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F-MANIFOLDS AND INTEGRABLE SYSTEMS OF HYDRODYNAMIC TYPE

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Abstract. We investigate the role of Hertling-Manin condition on the structure constants of an associative commutative algebra in the theory of integrable systems of hydrodynamic type. In such a framework we introduce the notion of F-manifold with compatible connection generalizing a structure introduced by Manin.

1. Introduction

In their seminal papers [6, 26], Dubrovin, Novikov, and Tsarev pointed out a deep relation between the integrability properties of systems of PDEs of hydrodynamic type

$$u^i_t = V^i_j u^j_x\,, \qquad i = 1, \dots, n\,, \tag{1}$$

(sum over repeated indices is understood) and geometrical — in particular, Riemannian — structures on the target manifold M, where $(u^1, \dots, u^n)$ play the role of coordinates. Probably, the most important of such structures is the notion of a Frobenius manifold, introduced by Dubrovin (see, e.g., [3]) in order to give a coordinate-free description of the famous WDVV equations. A crucial ingredient involved in the definition of Frobenius manifolds is a $(1,2)$-type tensor field c giving an associative commutative product on every tangent space:

$$(X \circ Y)^i := c^i_{jk} X^j Y^k\,,$$

where X and Y are vector fields. More recently [15], Hertling and Manin showed that this product satisfies the condition

$$\begin{aligned}
&- [X \circ Y, Z] \circ W - [X \circ Y, W] \circ Z - X \circ [Y, Z \circ W] \\
&+ X \circ [Y, Z] \circ W + X \circ [Y, W] \circ Z - Y \circ [X, Z \circ W] \\
&+ Y \circ [X, Z] \circ W + Y \circ [X, W] \circ Z = 0\,,
\end{aligned} \tag{2}$$

or, in terms of the components of c,

$$(\partial_s c^k_{im}) c^s_{jl} - (\partial_s c^k_{jl}) c^s_{im} + (\partial_i c^s_{jl}) c^k_{sm} + (\partial_m c^s_{jl}) c^k_{si} - (\partial_l c^s_{im}) c^k_{js} - (\partial_j c^s_{im}) c^k_{ls} = 0\,. \tag{3}$$

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They called a manifold endowed with an associative commutative multiplicative structure satisfying condition (2), an F-manifold.

The aim of this paper is to study the properties of the PDEs of hydrodynamic type associated with F-manifolds. The system (3) and its relation with integrable systems has been considered from a different point of view in [16], where it is shown to describe the evolution of the dispersionless KP hierarchy. Here, following the insights coming from the case of the principal hierarchy in the context of Frobenius manifolds, we will consider the PDEs of the form

$$u^i_t = (V_X)^i_j u^j_x, \;\; i = 1, \dots, n \;, \qquad (V_X)^i_j := c^i_{jk} X^k \;, \tag{4}$$

where X is a vector field on M and c satisfies associativity, commutativity, and (2). These assumptions have two important consequences, spelled out respectively in Section 2 and 3:

1. For any choice of the vector field X, the Haantjes tensor associated with the (1,1) tensor field V_X vanishes.
2. They allow one to write the condition of commutativity of two flows of the form (4) as a simple requirement on the corresponding vector fields on M.

Starting from Section 4, we put into the game an additional structure, namely a connection ∇ satisfying the symmetry condition

$$\left(\nabla_X c\right)(Y, Z) = \left(\nabla_Y c\right)(X, Z) \;, \tag{5}$$

for all vector fields X, Y, and Z. Remarkably, as shown by Hertling [14], condition (2) follows from (5). According to Manin [18], we study the special case where the connection ∇ is flat and we show how to construct an integrable hierarchy of hydrodynamic type. The construction is divided in two steps. First — using a basis of flat vector fields — one defines a set of flows, known as *primary flows*. Then, from these flows one can define recursively the higher flows of the hierarchy. In this way, each primary flow turns out to be the starting point of a hierarchy. This construction is a straightforward generalization of the principal hierarchy defined by Dubrovin in the case of Frobenius manifolds [3]. Notice however that we do not need the connection to be necessarily metric.

The general (non-flat) case is studied in Section 5, where we introduce the notion of F-manifold with compatible (non-flat) connection ∇ and we show that the associated integrable systems of hydrodynamic type are defined by a family of vector fields satisfying the following condition:

$$c^i_{jm} \nabla_k X^m = c^i_{km} \nabla_j X^m \,. \tag{6}$$

In the non-flat case the existence of solutions of the above system is not guaranteed. Indeed we prove that every solution X of (6) satisfies the condition

$$(R^k_{lmi} c^n_{pk} + R^k_{lip} c^n_{mk} + R^k_{lpm} c^n_{ik}) X^l = 0 \,,$$

where R is the curvature tensor of ∇. It is thus natural to introduce the following requirement on the curvature:

$$R^k_{lmi} c^n_{pk} + R^k_{lip} c^n_{mk} + R^k_{lpm} c^n_{ik} = 0 \,. \tag{7}$$

If the structure constants c^i_{jk} admit canonical coordinates (i.e., coordinates such that $c^k_{ij} = \delta^k_i \delta^k_j$), then condition (7) is related to the well-known semi-Hamiltonian property introduced by Tsarev [26] as compatibility condition for the linear system providing the symmetries of a diagonal system of hydrodynamic type.

In Section 6, motivated by the Hamiltonian theory of systems of hydrodynamic type, we consider the case of metric connections and we introduce the notion of Riemannian F-manifold.

We stress that the main novelties of the paper are contained in Section 5 and 6, where the known relations between integrable PDEs of hydrodynamic type and Frobenius manifolds are extended to the non-flat case. In particular, the compatibility condition (7) between the product c and the connection ∇ turns out to be the geometric counterpart of the above mentioned semi-Hamiltonian property.

2. The Haantjes tensor

An important class of systems of hydrodynamic type, widely studied in the literature, consists in those systems which admit diagonal form. In this section we review some diagonalizability properties from our point of view, and we show — under a suitable hypothesis — the existence of canonical coordinates for the corresponding F-manifold.

Recall that a system (1) is said to be diagonalizable if there exists a set of coordinates $\left(r^1, \ldots, r^n\right)$ — usually called *Riemann invariants* — such that the tensor V^i_j is diagonal in these coordinates: $V^i_j(r) = v^i\, \delta^i_j$. Then the system takes the (diagonal) form

$$r^i_t = v^i(r^1, \ldots, r^n) r^i_x, \;\; i = 1, \ldots, n\,.$$

It is important to recall that there exists an invariant criterion for the diagonalizability. One first introduces the *Nijenhuis tensor* of V as

$$N_V(X,Y) = [VX, VY] - V\,[X, VY] - V\,[VX, Y] + V^2\,[X,Y]\,,$$

where X and Y are arbitrary vector fields, and then defines the *Haantjes tensor* as

$$H_V(X,Y) = N_V(VX, VY) - V\,N_V(X, VY) - V\,N_V(VX, Y) + V^2 N_V(X,Y)\,.$$

In the case when V has mutually distinct eigenvalues, then V is diagonalizable if and only if its Haantjes tensor is identically zero. In this section, we consider the Haantjes tensor of

$$(V_Z)^i_j = c^i_{jk} Z^k\,, \tag{8}$$

where c satisfies associativity, commutativity, and the Hertling-Manin condition (2). For a $(1,1)$- type tensor field of the form (8), the Nijenhuis tensor reads

$$N_{V_Z}(X,Y) = [Z \circ X, Z \circ Y] + Z^2 \circ [X,Y] - Z \circ [X, Z \circ Y] - Z \circ [Z \circ X, Y]\,.$$

By using the Hertling-Manin condition (2) evaluated at $X = Z$, this can be written as

$$N_{V_Z}(X,Y) = [X \circ Z, Z] \circ Y - [X,Z] \circ Z \circ Y + [Z, Y \circ Z] \circ X - [Z,Y] \circ X \circ Z,$$

using this identity it is easy to prove the following

Theorem 1. *The Haantjes tensor associated with V_Z vanishes for any choice of the vector field Z.*

Proof. Let us write for simplicity N in place of N_{V_Z}. Then, we have that

$$\begin{aligned} H_{V_Z}(X,Y) &= N(Z\circ X, Z\circ Y)+Z^2\circ N(X,Y)-Z\circ N(X,Z\circ Y)-Z\circ N(Z\circ X,Y)\\ &= [X\circ Z^2,Z]\circ Y\circ Z-[X\circ Z,Z]\circ Z^2\circ Y+[Z,Y\circ Z^2]\circ X\circ Z\\ &\quad -[Z,Y\circ Z]\circ X\circ Z^2+[X\circ Z,Z]\circ Y\circ Z^2-[X,Z]\circ Z^3\circ Y\\ &\quad +[Z,Y\circ Z]\circ X\circ Z^2-[Z,Y]\circ X\circ Z^3-[X\circ Z,Z]\circ Z^2\circ Y\\ &\quad +[X,Z]\circ Z^3\circ Y-[Z,Y\circ Z^2]\circ X\circ Z+[Z,Y\circ Z]\circ X\circ Z^2\\ &\quad -[X\circ Z^2,Z]\circ Y\circ Z+[X\circ Z,Z]\circ Z^2\circ Y-[Z,Y\circ Z]\circ X\circ Z^2\\ &\quad +[Z,Y]\circ X\circ Z^3=0\,, \end{aligned}$$

where $Z^2 = Z\circ Z$ and $Z^3 = Z\circ Z\circ Z$. □

Suppose now that X is a vector field such that V_X has everywhere distinct real eigenvalues $(v^1,\dots,v^n)$. Since the Haantjes tensor of V_X vanishes, there exist local coordinates $(r^1,\dots,r^n)$ such that $(V_X)^i_j=\delta^i_j v^i$. These coordinates are Riemann invariants of the corresponding system of hydrodynamic type. Moreover, we have

Proposition 2. *The components of the tensor field c in the coordinates $(r^1,\dots,r^n)$ are given by*

$$c^k_{ij}=f_i\delta^k_i\delta^k_j\,,$$

where every f_i depends on the variable r^i only.

Proof. In diagonal coordinates we have

$$(V_X)^i_j=c^i_{jk}X^k=v^i\delta^i_j\,,$$

hence, we get

$$c^j_{pq}c^i_{jk}X^k=c^j_{pq}v^i\delta^i_j=c^i_{pq}v^i\,.$$

On the other hand, due to the associativity of the algebra, we can also write

$$c^j_{pq}c^i_{jk}X^k=c^j_{pk}c^i_{jq}X^k=c^i_{jq}v^j\delta^j_p=c^i_{pq}v^p\qquad\text{(no sum over } p\text{)},$$

and therefore,

$$c^i_{pq}\left(v^i-v^p\right)=0\,.$$

Since the algebra is commutative and the eigenvalues of V_X are pairwise distinct, this means that the structure constants, in the coordinates $(r^1,\dots,r^n)$, take the form

$$c^i_{jk}=f_i\delta^i_j\delta^i_k\,,\tag{9}$$

where the f_i are arbitrary functions, depending in principle on all the variables $r^1,\dots,r^n$. The requirement on the structure constants c to satisfy the Hertling-Manin condition (3) implies further constraints on the functions f_i. Indeed, substituting

(9) into (3), we get a set of equations which the f_i must satisfy; considering for instance the case $i = k = l = m \neq j$, we get

$$f_i\, \partial_j f_i = 0\,,$$

which means that every f_i is either constant or it depends on r^i only. It is easy to check that conditions (3) give no further restrictions on the f_i; the proposition is proved. □

If the functions f_i are everywhere different from zero, then it is easy to show that there exist local coordinates, called *canonical coordinates*, such that $c^k_{ij} = \delta^k_i \delta^k_j$. Moreover, in this case, the vector field

$$e = \sum_{i=1}^{n} \frac{1}{f_i} \frac{\partial}{\partial r^i}$$

is globally defined and is the unit of the algebra.

Remark 3. If the algebra has a unity e, then the Hertling-Manin condition implies

$$\mathrm{Lie}_e c = 0\,.$$

Indeed, for $X = Y = e$ the Hertling-Manin condition becomes

$$-[e, Z \circ W] + [e, Z] \circ W + [e, W] \circ Z = 0\,.$$

Remark 4. An alternative proof of the existence of canonical coordinates has been given in [15] under the assumption of semisimplicity of the algebra, that is, the existence of a basis of idempotents. Here we assume the existence of a vector field X such that V_X has everywhere distinct real eigenvalues.

3. Commutativity of the flows

As a consequence of the Hertling-Manin condition, the conditions for the commutativity of two hydrodynamical flows take a rather simple form.

Proposition 5. *The flows*

$$u^i_t = [V_X]^i_j u^j_x = c^i_{jk} X^j u^k_x \tag{10}$$

and

$$u^i_\tau = [V_Y]^i_j u^j_x = c^i_{jk} Y^j u^k_x \tag{11}$$

commute if and only if the vector fields X and Y satisfy the condition

$$\Big((\mathrm{Lie}_X c)\,(Y, Z) - (\mathrm{Lie}_Y c)\,(X, Z) + [X, Y] \circ Z \Big) \circ Z = 0\,,$$

for any vector field Z. Equivalently,

$$\begin{aligned} &\Big((\mathrm{Lie}_X c)\,(Y, Z) - (\mathrm{Lie}_Y c)\,(X, Z) + [X, Y] \circ Z \Big) \circ W \\ &\qquad + \Big((\mathrm{Lie}_X c)\,(Y, W) - (\mathrm{Lie}_Y c)\,(X, W) + [X, Y] \circ W \Big) \circ Z = 0 \end{aligned}$$

for all pairs (Z, W) of vector fields. In local coordinates this means that

$$\begin{aligned} &c^r_{is} \big[(\mathrm{Lie}_X c)^i_{jq} Y^q - (\mathrm{Lie}_Y c)^i_{jq} X^q + c^i_{jq} [X, Y]^q \big] \\ &\qquad + c^r_{ij} \big[(\mathrm{Lie}_X c)^i_{sq} Y^q - (\mathrm{Lie}_Y c)^i_{sq} X^q + c^i_{sq} [X, Y]^q \big] = 0\,. \end{aligned}$$

Proof. It is well-known [23] that the commutativity of the flows (10) and (11) is equivalent to the following requirements:

1. The $(1,1)$-tensor fields V_X and V_Y (seen as endomorphism of the tangent bundle) commute.
2. For any vector field Z the following condition is satisfied:

$$[V_X(Z), V_Y(Z)] - V_X\big([Z, V_Y(Z)]\big) + V_Y\big([Z, V_X(Z)]\big) = 0\,,$$

that is to say,

$$[Z \circ X, Z \circ Y] - X \circ [Z, Z \circ Y] + Y \circ [Z, Z \circ X] = 0\,.$$

The first requirement is automatically verified due to the associativity of the algebra. Making use of identity (2), the second one becomes

$$([Z \circ X, Y] + [X, Z \circ Y] - [X, Z] \circ Y - [X, Y] \circ Z - X \circ [Z, Y]) \circ Z = 0\,. \tag{12}$$

A simple calculation shows that the quantity in the bracket, namely

$$[Z \circ X, Y] + [X, Z \circ Y] - [X, Z] \circ Y - [X, Y] \circ Z - X \circ [Z, Y]\,,$$

is equal to

$$(\mathrm{Lie}_X c)\,(Y, Z) - (\mathrm{Lie}_Y c)\,(X, Z) + [X, Y] \circ Z\,. \tag{13}$$

Substituting (13) into (12), we get the result. □

Corollary 6. *A sufficient condition for the commutativity of the hydrodynamic flows* (10) *and* (11) *is that*

$$(\mathrm{Lie}_X c)\,(Y, Z) - (\mathrm{Lie}_Y c)\,(X, Z) + [X, Y] \circ Z = 0 \tag{14}$$

for all vector fields Z, that is,

$$(\mathrm{Lie}_X c)^i_{pq}\, Y^q - (\mathrm{Lie}_Y c)^i_{pq}\, X^q + c^i_{pq}[X, Y]^q = 0 \tag{15}$$

or, equivalently,

$$\mathrm{Lie}_X V_Y - \mathrm{Lie}_Y V_X - V_{[X,Y]} = 0\,. \tag{16}$$

4. Dubrovin principal hierarchy

In this section, we adapt Dubrovin's construction of the principal hierarchy [3] to the case of F-manifolds with compatible flat connection introduced by Manin in [18].

Definition 7. An F-manifold with compatible flat connection is a manifold endowed with an associative commutative multiplicative structure given by a $(1,2)$-tensor field c and a flat torsionless connection ∇ satisfying the symmetry condition

$$\nabla_l c^i_{jk} = \nabla_j c^i_{lk}\,, \tag{17}$$

meaning that ∇c is totally symmetric:

$$(\nabla_X c)\,(Y, Z) = (\nabla_Y c)\,(X, Z)\ , \tag{18}$$

for all vector fields X, Y, and Z.

Notice that Hertling-Manin condition (2) does not appear in the above definition. Indeed, as proved by Hertling in [14], it is a consequence of the existence of a torsionless (even non-flat) connection ∇ satisfying (17).

Remark 8. Notice that in flat coordinates condition (17) reads

$$\partial_l c^i_{jk} = \partial_j c^i_{lk}.$$

This, together with the commutativity of the algebra, implies that

$$c^i_{jk} = \partial_j C^i_k = \partial_j \partial_k C^i\,.$$

Therefore, condition (17) is equivalent to the local existence of a vector field C satisfying, for any pair (X, Y) of flat vector fields, the condition

$$X \circ Y = [X, [Y, C]]\,.$$

The above condition appears in the original definition of Manin [18].

Let us construct now the principal hierarchy. In order to do so, the first step consists in defining the primary flows. Since the connection is flat, we can consider a basis $(X_{(1,0)}, \dots, X_{(n,0)})$ of flat vector fields; the primary flows are thus defined as

$$u^i_{t_{(p,0)}} = c^i_{jk} X^k_{(p,0)} u^j_x\,. \tag{19}$$

Proposition 9. *The primary flows* (19) *commute.*

Proof. Since the $X_{(p,0)}$ are flat and the torsion vanishes, they commute and

$$\mathrm{Lie}_{X_{(p,0)}} c = \nabla_{X_{(p,0)}} c\,.$$

Therefore, the commutativity condition (14) for the vector fields $X = X_{(p,0)}$ and $Y = X_{(q,0)}$ follows from condition (17). □

Starting from the primary flows (19) one can introduce the "higher flows" of the hierarchy, defined as

$$u^i_{t_{(p,\alpha)}} = c^i_{jk} X^j_{(p,\alpha)} u^k_x\,, \tag{20}$$

by means of the following recursion relations:

$$\nabla_j X^i_{(p,\alpha)} = c^i_{jk} X^k_{(p,\alpha-1)}\,. \tag{21}$$

Remark 10. The flatness of the connection ∇, the symmetry of the tensor ∇c (condition (17)) and the associativity of the algebra with structure constants c^i_{jk} are equivalent to the flatness of the one-parameter family of connections defined, for any pair of vector fields X and Y, by

$$\tilde{\nabla}_X Y = \nabla_X Y + z X \circ Y, \qquad z \in \mathbb{C}\,.$$

The vector fields obtained by means of the recursive relations (21) are nothing but the z-coefficients of a basis of flat vector fields of the deformed connection [3].

In order to show that the higher flows (20) are well-defined, it is necessary to prove the following

Proposition 11. *The recursive relations* (21) *are compatible.*

Proof. We note that the recursive relations (21) can be written in the form

$$\partial_j X^i_{(p,\alpha)} = -\Gamma^i_{jk} X^k_{(p,\alpha)} + c^i_{kj} X^k_{(p,\alpha-1)},$$

thus, we have

$$\begin{aligned}(\partial_j\partial_m - \partial_m\partial_j) X^i_{(p,\alpha)} &= \left[\partial_m \Gamma^i_{jl} - \partial_j \Gamma^i_{ml} - \Gamma^i_{jk}\Gamma^k_{ml} + \Gamma^i_{mk}\Gamma^k_{jl}\right] X^l_{(p,\alpha)} \\ &\quad - \left[\partial_m c^i_{jl} - \partial_j c^i_{ml} - \Gamma^i_{kj} c^k_{ml} - \Gamma^k_{lm} c^i_{jk} + \Gamma^i_{km} c^k_{jl} + \Gamma^k_{lj} c^i_{mk}\right] \\ &\quad \times X^l_{(p,\alpha-1)} + \left[c^i_{jk} c^k_{ml} - c^i_{mk} c^k_{jl}\right] X^l_{(p,\alpha-2)}\,.\end{aligned}$$

The flatness of the connection ∇, together with identity (17) and the associativity of the algebra, implies the vanishing of the quantity above. Therefore, relations (21) are compatible. □

Since the primary flows (19) commute and the recursive relations (21) are compatible, it only remains to prove the following

Theorem 12. *The flows of the principal hierarchy commute.*

Proof. Let us consider the hydrodynamic flows associated with the vector fields $X_{(p,\alpha)}$ and $X_{(q,\beta)}$. In order to show that these flows commute, we prove that they satisfy the sufficient condition (15). In local coordinates it reads:

$$\begin{aligned} &X^m_{(p,\alpha)}(\partial_m c^i_{jk}) X^k_{(q,\beta)} - X^m_{(q,\beta)}(\partial_m c^i_{jk}) X^k_{(p,\alpha)} + \\ &-c^l_{jk}(\partial_l X^i_{(p,\alpha)}) X^k_{(q,\beta)} + c^i_{lk}(\partial_j X^l_{(p,\alpha)}) X^k_{(q,\beta)} + \\ &+c^i_{jl}(\partial_k X^l_{(p,\alpha)}) X^k_{(q,\beta)} + c^l_{jk}(\partial_l X^i_{(q,\beta)}) X^k_{(p,\alpha)} + \\ &-c^i_{lk}(\partial_j X^l_{(q,\beta)}) X^k_{(p,\alpha)} - c^i_{jl}(\partial_k X^l_{(q,\beta)}) X^k_{(p,\alpha)} + \\ &-c^i_{jk}\left((\partial_l X^k_{(p,\alpha)}) X^l_{(q,\beta)} + (\partial_l X^k_{(q,\beta)}) X^l_{(p,\alpha)}\right) = 0\,. \end{aligned}$$

In particular, if the coordinates are flat, the first row vanishes due to the symmetry of the tensor ∇c. Moreover, using the recursive relations (21) we obtain

$$\begin{aligned} &- c^l_{jk} c^i_{ln} X^n_{(p,\alpha-1)} X^k_{(q,\beta)} + c^i_{lk} c^l_{jn} X^n_{(p,\alpha-1)} X^k_{(q,\beta)} + \\ &+ c^i_{jl} c^l_{kn} X^n_{(p,\alpha-1)} X^k_{(q,\beta)} + c^l_{jk} c^i_{ln} X^n_{(q,\beta-1)} X^k_{(p,\alpha)} + \\ &- c^i_{lk} c^l_{jn} X^n_{(q,\beta-1)} X^k_{(p,\alpha)} - c^i_{jl} c^l_{kn} X^n_{(q,\beta-1)} X^k_{(p,\alpha)} + \\ &- c^i_{jk} c^k_{mn} X^n_{(p,\alpha-1)} X^m_{(q,\beta)} + c^i_{jk} c^k_{mn} X^n_{(q,\beta-1)} X^m_{(p,\alpha)} \end{aligned}$$

which vanishes due to the associativity of the algebra. □

Remark 13. The flows of the principal hierarchy are well-defined even in the case when the torsion of the flat connection ∇ does not vanish. However, their commutativity depends crucially on this additional assumption.

5. F-MANIFOLDS WITH COMPATIBLE CONNECTION AND RELATED INTEGRABLE SYSTEMS

From the point of view of the theory of integrable systems of hydrodynamic type, the "flat case" and its associated principal hierarchy are exceptional. Therefore, it is quite natural to extend the notion of F-manifolds with compatible flat connection to the non-flat case. As a starting point, we consider an F-manifold endowed with a connection ∇ satisfying (17). If ∇ is flat, we know how to construct integrable systems of hydrodynamic type. Indeed, the starting point of the construction of the previous section is a basis of *flat* vector fields, and the recursive procedure (21) defining the "higher" vector fields and the corresponding flows is well-defined as a consequence of the vanishing of the curvature. In the non-flat case, in order to define integrable systems of hydrodynamic type one needs to find an alternative way to select the vector fields.

5.1. **Hydrodynamic-type systems associated with F-manifolds.** In the flat case, the vector fields X defining the principal hierarchy satisfy the condition

$$(\nabla_Z X) \circ W = (\nabla_W X) \circ Z \tag{22}$$

for all pairs (Z, W) of vector fields, that is, in local coordinates,

$$c^i_{jm} \nabla_k X^m = c^i_{km} \nabla_j X^m \,. \tag{23}$$

Indeed, in the case of the flat vector fields $X_{(p,0)}$ defining the primary flows, both sides of (23) vanish due to

$$\nabla_k X^m_{(p,0)} = 0, \;\; p = 1, \ldots, n \,,$$

while the vector fields defining the higher flows of the hierarchy satisfy (23) due to the associativity of the algebra:

$$c^i_{jm} \nabla_k X^m_{(p,\alpha)} = c^i_{jm} c^m_{kl} X^l_{(p,\alpha-1)} = c^i_{km} c^m_{jl} X^l_{(p,\alpha-1)} = c^i_{km} \nabla_j X^m_{(p,\alpha)} \,.$$

A crucial remark is the following: if ∇ satisfies condition (17), then *any pair of solutions of* (23) *defines commuting flows even if the connection* ∇ *is not flat.* More precisely, we have the following

Proposition 14. *If X and Y are two vector fields satisfying condition* (22)*, then the associated flows*

$$u^i_t = c^i_{jk} X^k u^j_x \tag{24}$$

and

$$u^i_\tau = c^i_{jk} Y^k u^j_x \tag{25}$$

commute.

Proof. Recall from Proposition 5 that the flows (24) and (25) commute if and only if

$$\left((\mathrm{Lie}_X c)\,(Y, Z) - (\mathrm{Lie}_Y c)\,(X, Z) + [X, Y] \circ Z \right) \circ Z = 0 \tag{26}$$

for any vector field Z. On the other hand, the vanishing of the torsion of ∇ gives the identity

$$(\mathrm{Lie}_X c)\,(Y, Z) = (\nabla_X c)\,(Y, Z) - \nabla_{c(Y,Z)} X + c(Y, \nabla_Z X) + c(\nabla_Y X, Z) \,,$$

and this, together with the symmetry (18) of ∇c, can be used to write the term in the bracket of (26) as

$$-\nabla_{Y\circ Z}X + \nabla_{X\circ Z}Y + [Y,X]\circ Z\,.$$

Multiplying the above identity by Z, and using property (22) for the vector fields X and Y, we obtain

$$\begin{aligned} &-(\nabla_{Y\circ Z}X)\circ Z + (\nabla_{X\circ Z}Y)\circ Z + [Y,X]\circ Z^2\\ =&-(\nabla_Z X)\circ(Y\circ Z) + (\nabla_Z Y)\circ(X\circ Z) + [Y,X]\circ Z^2\\ =&-(\nabla_Y X)\circ Z^2 + (\nabla_X Y)\circ Z^2 + [Y,X]\circ Z^2 = 0\,. \end{aligned}$$

The proposition is proved. □

Remark 15. From (17) and (22) it follows that the (1,1)-tensor field

$$(V_X)^i_j = c^i_{jk}X^k$$

satisfies the condition

$$\nabla_k (V_X)^i_j = \nabla_j (V_X)^i_k\,,$$

which is well-known in the Hamiltonian theory of systems of hydrodynamic type [6].

In the flat case, we have seen that system (23) admits a set of solutions, given by the vector fields of the principal hierarchy. However, if ∇ is non-flat, existence of solutions for system (23) is not guaranteed; additional constraints have to be imposed on the curvature R of the connection ∇.

Proposition 16. *If X is a solution of* (22)*, then the identity*

$$Z\circ R(W,Y)(X) + W\circ R(Y,Z)(X) + Y\circ R(Z,W)(X) = 0\,, \tag{27}$$

holds for any choice of the vector fields (Y,W,Z)*.*

Proof. Condition (22) implies

$$\nabla_W(Z\circ\nabla_Y X - Y\circ\nabla_Z X) + \nabla_Y(W\circ\nabla_Z X - Z\circ\nabla_W X) + \nabla_Z(Y\circ\nabla_W X - W\circ\nabla_Y X) = 0\,.$$

Using the symmetry condition (17) written in the form

$$\nabla_Y(X\circ Z) - \nabla_X(Y\circ Z) + Y\circ\nabla_X Z - X\circ\nabla_Y Z - [Y,X]\circ Z = 0$$

we obtain identity (27). □

Condition (27) must be satisfied for *any* solution X of the system (23). Since we are looking for a family of vector fields satisfying (23), it is natural to require that (27) holds true for an *arbitrary* vector field X.

Definition 17. An *F-manifold with compatible connection* is a manifold endowed with an associative commutative multiplicative structure given by a $(1,2)$-tensor field c and a torsionless connection ∇ satisfying condition (18) and condition

$$Z \circ R(W,Y)(X) + W \circ R(Y,Z)(X) + Y \circ R(Z,W)(X) = 0\,, \tag{28}$$

for any choice of the vector fields (X, Y, W, Z). In local coordinates this means that

$$R^k_{lmi} c^n_{pk} + R^k_{lip} c^n_{mk} + R^k_{lpm} c^n_{ik} = 0\,. \tag{29}$$

Remark 18. An equivalent form of condition (28) can be easily obtained using the (second) Bianchi identity for the deformed connection

$$\tilde{\nabla}_X Y = \nabla_X Y + zX \circ Y\,, \qquad z \in \mathbb{C},$$

where X and Y are arbitrary vector fields. Indeed, by associativity and symmetry condition (17), the Riemann tensor of this connection does not depend on z [25]. Using this fact it is easy to see that the Bianchi identity reduces to

$$\begin{aligned} 0 &= \tilde{\nabla}_X R(Y,Z)(W) + \tilde{\nabla}_Z R(X,Y)(W) + \tilde{\nabla}_Y R(Z,X)(W) \\ &= X \circ R(Y,Z)(W) + Z \circ R(X,Y)(W) + Y \circ R(Z,X)(W) \\ &\quad - R(Y,Z)(X \circ W) - R(X,Y)(Z \circ W) - R(Z,X)(Y \circ W) \end{aligned}$$

for any choice of the vector fields (X, Y, W, Z). Hence, condition (28) is equivalent to

$$R(Y,Z)(X \circ W) + R(X,Y)(Z \circ W) + R(Z,X)(Y \circ W) = 0\,,$$

for every (X, Y, W, Z).

From now on we will assume the existence of canonical coordinates $(r^1, \dots, r^n)$, discussing the meaning of condition (29) under this additional assumption.

Proposition 19. *In canonical coordinates, system* (23) *reduces to*

$$\partial_k v^i = \Gamma^i_{ki}(v^k - v^i)\,, \qquad i \neq k\,, \tag{30}$$

where v^i are the components of X in such coordinates.

Proof. Writing (23) in canonical coordinates, we get

$$\delta^i_j(\partial_k v^i + \Gamma^i_{kl} v^l) = \delta^i_k(\partial_j v^i + \Gamma^i_{jl} v^l)\,.$$

In the case $i = j \neq k$, using the identities

$$\Gamma^i_{kk} = -\Gamma^i_{ki} \tag{31}$$

and

$$\Gamma^i_{kl} = 0\,, \qquad i \neq k \neq l \neq i\,, \tag{32}$$

which follow from (17), we obtain system (30). The remaining conditions give no further constraints. □

Remark 20. We recall that, in canonical coordinates, the components of the vector field X coincide with the characteristic velocities of the associated system of hydrodynamic type:

$$r^i_t = c^i_{jk} v^k r^j_x = v^i r^i_x \,, \qquad i = 1, \dots, n\,.$$

Compatibility conditions of system (30) are well-known in the literature [26], and are given by the following condition:

$$\partial_i \Gamma^k_{km} - \Gamma^k_{km}\Gamma^m_{im} + \Gamma^k_{ik}\Gamma^k_{km} - \Gamma^k_{ik}\Gamma^i_{im} = 0\,, \tag{33}$$

for pairwise distinct indices k, i, m. Notice that (33) implies

$$\partial_i \Gamma^k_{mk} - \partial_m \Gamma^k_{ik} = 0\,, \tag{34}$$

for pairwise distinct indices k, i, m.

Proposition 21. *Condition* (29) *is equivalent to condition* (33)*.*

Proof. In canonical coordinates, condition (29) reads

$$\begin{aligned} &R^k_{lmi} c^n_{pk} + R^k_{lip} c^n_{mk} + R^k_{lpm} c^n_{ik} = \\ &R^k_{lmi} \delta^n_p \delta^n_k + R^k_{lip} \delta^n_m \delta^n_k + R^k_{lpm} \delta^n_i \delta^n_k = \\ &R^n_{lmi} \delta^n_p + R^n_{lip} \delta^n_m + R^n_{lpm} \delta^n_i = 0\ . \end{aligned}$$

If all the indices m, i, p, n are distinct the above condition is trivially satisfied. Let us consider the case $n = p$ (the case $n \neq p$ can be treated in the same way and does not add further condition). If $n = i$, we obtain

$$R^n_{lmn} + R^n_{lnm} + \delta^n_m\, R^n_{lnn} = 0\ ,$$

that is satisfied due to the skew-symmetry of the Riemann tensor with respect to the second and third lower indices. The same if $n = m$. For $n \neq i, m$, we obtain

$$R^n_{nmi} = 0, \tag{35}$$

if $l = n$ and

$$R^n_{lmi} = 0, \tag{36}$$

if $l \neq n$. Since, due to (31), the components of the Riemann tensor vanish if all the indices are distinct, condition (36) reduces to

$$R^n_{mmi} = 0, \qquad n \neq m \neq i \neq n. \tag{37}$$

Finally, using (31) and (32), it is easy to check that conditions (35) and (37) are equivalent to conditions (34) and (33) respectively. This proves the proposition. $\square$

Remark 22. If the compatibility conditions (34) and (33) are satisfied, the general solution of the system (30) depends on n arbitrary functions of a single variable. Moreover, due to (34), any solution $(v^1, \dots, v^n)$ of (30) satisfies the condition

$$\partial_k \left(\frac{\partial_j v^i}{v^j - v^i} \right) - \partial_j \left(\frac{\partial_k v^i}{v^k - v^i} \right) = 0\,, \qquad i \neq j \neq k \neq i\,, \tag{38}$$

known in literature as semi-Hamiltonian property [26]. An invariant and highly non trivial formulation of such a property was found in [23].

Due to the above remark, under the assumption of existence of canonical coordinates we have a set of solutions of (30) leading to a family of commuting systems of hydrodynamic type, depending on n arbitrary functions. This result shows the deep relation between F-manifold with compatible connection (Definition 17) and integrable systems of PDEs.

Remark 23. A generalization of Dubrovin's principal hierarchy of Section 4 has been given by Mokhov [19, 20] in the case of flat submanifolds with flat normal bundle in pseudo-Euclidean spaces. It would be interesting to extend Mokhov's approach also to the non-flat case, and to relate it with the construction considered above.

6. Riemannian F-manifolds and Egorov metrics

In this section we consider the special case where the connection ∇ is a metric connection. This assumption plays an important role in the Hamiltonian theory of systems of hydrodynamic type (see for instance [7, 22, 24] and references therein), as well as in the theory of Frobenius manifolds [3, 4].

Definition 24. A *Riemannian F-manifold* is an F-manifold with a compatible connection ∇ satisfying the following additional conditions:
1. The connection is metric:

$$\nabla g = 0\,.$$

2. The inner product $\langle \cdot, \cdot \rangle$ defined by the metric g is invariant with respect to the product $\circ$:

$$\langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle\,. \tag{39}$$

In local coordinates, condition (39) reads

$$g_{iq} c^q_{lp} = g_{lq} c^q_{ip}, \quad \text{or} \quad g^{iq} c^l_{qp} = g^{lq} c^i_{qp}\,, \tag{40}$$

where g_{ij} and g^{ij} are respectively the covariant and the contravariant components of the metric g.

If there exist canonical coordinates, the metric g entering the definition of Riemannian F-manifold is an Egorov metric. Let us recall the definition of this special class of metrics.

Definition 25. A metric is called *Egorov* if there exist coordinates $(r^1,\dots,r^n)$ such that it is diagonal and *potential*:

$$g_{ij} = \delta^i_j\, g_{ii}(r^1,\dots,r^n) = \delta^i_j\, \partial_i F\,,$$

for a certain function F.

It is clear by this definition that a flat metric is trivially of Egorov type. The converse statement is, of course, false.

Now, if we assume the existence of canonical coordinates, condition (40) tells us that the metric g is diagonal in such coordinates, while condition (31) — which follows from (17) — implies that the metric is potential. Therefore, g is an Egorov metric. Conversely, given an Egorov metric g whose curvature tensor satisfies condition (37), we can locally construct a Riemannian F-manifold. More precisely, let $\left(r^1,\dots,r^n\right)$ be the coordinates where g is diagonal and potential. Then, the metric g and the structure constants

$$c^i_{jk}(r) = \delta^i_j\delta^i_k$$

endow the open set where the coordinates $\left(r^1,\dots,r^n\right)$ are defined with the structure of a Riemannian F-manifold.

We point out that condition (29) is far from being trivial. Indeed, using the above remark, it is easy to construct examples of metrics satisfying properties (39) and (17). Much more difficult is the problem of finding Egorov metrics which satisfy also condition (29), since the potential has to fulfill (37). However, there exists an important class of metrics, appearing in the Hamiltonian theory of integrable hierarchies of hydrodynamic type (not necessarily of Egorov type) whose curvature satisfies (29). These are the metrics whose Riemann tensor admits “a quadratic expansion” in terms of the flows of the hierarchy [8, 21]:

$$u^i_{t_\alpha} = c^i_{jk}X^k_{(\alpha)}u^j_x\,,\qquad i=1,\dots,n\,.$$

This means that

$$R^{sk}_{mi} = \left(c^s_{ml}c^k_{iq} - c^s_{il}c^k_{mq}\right)\sum_\alpha \epsilon_\alpha X^l_{(\alpha)}X^q_{(\alpha)},\qquad \epsilon_\alpha = \pm 1, \tag{41}$$

where the index α can take value on a finite or infinite — even continuous — set.

Proposition 26. *Suppose that* ∇ *is the Levi-Civita connection of a metric* g*, and that its curvature satisfies condition* (41)*. In this case, condition* (29) *is automatically satisfied.*

Proof. We have that

$$R^{sk}_{mi}c^n_{pk}+R^{sk}_{ip}c^n_{mk}+R^{sk}_{pm}c^n_{ik}=\sum_\alpha\epsilon_\alpha[(c^s_{mr}c^k_{iq}-c^s_{ir}c^k_{mq})c^n_{pk}$$
$$+(c^s_{ir}c^k_{pq}-c^s_{pr}c^k_{iq})c^n_{mk}+(c^s_{pr}c^k_{mq}-c^s_{mr}c^k_{pq})c^n_{ik}]X^r_{(\alpha)}X^q_{(\alpha)}=$$
$$=\sum_\alpha\epsilon_\alpha[(c^k_{iq}c^n_{pk}-c^k_{pq}c^n_{ik})c^s_{mr}+(c^k_{pq}c^n_{mk}-c^k_{mq}c^n_{pk})c^s_{ir}$$
$$+(c^k_{mq}c^n_{ik}-c^k_{iq}c^n_{mk})c^s_{pr}]X^r_{(\alpha)}X^q_{(\alpha)}\,,$$

which vanishes due to associativity. □

Remark 27. If the functions

$$g^{lq}:=\sum_\alpha\epsilon_\alpha X^l_{(\alpha)}X^q_{(\alpha)}$$

define the contravariant components of a metric satisfying condition (40), then the operator

$$\sum_\alpha\epsilon_\alpha\,(w_\alpha)^i_k\,u^k_x\left(\frac{d}{dx}\right)^{-1}(w_\alpha)^j_h\,u^h_x,\qquad (w_\alpha)^i_j:=c^i_{jk}X^k_{(\alpha)}$$

is a purely nonlocal Poisson operator (see [10] for details).

7. Conclusions and open problems

In this paper, extending a construction of Manin, we have introduced a special class of F-manifolds, called *F-manifolds with compatible connection*. On the loop-space of such manifolds, we have defined a class of integrable systems of hydrodynamic type that, in the flat case, reduces to the Dubrovin principal hierarchy (see Section 5). Alternatively, one can follow the opposite path, moving from integrable systems of hydrodynamic type to F-manifolds. The point is that there are many possible "factorizations" $V^i_j=c^i_{jk}X^k$ of the (1,1)-type tensor fields that define the flows (1), leading to F-manifolds of our class. For instance, in the case of semi-Hamiltonian systems

$$r^i_t=v^i(r)\,r^i_x\,,\qquad i=1,\dots,n\,,$$

one can define the structure constants in the coordinates given by the Riemann invariants $(r^1,\dots,r^n)$ as

$$c^i_{jk}=\delta^i_j\delta^i_k\,,$$

and the Christoffel symbols of the torsion-free connection ∇ as

$$\Gamma^i_{jk}=0\qquad\text{for } i\neq j\neq k\neq i\,,\tag{42}$$

$$\Gamma^i_{jj}=-\Gamma^i_{ji}\qquad\text{for } i\neq j\,,\tag{43}$$

$$\Gamma^i_{ij}=\Gamma^i_{ji}=\frac{\partial_j v^i}{v^j-v^i}\qquad\text{for } i\neq j\,,\tag{44}$$

leaving Γ^i_{ii} arbitrary. Conditions (42–43) are equivalent to the symmetry of ∇c, and therefore from the proof of Proposition 21 it is clear that — on the open set where the coordinates $(r^1, \ldots, r^n)$ are defined — the pair (c,∇) satisfies all conditions of Definition 17. Moreover, relation (44) says that the integrable PDEs associated with this F-manifold are the semi-Hamiltonian systems we have started with. Notice that, in general, the connection ∇ *does not* coincide with the Levi-Civita connection of the diagonal metrics solutions of the system

$$\partial_j \ln \sqrt{g_{ii}} = \frac{\partial_j v^i}{v^j - v^i}\,, \qquad i \neq j\,. \tag{45}$$

They coincide, for a suitable choice of the Γ^i_{ii}, if the solution of (45) we are considering is potential in the coordinates $(r^1, \ldots, r^n)$. In this case, the procedure we just described coincides with the procedure mentioned in Section 6, and leads to a (locally defined) structure of *Riemannian* F-manifold. Otherwise, we are in the more general case of non metric connections treated in Section 5.

It is an interesting open problem to understand the consequences of the freedom in the choice of the Γ^i_{ii}. This freedom might be exploited to obtain useful information on the integrable systems of hydrodynamic type.

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