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AN OBSERVATION ON KRULL AND DERIVED DIMENSIONS OF SOME TOPOLOGICAL LATTICES

M. Rostami* and Ilda I. Rodrigues†

Abstract. Let $(L, \leq)$, be an algebraic lattice. It is well-known that $(L, \leq)$ with its topological structure is topologically scattered if and only if $(L, \leq)$ is ordered scattered with respect to its algebraic structure. In this note we prove that, if L is a distributive algebraic lattice in which every element is the infimum of finitely many primes, then L has Krull-dimension if and only if L has derived dimension. We also prove the same result for *spec* L, the set of all prime elements of L. Hence the dimensions on the lattice and on the spectrum coincide.

1. Preliminaries

In this paper we use the notation and terminology of [4] and [9] throughout.

Let $(P, \leq)$ be a poset. For $x \in P$, we define $\downarrow x = \{y \in P : y \leq x\}$ and $\uparrow x = \{y \in P : x \leq y\}$. For $A \subseteq P$, $\downarrow A = \bigcup\{\downarrow a : a \in A\}$ and $\uparrow A = \bigcup\{\uparrow a : a \in A\}$. The set A is called a *lower set* if $A = \downarrow A$ and it is called an *upper set* if $A = \uparrow A$. Let us recall that a lattice $(L, \vee, \wedge, \leq)$ is a poset in which every pair of elements has a join and a meet. We denote a lattice simply by L or by $(L, \leq)$ if we want to emphasize on order structure of L. A lattice is *complete* if any of its subsets has join and meet.

For a lattice $(L, \leq)$ and $D \subseteq L$, D is *directed* if $D \neq \emptyset$ and for $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$. Let $(L, \leq)$ be a complete lattice. Then, x is said to be *way below* y, written $x \ll y$, if for every directed subset $D \subseteq L$

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the relation $y \leq \sup D$ always implies the existence of $d \in D$ such that $x \leq d$. An element $x \in L$ is called *compact* if $x \ll x$. We denote the set of all compact elements of L by $K(L)$. For example, in the lattice [0, 1], if $x \ll y$ then either $x < y$ or $x = 0$. Therefore 0 is the compact element of L. In a lattice, if $x, y \ll z$, then $x \vee y \ll y$ is directed for all y. A complete lattice $(L, \leq)$ is called *continuous* if $x = \sup\{y \in L : y \ll x\}$, $x \in L$. The lattice $(L, \leq)$ is called *algebraic* if it is complete and satisfies the following axiom: $x = \sup\{\downarrow x \cap K(L)\}$, for every $x \in L$. Clearly, algebraic lattices are continuous.

2. The spectral theory of algebraic lattices

Let us recall that a Boolean algebra is a distributive lattice with top and bottom elements and in which every element has a complement. The classical Stone Representation Theorem for Boolean algebra states that any Boolean algebra B is associated to a totally disconnected (zero-dimensional) compact Hausdorff topological space *spec* B, called *Stone space* and conversely, the lattice of *clopen* (closed-open) subsets of any topological space X is a Boolean algebra, *clop* X. Moreover, a Boolean algebra B and its counterpart, *clop* (*spec* B), are dually isomorphic.

An interpretation of the Stone duality theorem can be found in spectral theory of continuous lattices which seeks to represent a lattice as the lattice of open (or closed) subsets of a topological space. According to the Stone duality of Boolean algebras and Stone spaces, it is possible to recover any Boolean algebra B from the clopen subsets of the space of maximal ideals of B. For more details, we refer to [6].

For an algebraic lattice L, with compact elements $K(L)$, we recall the following topologies (see [4]).

(1) Lower topology: $\omega(L)$ has a closed sub-basis, namely the sets $\uparrow x$, $x \in L$;

(2) Scott topology: $\sigma(L)$ has an open basis, the sets k, $k \in K(L)$;

(3) Lawson topology: $\lambda(L)$ is the join of $\omega(L)$ and $\sigma(L)$.

The Lawson topology or λ-topology has a basis, the sets $U \setminus \uparrow F$ where $U \in \sigma(L)$ and F is finite in L. The following theorem, in [9], relates the algebraic lattices to the λ-topology.

Theorem 2.1. *Let L be an algebraic lattice. Then*

(1) *The λ-topology with basis all sets of the form $\uparrow k \setminus \bigcup_{i=1}^{n}$, where $k \in K(L)$ and $x_1, x_2, \ldots, x_n \in L$, is a compact Hausdorff topology on L relative to which L is* totally disconnected *and, as such, is isomorphic to the lattice of Lawson clopen upper sets of L.*

(2) *With respect to this λ-topology, the map $(x, y) \to inf\{x, y\}$, $L \times L \to L$, is continuous. In fact, relative to λ-topology, L (algebraic) is a compact zero-dimensional semilattice (see* [4]*). The converse of the above theorem is also true, namely, if L is a compact totally disconnected topological semilattice, then L is Lawson semilattice.*

We now turn our attention to distributive algebraic lattices.

Definition 2.1. In a complete distributive algebraic lattice $(L, \leq)$, an element $x \in L$ is called prime if for every $a, b \in L$, $a \wedge b \leq x$ implies that $a \leq x$ or $b \leq x$. An element $x \in L$ is called (join) irreducible if $x = a \vee b$ implies that $x = a$ or $x = b$. Denote by *spec* L the set of all prime elements of L other than 1. We have $y = \inf(\uparrow y \cap spec\ L)$, $y \in L$. Hence algebraic lattices have an abundance of primes. This implies that *spec* L order generates L.

Now we can ask the following natural question: Is it possible to recover a distributive algebraic lattice L from *spec* L?
To answer this question, we need a short review of the spectral theory of these lattices. The spectral theory, which plays an important role in commutative ring theory, algebraic geometry and $C^* - algebra$, simply serves the purpose of representing the lattice L (or any other algebraic structure) as a lattice of open (closed) subsets of a topological space X. As we have already remarked, the idea originates from the Stone representation theory of Boolean algebra.

Let $h(a) = \{p \in spec\ L : a \leq p\} = \uparrow a \cap spec\ L$, then $h(a)$ is called the *hull* of a. Let *spec* L be given a topology with closed sets $\{h(a) : a \in L\}$. This topology is called *hull-kernel topology* on *spec* L (in [4]). With respect to this topology there is a lattice isomorphism

$$\begin{aligned} &L \to \Gamma(spec\ L) = \text{ closed subsets of } spec\ L \\ &x \mapsto \uparrow x \cap spec\ L\,. \end{aligned}$$

So, if $X = spec\ L$ is equipped with hull-kernel topology then L and $\Gamma(X)$ are isomorphic lattices. Hence every distributive algebraic lattice can be recovered from hull-kernel closed subsets of its spectral space and it has the structure of a topology. In other words, *spec* L induces a topological structure over the algebraic lattice L. For the order-theoretic topology and its applications, we refer to [10].

Let $(P, \leq)$ be a poset and $A \subseteq P$. A is called *order-dense* if, given $a, b \in P$ $(a < b)$, there exists $c \in A$ such that $a < c < b$. If a poset $(P, \leq)$ has no order-dense chain then P is said to be *order-scattered*.

A topological space X is called *topologically scattered* if every subset A of X has a relatively isolated point.

These concepts have a long history dating back to [5]. For their confrontation see [8] and [9] (see Theorem 3.1 below); further, see e.g. [2], [12] or [14].

3. Krull and derived dimensions of L

In this section we first define the Krull-dimension and the derived dimension in algebraic lattices. Then by reformulating a theorem of Mislove (Theorem 3.1) and we show that, in an algebraic lattice in which every element is the infimum of finitely many primes, the notion of dimensions on the lattice and on the spectrum coincide.

Definition 3.1. The Krull-dimension, $K(\dim L)$, of a distributive lattice L is defined by transfinite induction as follows:

$K(\dim L) = -1$ if and only if $L = 0$; if α is an ordinal number and $K(\dim L) \nless \alpha$, then $K(\dim L) = \alpha$ provided that for every descending chain $x_1 \geq x_2 \geq \ldots$ of elements of L there is a natural number n such that $K\left(\dim\left[x_{i+1}, x_i\right]\right) < \alpha$ for $i \geq n$ (see [13]).

Now we quote from [7] the following definition.

Definition 3.2. Let X be a topological space. For any ordinal α define derived set of order α by $X_0 = X$, $X_1 = X'$, the limit points of X, and $X_{\alpha+1} = X_\alpha^{'}$; if α is limit ordinal then $X_\alpha = \bigcap\limits_{\beta<\alpha} X_\beta$. Hence $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_\alpha \supseteq X_{\alpha+1} \supseteq \ldots$. The smallest ordinal α such that $X_\alpha = \emptyset$ is called derived dimension of X and it is denoted by $d'(X)$.

Example 3.1. Let $A = \left\{\left(\frac{1}{n}, \frac{1}{m}\right) : n, m \in \mathbb{N}\right\}$, $B = \left\{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\right\}$, $C = \left\{\left(0, \frac{1}{m}\right) : m \in \mathbb{N}\right\}$ and $X = A \cup B \cup C$ be subsets of Euclidean plane with usual topology. Then, the set of all isolated points of X, $\mathit{Iso}\,(X) = A$. Now clearly, we have that $X_2 = \{(0,0)\}$ and $X_3 = \emptyset$. Hence $d'(X) = 3$.

We now turn our attention to *spec* L. Recall that the lattice L is indeed the topology of *spec* L (with hull-kernel topology). But *spec* L has also order structure, so the notions of topologically scattered and order scattered both make sense. As the following example shows, *spec* L is independent of L in having the above dimensions.

Example 3.2. Let $L = \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{0\}$. Then L is a distributive algebraic lattice and *spec* $L = L \setminus \{1\}$. Now the hull-kernel topology on *spec* L is the set $\{\mathit{spec}\ L \setminus \{x\} : x \in L\}$, which is the same as the family $\left\{\left[0, \frac{1}{n}\right] \cap L : n > 1\right\}$. Then *spec* L does not have any isolated points in this topology. Consequently, *spec* L cannot have derived dimension. But clearly L has both of the above mentioned dimensions.

However, there is another topology on *spec* L which *spec* L inherits from λ-topology (Lawson topology). This topology, which arises as the join of the topology and its "complement", is called the *patch topology*. With respect to this topology we have the following theorem [9].

Theorem 3.1 (Mislove)**.** *Let L be a distributive algebraic lattice in which every point is the infimum of finitely many primes. Then the following statements are equivalents.*

(1) *L is topologically scattered*

(2) *L is order scattered*

(3) *spec L is topologically scattered*

(4) *spec L is order scattered*

Remark 3.1. In this theorem, if the distributive lattice L is not algebraic, or if in L some point is not the infimum of finitely many primes, then L is neither order scattered nor topologically scattered ([9]).

This is rather deep and fundamental theorem in order topology and a lot of effort has been made for its demonstration.

Now, our objective is the reformulation of this important theorem in the language of Krull and derived dimensions. We believe that for algebraist this *dimension modification* of the theorem is even more useful than its original form.

Theorem 3.2. *Suppose that L is a distributive algebraic lattice in which every point is the infimum of finitely many primes. Then the following are equivalent.*

(1) *L has derived dimension*

(2) *L has Krull-dimension*

(3) *spec L has derived dimension*

(4) *spec L has Krull-dimension*

Note that the restriction on algebraic lattice L is exactly the same as saying that the Boolean lattice $2^{\mathbb{N}}$ is not contained in L.

Proof. We only need to show that (1) $\Leftrightarrow$ (2). Then by Theorem 3.2 all four conditions are equivalent.

Let $D = \{r = \frac{n}{2^m} : n, m \in \mathbb{N}, n \leq 2^m\}$ be the set of all dyadic rationales between 0 and 1. Let P be a poset and $a, b \in P$. A dyadic chain from a to b is an injective map $\varphi\colon D \to P$ such that $r < s \to \varphi(r) < \varphi(s)$, $r, s \in D$; $\varphi(0) = a$ and $\varphi(1) = b$.

Now if P fails to have Krull-dimension, a totally ordered subset of P which is order-dense in itself may be built up using the observation that whenever an interval $[x,\, z]$ in P does not have Krull dimension, there exists an element $y \in P$ such that $x < y < z$ and the intervals $[x,\, y]$ and $[y,\, z]$ both fail to have Krull dimensions. In fact by the theorem of Lemonnier [9, Theorem 3.1.10, pp.125], the poset P has Krull-dimension if and only if P has no subset order isomorphic to the set of all dyadic rationals D. By [4], there exists a dyadic chain from a to b in P if and only if there is a subset $S \subseteq P$ sucha that $a, b \in S$ and S is order-dense in P. Recall that, by definition, a poset P is order scattered if and only if P has no order dense chain. This is equivalent to the fact that the chain D is not embeddable in it. Therefore P is order scattered if and only if P has Krull-dimension.

Now let X be a topological space. Since X is a set there exists a least ordinal α with $X_\alpha = X_{\alpha+1} = \ldots$. But X_α is a perfect set, this means that it is closed without having any isolated points. Hence $X_\alpha = \emptyset$ if and only if $X \setminus X_\alpha = X$; or, equivalently, every subset of X has an isolated point. this implies that X is scattered if and only if X has derived dimension. But since in distributive algebraic lattice L the notions order scattered and topologically scattered are equivalent, we have L has Krull-dimension if and only if L has derived dimension. Hence if L is an algebraic lattice in which every element is the infimum of finitely many elements, the above dimensions on L and on *spec* L coincide. This completes the proof of the theorem. $\square$

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Departamento de Matemática, Universidade da Beira Interior,
6200 Covilhã, Portugal
E-mail: rostami@ubi.pt ilda@ubi.pt

$*$ $\dagger$