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A CERTAIN INTEGRAL-RECURRENCE EQUATION WITH DISCRETE-CONTINUOUS AUTO-CONVOLUTION

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Abstract. Laplace transform and some of the author's previous results about first order differential-recurrence equations with discrete auto-convolution are used to solve a new type of non-linear quadratic integral equation. This paper continues the author's work from other articles in which are considered and solved new types of algebraic-differential or integral equations.

1. Introduction

In the earlier paper [4], N. M. Flaisher solved by Fourier transform method a second order differential-recurrence equation.

The present author used in [2] the Laplace transform to derive Newton's formulas about the sums of powers of the roots of a polynomial.

In this paper, the Laplace transform will be used to solve an integral-recurrence equation on semi-axis, with discrete-continuous auto-convolution of its unknowns. Namely, applying the Laplace transform on considered equation, we obtain for the transforms of unknowns a first order differential-recurrence equation with discrete auto-convolution of the type studied in [3] and [1]. Using for this equation the general theory given in [3], we find the transforms of unknowns in convenient assumptions, the solutions of the initial equation being obtained by inverse Laplace transform.

2. Convolution products

Two convolution products have been imposed over the time. The first, in continuous-variable case, is *the bilateral convolution* of two integrable functions $u(x)$ and $v(x)$ on real axis, given by formula

$$u(x) \star v(x) = \int_{-\infty}^{\infty} u(t)v(x-t)\,dt\,,$$

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that is reduced to *the unilateral convolution* on real semi-axis or *causal convolution*,

$$u(x) \star v(x) = \int_0^x u(t)v(x-t)\,dt\,, \quad \forall x \geq 0$$

and $u(x) \star v(x) = 0$, $\forall x < 0$, when the factors are causal functions, i.e. $u(x) = v(x) = 0$, $\forall x < 0$. This integral appears, without the notion to be effective considered, in the second half of eighteenth century in independent work of J. R. d'Alembert and P. S. Laplace about Taylor expansion and also in probability theory, by last. The name of *convolution* was firstly used in the paper [6]. It is also called *composition product* after the French and *Faltung* in German literature.

The second, in discrete-variable case, is *the discrete convolution* of two numerical bilateral sequences $(a_n : n \in \mathbb{Z})$ and $(b_n : n \in \mathbb{Z})$, given by formula

$$(a_n) \star (b_n) = \Big(\sum_{k=-\infty}^{\infty} a_k b_{n-k} \Big)\,,$$

and *the causal discrete convolution* or *Cauchy product*,

$$(a_n) \star (b_n) = \Big(\sum_{k=0}^{n} a_k b_{n-k} \Big)\,,$$

when the sequences are unilateral, with indices $n = 0, 1, 2, \ldots$. The last type of sum was used by A. L. Cauchy in connection with the multiplication of numerical and power series, in his famous book *Cours d'Analyse de l'École Royale Polytechnique* which appeared at Paris in 1821. Closely related to this product is *the combinatorial discrete convolution*, defined by formula

$$(a_n) \star_C (b_n) = \Big(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \Big) = (n!)\Big(\Big(\frac{1}{n!} a_n\Big) \star \Big(\frac{1}{n!} b_n\Big) \Big)\,,$$

where the product between the sequence of factorials and the causal discrete convolution must be performed terms by terms.

The causal and combinatorial discrete convolutions can be considered when the terms of the factor sequences belong to a commutative ring. If this ring is composed by functions with continuous-variable convolution as multiplicative product, we get a new product called *discrete-continuous convolution*, given by formula

$$(u_n(x)) \star (v_n(x)) = \Big(\sum_{k=0}^{n} \int_{-\infty}^{\infty} u_k(t) v_{n-k}(x-t)\,dt \Big)\,,$$

for any two sequences with indices $n = 0, 1, 2, \ldots$, of complex-valued integrable functions on real axis. If these functions are causal, then the product take the form

$$(u_n(x)) \star (v_n(x)) = \Big(\sum_{k=0}^{n} \int_{0}^{x} u_k(t) v_{n-k}(x-t)\,dt \Big)\,.$$

In appropriate circumstances, applying Fourier, respective Laplace transform, these convolutions are reduced to discrete convolution $\sum_{k=0}^{n} \hat{u}_k \hat{v}_{n-k}$ between the sequences of transforms.

We also name *combinatorial discrete-continuous convolution* of sequences of causal functions $(u_n(x))$ and $(v_n(x))$, the sequence of functions

$$(u_n(x)) \star_C (v_n(x)) = \Big(\sum_{k=0}^{n} \binom{n}{k} \int_0^x u_k(t) v_{n-k}(x-t)\, dt \Big) .$$

The integral equations considered in this paper will contain these new types of convolution products.

3. First order differential-recurrence equations

In [3] (see also [1] for some particular cases) we considered first order differential--recurrence equations with discrete auto-convolution of the form

$$w_n'(s) = a(s) \sum_{k=0}^{n} w_k(s) w_{n-k}(s) , \qquad n = 0, 1, 2, \dots , \tag{1}$$

where $a(s)$ is a given integrable function and $w_n(s)$, $n = 0, 1, 2, \dots$, is the sequence of unknown differentiable functions of the real variable s. If the initial values $w_n(s_0) \neq 0$, $n = 0, 1, 2, \dots$, in a given real number s_0, are in geometric progression, was proved in [3] that the equation (1) has the solutions

$$w_n(s) = \frac{w_n(s_0)}{B^{n+1}(s)} , \qquad n = 0, 1, 2, \dots , \tag{2}$$

where $B(s) = 1 + w_0(s_0)A(s_0) - w_0(s_0)A(s)$ and $A(s) = \int a(s)\, ds$.

We also considered in [3] a second initial-value problem, where the initial values $w_n(s_0) \neq 0$, $n = 1, 2, \dots$, are in geometric progression, while $w_0(s_0)$ is given by formula

$$w_0(s_0) = \frac{w_1^2(s_0)}{w_2(s_0)} - \frac{1}{A(s_0)} \neq 0 , \tag{3}$$

for $A(s_0) \neq 0$. In this case, the equation (1) has the solutions

$$w_0(s) = \frac{w_0(s_0)}{B(s)} , \tag{4}$$

and

$$w_n(s) = \frac{w_n(s_0) A^{n-1}(s)}{A^{n-1}(s_0) B^{n+1}(s)} , \qquad n = 1, 2, \dots . \tag{5}$$

By analytic extension, follow that these results are true for analytic functions in the case when s is a complex variable.

4. Integral-recurrence equation with discrete-continuous auto-convolution

Theorem 1. *Let $a, b \neq 0$ and s_0 be given complex numbers. The Laplace transformable solutions of the integral-recurrence equation with discrete-continuous* *auto-convolution*

$$xu_n(x) + \sum_{k=0}^{n} \int_0^x u_k(t)u_{n-k}(x-t)\,dt = 0\,, \quad \forall x \geq 0\,,\ n = 0,1,2,\dots\,, \tag{6}$$

satisfying, for $n = 0,1,2,\dots$, *one of the conditions:*

(i)

$$\int_0^\infty e^{-s_0 x} u_n(x)\,dx = ab^n\,, \tag{7}$$

for $\mathrm{Re}(a) < 0$, *respectively*

(ii)

$$\int_0^\infty e^{-s_0 x} u_0(x)\,dx = a - \frac{1}{s_0}\,, \quad \int_0^\infty e^{-s_0 x} u_n(x)\,dx = ab^n\,, \quad \forall n \geq 1\,, \tag{8}$$

for $s_0 \neq 0$ *and* $\mathrm{Re}(a) < \mathrm{Re}\left(\frac{1}{s_0}\right)$, *are given by the formulas:*

(i)

$$u_n(x) = \frac{(-1)^{n+1}}{n!}\left(\frac{b}{a}\right)^n x^n e^{\frac{(as_0+1)x}{a}}\,, \quad \forall x \geq 0\,,\ n = 0,1,2,\dots\,, \tag{9}$$

respectively

(ii)

$$u_0(x) = -e^{\frac{as_0^2 x}{as_0 - 1}}\,, \quad \forall x \geq 0\,, \tag{10}$$

and

$$u_n(x) = \frac{(-1)^{n+1}(ab)^n s_0^{2n}}{(as_0-1)^{2n}} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(as_0-1)^k x^{n-k}}{(n-k)!\,a^k s_0^{2k}}\, e^{\frac{as_0^2 x}{as_0-1}}\,, \quad \forall x \geq 0\,,\ n \geq 1\,. \tag{11}$$

Proof. Applying Laplace transform $L(u(x)) = \hat{u}(s) = \int_0^\infty e^{-sx}u(x)\,dx$ on the equation (6) and taking into account the formulas that give Laplace transforms of the first order derivative and causal convolution product (see [5]), the equation (6) gets the form

$$\hat{u}_n'(s) = \sum_{k=0}^{n} \hat{u}_k(s)\hat{u}_{n-k}(s)\,, \tag{12}$$

for every complex number s with $\mathrm{Re}(s) > r$, where r is a given real number. The equation (12) is of form (1), with $a(s) = 1$, hence $A(s) = \int a(s)\,ds = s$.

(i) From $\mathrm{Re}(a) < 0$, it results $\mathrm{Re}\left(\frac{as_0+1}{a}\right) < \mathrm{Re}(s_0)$, hence we can take $r \in \left(\mathrm{Re}\left(\frac{as_0+1}{a}\right), \mathrm{Re}(s_0)\right)$. Because $\mathrm{Re}(s_0) > r$, from (7) results $\hat{u}_n(s_0) = \int_0^\infty e^{-s_0 x}u_n(x)\,dx = ab^n$, $n = 0,1,2,\dots$, so the initial values are in geometric progression. Also $\hat{u}_0(s_0) = a$, hence $B(s) = 1 + \hat{u}_0(s_0)A(s_0) - \hat{u}_0(s_0)A(s) = 1 + as_0 - as$.

Using formula (2) it results that the equation (12) has for every complex number s, with $\mathrm{Re}(s) > r > \mathrm{Re}\left(\dfrac{as_0+1}{a}\right)$, the solutions

$$(13)\quad \hat{u}_n(s) = \frac{\hat{u}_n(s_0)}{B^{n+1}(s)} = \frac{ab^n}{(1+as_0-as)^{n+1}} = (-1)^{n+1}\left(\frac{b}{a}\right)^n \frac{1}{\left(s-\frac{as_0+1}{a}\right)^{n+1}},$$

for $n = 0, 1, 2, \ldots$. Using inverse Laplace transform, from (13) results (9).

(ii) From $\mathrm{Re}(a) < \mathrm{Re}\left(\frac{1}{s_0}\right)$, it results $\mathrm{Re}\left(\frac{as_0^2}{as_0-1}\right) < \mathrm{Re}(s_0)$, hence we can take $r \in \left(\mathrm{Re}\left(\frac{as_0^2}{as_0-1}\right), \mathrm{Re}(s_0)\right)$. Because $\mathrm{Re}(s_0) > r$, from (8) results $\hat{u}_n(s_0) = \int_0^\infty e^{-s_0x}u_n(x)dx = ab^n$, $n = 1, 2, \ldots$, so these initial values are in geometric progression, and $\hat{u}_0(s_0) = \int_0^\infty e^{-s_0x}u_0(x)\,dx = a - \frac{1}{s_0} = \frac{\hat{u}_1^2(s_0)}{\hat{u}_2(s_0)} - \frac{1}{A(s_0)} \neq 0$. Now, $B(s) = 1 + \left(a - \frac{1}{s_0}\right)(s_0 - s) = \frac{as_0^2-(as_0-1)s}{s_0}$.

In conformity with formulas (4) and (5), the equation (12) has for every complex number s, with $\mathrm{Re}(s) > r > \mathrm{Re}\left(\frac{as_0^2}{as_0-1}\right)$, the solutions

$$(14)\qquad \hat{u}_0(s) = \frac{\hat{u}_0(s_0)}{B(s)} = \frac{as_0-1}{as_0^2-(as_0-1)s} = -\frac{1}{s-\frac{as_0^2}{as_0-1}},$$

and

$$\hat{u}_n(s) = \frac{\hat{u}_n(s_0)A^{n-1}(s)}{A^{n-1}(s_0)B^{n+1}(s)} = \frac{ab^ns_0^2s^{n-1}}{[as_0^2-(as_0-1)s]^{n+1}}$$

$$(15)\qquad = \frac{(-1)^{n+1}ab^ns_0^2}{(as_0-1)^{n+1}}\frac{s^{n-1}}{(s-\frac{as_0^2}{as_0-1})^{n+1}},\quad \forall n \geq 1.$$

For $n \geq 1$ arbitrary fixed, if z is a complex number, we have following partial fraction expansion

$$(16)\qquad \frac{s^{n-1}}{(s-z)^{n+1}} = \sum_{k=0}^{n}\frac{A_k}{(s-z)^{n-k+1}},$$

where A_k are complex numbers that will be determined from the identity

$$(17)\qquad s^{n-1} = \sum_{k=0}^{n}A_k(s-z)^k.$$

From (17), first results $A_n = 0$ and $A_0 = z^{n-1}$. For $n = 1$ the equality (16) is obviously. For $n \geq 2$, derivating (17) of $j = 1, 2, \ldots, n-1$ times, is obtained the identity

$$(18)\quad (n-1)(n-2)\ldots(n-j)s^{n-j-1} = \sum_{k=j}^{n-1}k(k-1)\ldots(k-j+1)A_k(s-z)^{k-j}.$$

For $s = z$, from (18) results

$$A_j = \frac{(n-1)(n-2)\ldots(n-j)}{j!}z^{n-j-1} = \binom{n-1}{j}z^{n-j-1},\quad j = 1, 2, \ldots, n-1.$$

Therefore, the identity (16) takes the form

$$\frac{s^{n-1}}{(s-z)^{n+1}} = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{z^{n-k-1}}{(s-z)^{n-k+1}}\,, \quad n \geq 1\,. \tag{19}$$

From (15) and (19), in which $z = \frac{as_0^2}{as_0-1}$, it results for $n \geq 1$, that

$$\hat{u}_n(s) = \frac{(-1)^{n+1}(ab)^n s_0^{2n}}{(as_0-1)^{2n}} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(as_0-1)^k}{a^k s_0^{2k}(s-\frac{as_0^2}{as_0-1})^{n-k+1}}\,. \tag{20}$$

Using inverse Laplace transform, from (14) and (20) results (10) and (11). □

Remark 1. The integral-recurrence equation with combinatorial discrete-continuous auto-convolution

$$xv_n(x) + \sum_{k=0}^{n} \binom{n}{k} \int_0^x v_k(t)v_{n-k}(x-t)\,dt = 0\,, \quad n = 0,1,2,\dots\,,$$

is reduced to equation (6) by change of unknown $v_n(x) = n!u_n(x)$, $n = 0,1,2,\dots$.

Remark 2. Using Fourier transform instead of Laplace transform, the corresponding bilateral integral-recurrence equations on entire real axis can be solved in the same manner.

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