\,;\ \ k = 1,2,\dots, \end{cases}
$$

with boundary condition

$$
(27)\qquad u(t,x) = 0\,, \qquad t > 0\,,\ t \neq k\pi\,,\ k = 1,2,\dots\,,\ x \in \{-1,1\}\,,
$$

and the delayed values of $u(t,x)$ for $(t,x) \in \left[-\frac{11}{2}\pi, 0\right] \times [-1,1]$ are given by $u(t,x) = t(1+x^2)$.

One can easily check hypotheses (H1)–(H4), for

$$
b_j\,(t) = \frac{|\sin t|}{j}\,, \quad \rho_j(t) = \frac{t}{j} \leq t\,, \quad j = 1,\dots,l\,,
$$

$$
\tau(t) = \frac{t}{2}\,, \quad a_i(t) = 2(1+t)^i \geq a_0 = 2\,, \quad i = n = 1
$$

$$
g_r(t,x) = \frac{4\pi\,|\sin t|\,(x^2+1)}{\sqrt{r}}\,, \quad h_r(u) = \frac{|u|}{\sqrt{r}},\ \eta_r = \frac{11}{2r}\pi\,, \quad r = 1,\dots,m
$$

$$
f(t,x) = 0\,; \quad I\big(t_k, x, u(t_k,x)\big) = \frac{u(t_k,x)}{t_k}(1-\cos x)\,, \quad t_k = k\pi\,;\ k = 1,2,\dots\,.
$$

Since h_r, $r = 1,\dots,m$ are positive and convex in $\mathbb{R}_+$, then there exist positive constants $\alpha_k = \dfrac{1}{k\pi}$, $k = 1,2,\dots$, such that for any function $u\colon \mathbb{R}_+ \times [-1,1] \to \mathbb{R}$, we have by assumption (H5),

$$
\int_{-1}^{1} \frac{u(k\pi,x)}{k\pi}(1-\cos x)\,dx \leq \frac{1}{k\pi}\int_{-1}^{1} u(k\pi,x)\,dx\,, \quad k = 1,2,\dots
$$

So, according to theorem 1, if $u(t,x)$ is a positive solution to the problem (26)–(27) in $[\sigma, +\infty[\times (-1,1)$, $\sigma > \frac{11}{2r}\pi$, then the corresponding impulsive differential inequality

$$
\begin{cases} \dfrac{d}{dt}\Big(U(t) - \sum\limits_{j=1}^{l} \dfrac{|\sin t|}{j}\, U\Big(\dfrac{t}{j}\Big)\Big) + \lambda_1 U\Big(\dfrac{t}{2}\Big) + \dfrac{4\pi|\sin t|}{\sqrt{m}} \sum\limits_{r=1}^{m} \dfrac{\left|U\left(t - \frac{11}{2r}\pi\right)\right|}{r} \leq 0\,, \\ \qquad t \neq t_k\,;\ t \geq \sigma \\ U(t_k^+) \leq \Big(1 + \dfrac{1}{k\pi}\Big)\, U(t_k)\,, \qquad t_k = k\pi\,,\ \ k = 1,2,\dots \end{cases}
$$

has a positive solution $U(t) = K_\Phi^{-1} \int_\Omega u(t,x)\Phi(x)\,dx$, where

$$K_\Phi = \int_\Omega \Phi(x)\,dx\,, \quad G(t) = \min_{1\le r\le m}\Big(\inf_{x\in(-1,1)} \frac{4\pi|\sin t|(x^2+1)}{\sqrt{r}}\Big) = \frac{4\pi|\sin t|}{\sqrt{m}}\,.$$

As the assumptions (H6)–(H7) are also satisfied, then Theorem 2 can be applied. On the other hand, the hypotheses (H8)–(H9) are satisfied and we have,

$$b_j\,(t_k) = \frac{|\sin t_k|}{j} = 0\,, \quad \text{for} \quad t_k = k\pi\,,$$

$$t_{k+1} - t_k \ge \pi\,, \quad 0 < \alpha_k = 1/k\pi < \alpha = 1\,, \quad k = 1,2,\dots\,,$$

$$h_r\,(u) = \frac{|u|}{\sqrt{r}} \ge \overline{h}(u) = \frac{|u|}{\sqrt{m}}\,, \quad r = 1,\dots,m\,.$$

$\overline{h}$ is nondecreasing and there exists $K > 0$ such that $\frac{\overline{h}(u)}{u} = \frac{1}{\sqrt{m}} > K$, for every $u > 0$. So, for $\eta = \frac{11}{2}\pi = \max_{1\le r\le m} \frac{11}{2r}\pi$, we have for $p = 1 + [\eta/\pi] = 6$ and each $m \ge 1$

$$\limsup_{k\to\infty} \int_{k\pi}^{k\pi+\eta} \frac{4\pi|\sin s|}{\sqrt{m}}\,ds = \limsup_{k\to\infty} \frac{4\pi}{\sqrt{m}} 11 = \frac{44\pi}{\sqrt{m}} > \frac{2^6}{mK}\,.$$

Thus we have to take K and m such that $1 > K\sqrt{m} > \frac{16}{11\pi} \simeq 0.4629$.

We conclude by Theorem 4 that each non trivial solution to the problem (26)–(27) is oscillatory in $\mathbb{R}_+ \times (-1,1)$.

References

[1] Bainov, D., Minchev, E., *Trends in theory of impulsive partial differential equations*, Nonlinear World **3** (3) (1996), 357–384.

[2] Bainov, D., Minchev, E., *Forced oscillations of solutions of impulsive nonlinear parabolic differential-difference equations*, J. Korean Math. Soc. **35** (4) (1998), 881–890.

[3] Cui, B., Deng, F. Q., Li, W. N., Liu, Y. Q., *Oscillation problems for delay parabolic systems with impulses*, Dyn. Contin. Discrete Impuls Syst. Ser. A Math. Anal. **12** (2005), 67–76.

[4] Fu, X., Liu, X., *Oscillation criteria for impulsive hyperbolic systems*, Dynam. Contin. Discrete Impuls. Systems **3** (2) (1997), 225–244.

[5] Fu, X., Liu, X., Sivaloganathan, S., *Oscillation criteria for impulsive parabolic differential equations with delay*, J. Math. Anal. Appl. **268** (2002), 647–664.

[6] Liu, A., Xiao, L., Liu, T., *Oscillation of nonlinear impulsive hyperbolic equations with several delays*, Electron. J. Differential Equations **2004** (24) (2004), 1–6.

LMA Lab, Department of Mathematics,
University Badji Mokhtar Annaba,
P.O.Box 12, Annaba 23000, Algeria.
E-mail: atmanira@yahoo.fr mazouzi_sa@yahoo.fr.