ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 251–256

ON LOCAL ISOMETRIC IMMERSIONS INTO COMPLEX AND QUATERNIONIC PROJECTIVE SPACES

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Abstract. We will prove that if an open subset of $\mathbb{C}P^n$ is isometrically immersed into $\mathbb{C}P^m$, with $m < (4/3)n - 2/3$, then the image is totally geodesic. We will also prove that if an open subset of $\mathbb{H}P^n$ isometrically immersed into $\mathbb{H}P^m$, with $m < (4/3)n - 5/6$, then the image is totally geodesic.

1. Introduction

It is a fundamental question in submanifold theory as to whether a given Riemannian manifold is (locally) isometrically immersible into another Riemannian manifold. A subsequent, and even more fundamental, question is whether this immersion, if it exists, is locally rigid.

Most such results have been on either isometric immersions into spaces of constant curvature, isometric immersions with some additional conditions, or isometric immersions of codimension 1, (cf. Tomter [10]).

Let $\mathbb{C}P^n \subset \mathbb{C}P^m$ be the standard embedding of the complex projective n-space into the complex projective m-space. In this article we will prove that, for low codimensions, each open region U of $\mathbb{C}P^n$ is rigid in $\mathbb{C}P^m$ in the class of real homothetic immersions. Those are real conformal immersions with constant nonzero conformality factor.

Let $\mathbb{H}P^n \subset \mathbb{H}P^m$ be the standard embedding of the quaternionic projective n-space into the quaternionic projective m-space. We will prove that, for low codimensions, each open region U of $\mathbb{H}P^n$ is rigid in $\mathbb{H}P^m$ in the class of real homothetic immersions.

Remark. The results in this paper are easily extended to rigidity in the class of conformal immersions, (e.g. see [9]).

The following definition will be central.

Definition. A totally real map is a real linear map between complex vector spaces with the property that all images of totally real subspaces are totally real. A totally

2010 *Mathematics Subject Classification*: primary 53C40.
Key words and phrases: submanifolds, homogeneous spaces, symmetric spaces.
This work was a part of the Dr. Scient. degree of the author. The work was financed by the Norwegian Research Council.
Received April 28, 2011. Editor J. Slovák.

real morphism is a diffeomorphism between almost complex manifolds such that its derivative is totally real at each point.

2. Isometric immersions into Complex projective space

Let $n < m$ be natural numbers. Let $M = \mathbb{C}P^n$ and $N = \mathbb{C}P^m$ be complex projective spaces, both equipped with the Fubini-Study metric. Further, let $f\colon U \subset M \to N$ be an isometric immersion, where U is an open set in M. The number $k = 2m - 2n$ denotes the real codimension of f. We let J^M and J^N be the almost complex structures on M and N respectively. Let f_* be the differential of f and let π_f be the projection of $T_{f(p)}N$ onto $f_{*(p)}(T_pM)$. Consider the following diagram:

$$\begin{tikzcd} & T_{f(p)}N \arrow[d, "\pi_f"] \\ T_pM \arrow[ur, "f_{*(p)}"] \arrow[r, "f_{*(p)}"] & f_{*(p)}(T_pM) \end{tikzcd} \tag{1}$$

Lemma 1. *If $f\colon M \to N$ is as above and $k < (1/2)m - 1$, then f is a totally real morphism.*

Proof. Since f is an immersion, the following is true for all $p \in U$

$$\operatorname{rank}\bigl(\pi_f \circ R_N(f_{*(p)}X, f_{*(p)}Y) \circ f_{*(p)} - f_{*(p)} \circ R_M(X,Y)\bigr) \le 2k\,. \tag{2}$$

For reference see Agaoka [2]. The curvature of $(\mathbb{C}P^n, J)$ is [3]

$$R(X,Y) = \alpha\bigl(X \wedge Y + JX \wedge JY + 2\langle X, JY\rangle J\bigr)\,, \tag{3}$$

where $X \perp Y$ and α is the holomorphic curvature. When $\langle X, J^N Y\rangle \neq 0$, the rank of $R_N(X,Y)$ is $2m$. So if $\langle f_{*(p)}X, J^N f_{*(p)}Y\rangle \neq 0$, the rank of $\pi_f \circ R_N\bigl(f_{*(p)}X, f_{*(p)}Y\bigr) \circ f_{*(p)}$ is at least $2m - 2k$. Let $Y \perp X, J^M X$. Then $\operatorname{rank} R_M(X,Y) = 4$. Thus, the rank of $\pi_f \circ R_N(f_{*(p)}X, f_{*(p)}Y) \circ f_{*(p)} - f_{*(p)} \circ R_M(X,Y)$ is at least $2m - 4 - 2k$. The inequality (2) gives $2m - 4 \le 4k$, we therefore have that $\langle f_{*(p)}X, J^N f_{*(p)}Y\rangle = 0$ whenever $k < (1/2)m - 1$ and $Y \perp X, J^M X$. □

Let ω be the symplectic form on TM, $\omega(X,Y) = \langle X, J^M Y\rangle$, and define the form $\tilde{\omega}(X,Y) = \langle f_* X, J^N f_* Y\rangle$.

Lemma 2. *If f is a totally real morphism, then $\tilde{\omega} = c\,\omega$, where c has its values in $[-1, 1]$.*

Proof. Per definition, the maximal subgroup of $SO(2n)$ that fixes ω is $U(n)$. Since $U(n)$ acts transitively on the family of all totally real subsets of T_pM, $U(n)$ will fix $\tilde{\omega}$ too. It is known that $SO(2n)/U(n)$ is an irreducible symmetric space [7], so we have that ω and $\tilde{\omega}$ are proportional. It is trivial to show that $|c| \le 1$. □

Lemma 3. *If f is a totally real morphism and $m < 2n$, then f_* is a complex linear map up to conjugation at each point. Thus, f is almost (anti) holomorphic.*

Proof. We only have to show that $|c| = 1$. If $|c| \neq 1$, we can construct $4n$ linearly independent vectors in $T_{f(p)}N$, but that makes $m \ge 2n$. □

Proposition 1. *If $k < (1/2)m - 1$ and f is an isometric immersion, then M and N have the same holomorphic curvatures.*

Proof. From Lemma 3 one knows that $f_*(J^M X) = J^N f_* X, \forall X \in T_pM$. We therefore have $R_N(f_*X, f_*(J^M X)) - f_* R_M(X, J^M X) = 2(\alpha_N - \alpha_M)[f_*X \wedge f_*(J^M X) + \langle X, X\rangle J^N]$. Agaoka's inequality (2) and $k < (1/2)m - 1$ implies that $\alpha_N - \alpha_M = 0$. □

We recall the notion of the subspace of nullity:

$$\Gamma(p) = \{X \in T_pM | R_M(X,Y) = (R_N(X,Y)|_{T_pM})^{\top}, \text{ for every } Y \in T_pM\}\,. \tag{4}$$

Notice that from Proposition 1 and Lemma 3, we have $\Gamma(p) = T_pM$ for all $p \in U$, (i.e. f is of full nullity).

Proposition 2. *If M and N are Kähler manifolds and f is an (anti) holomorphic isometric immersion of full nullity, then f is totally geodesic.*

Proof. The proposition is a special case of a result of Küpelî [8]. □

The following theorem follows from Proposition 1 and Proposition 2.

Theorem 1. *For an open set $U \subset \mathbb{C}P^n$, if $f\colon U \to \mathbb{C}P^m$ is an isometric immersion and $m < (4/3)n - 2/3$, then f is totally geodesic.*

3. Isometric immersions into Quaternionic projective space

Let $n < m$ be natural numbers, $M = \mathbb{H}P^n$, and $N = \mathbb{H}P^m$. I.e. M and N are quaternionic projective spaces of real dimensions $4n$ and $4m$ respectively. Let both spaces have the usual symmetric metric. Let $f : U \subset M \to N$ be an isometric immersion, where U is an open set in M. Let $k = 4m - 4n$ denote the real codimension of f. $\{I^M, J^M, K^M\}$ and $\{I^N, J^N, K^N\}$ are the quaternionic structures on M and N respectively. The curvature of $\mathbb{H}P^*$ is [3]

$$\begin{aligned} R(X,Y) = \alpha\big(&X \wedge Y + IX \wedge IY + JX \wedge JY + KX \wedge KY \\ &+ 2\langle X, IY\rangle I + 2\langle X, JY\rangle J + 2\langle X, KY\rangle K\big)\,, \end{aligned} \tag{5}$$

where $X \perp Y$ and α is a positive constant. Let $\langle X, aI^NY + bJ^NY + cK^NY\rangle \neq 0$, for some $a, b, c \in \mathbb{R}$. Then, the rank of $R_N(X,Y)$ is at least $4m-2$. So the rank of

$$\pi_f \circ R_N\big(f_{*(p)}X, f_{*(p)}Y\big) \circ f_{*(p)}$$

is at least $4m - 2 - 2k$.

Let $Y \perp X, I^MX, J^MX, K^MX$. Then $\mathrm{rank} R_M(X,Y) = 8$. Thus, we have that the rank of $\pi_f \circ R_N(f_{*(p)}X, f_{*(p)}Y) \circ f_{*(p)} - f_{*(p)} \circ R_M(X,Y)$ is at least $4m - 10 - 2k$. From the inequality (2) we have that $4m - 10 \leq 4k$. We therefore have that $\langle f_{*(p)}X, I^N f_{*(p)}Y\rangle = 0$, $\langle f_{*(p)}X, J^N f_{*(p)}Y\rangle = 0$, and $\langle f_{*(p)}X, K^N f_{*(p)}Y\rangle = 0$ whenever $k < m - 5/2$ and $Y \perp X$, I^MX, J^MX, K^MX. This establishes the following lemma.

Lemma 4. *If $f\colon M \to N$ is as above and $k < m - 5/2$, then f is a totally real morphism*[1] *with respect to I^M, J^M, and K^M.*

Let ω be the $\mathbb{R}^3$-valued "symplectic form" on TM,

$$\omega(X,Y) = (\langle X, I^M Y\rangle, \langle X, J^M Y\rangle, \langle X, K^M Y\rangle)\,,$$

and define the $\mathbb{R}^3$-valued form

$$\tilde{\omega}(X,Y) = (\langle f_* X, I^N f_* Y\rangle, \langle f_* X, J^N f_* Y\rangle, \langle f_* X, K^N f_* Y\rangle)\,.$$

Lemma 5. *If f_* is a totally real morphism, then $\tilde{\omega} = A \circ \omega$, where $A \in GL(R^3)$, with $\|A\| \leq 1$.*

Proof. $Sp(n)Sp(1)$ is the maximal subgroup of $SO(4n)$ that preserves the family of totally real subsets of $\mathbb{H}^n \simeq \mathbb{R}^{4n}$; $Sp(n)Sp(1)$ is the normalizer of $Sp(1)$ in $SO(4n)$, [6]. So the quotient space $SO(4n)/Sp(n)Sp(1)$ represents all orthogonal quaternionic structures on $\mathbb{R}^{4n}$. $\mathfrak{sp}(n)$ and $\mathfrak{sp}(1) \subset \mathfrak{so}(4n)$ are irreducible representations of $Sp(n)Sp(1)$ and it is known that $SO(4n)/Sp(n)Sp(1)$ is an isotropy irreducible homogeneous space for $n > 1$ [11], so we have the irreducible decomposition $\mathfrak{so}(4n) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \mathfrak{p}$ of non-equivalent $Sp(n)Sp(1)$-representations. The three components of ω may be viewed as a basis of $\mathfrak{sp}(1)$ in a natural way. ω is fixed by $Sp(n)Sp(1)$ and ω and $\tilde{\omega}$ are zero on all totally real subspaces of $\mathbb{H}^n$. The subrepresentation of $\mathfrak{so}(4n)$ generated by the components of $\tilde{\omega}$ must be $\mathfrak{sp}(1)$, since this is the only one that vanishes on totally real subspaces of $\mathbb{H}^n$. That is, $\tilde{\omega} = A \circ \omega$ where A is a 3×3 matrix. It is trivial to show that $\|A\| \leq 1$. ☐

Lemma 6. *If f_* is a totally real morphism and $m < 2n$, then f_* is a quaternionic linear map up to general conjugation. Thus, f is quaternionic analytic.*

Proof. We only have to show that $A \in O(3)$. If $A \notin O(3)$, we can construct $8n$ linearly independent vectors in $T_{f(p)}N$, but that makes $m \geq 2n$. ☐

Proposition 3. *If $k < m - 5/2$ and f is a local isometric immersion, then M and N have the same holomorphic curvatures.*

Proof. From Lemma 4 and Lemma 6 one knows that $f_*(I^M X) = I^N f_* X$, $f_*(J^M X) = J^N f_* X$, and $f_*(K^M X) = K^N f_* X$ $\forall X \in T_p M$. We therefore have $R_N(f_* X, f_*(I^M X)) - f_* R_M(X, I^M X) = 2(\alpha_N - \alpha_M)[f_* X \wedge f_*(I^M X) - f_*(J^M X) \wedge f_*(K^M X) + \langle X, X\rangle I^N]$. Agaokas inequality (2) and $k < m - 5/2$ implies that $\alpha_N - \alpha_M = 0$. ☐

A theorem of Alfred Gray [6, Theorem 5, page 127] is:

Theorem 2. *Let $\tilde{N}$ be a quaternionic Kähler manifold, and suppose M is a quaternionic submanifold of $\tilde{N}$. Let M have the induced Riemannian structure of $\tilde{N}$. Then M is a quaternionic Kähler manifold and M is totally geodesic in $\tilde{N}$.*

Thus from this theorem and Proposition 3 we have the following theorem.

Theorem 3. *Let $f\colon \mathbb{H}\mathrm{P}^n \longrightarrow \mathbb{H}\mathrm{P}^m$ be an isometric immersion. If $m < (4/3)n - 5/6$, then f is totally geodesic.*

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[1]A totally real morphism between quaternionic vector spaces is a real linear map where the images of totally real subspaces are totally real.

4. Discussion

Since $n < m < (4/3)n - 2/3$ we have that $m \geq 7$. So the examples of lowest dimension where Theorem 1 applies are

$$\begin{aligned} M = \mathbb{C}P^6 &\quad\to\quad N = \mathbb{C}P^7 \\ M = \mathbb{C}P^7 &\quad\to\quad N = \mathbb{C}P^8 \\ M = \mathbb{C}P^8 &\quad\to\quad N = \mathbb{C}P^9 \\ M = \mathbb{C}P^9 &\quad\to\quad N = \mathbb{C}P^{11} \\ M = \mathbb{C}P^{10} &\quad\to\quad N = \mathbb{C}P^{12} \end{aligned}$$

It would be of interest to improve the results in this article to local isometric immersions of spaces of lower dimensions such as $\mathbb{C}P^2$ into $\mathbb{C}P^3$. An article on this is in preparation.

From Proposition 1 we have that for $m < (4/3)n - 2/3$, there exists no local isometric immersions of $\mathbb{C}P^n$ into $\mathbb{C}P^m$ if these spaces have different holomorphic curvature. Dajczer and Rodriguez (see [4, 5]) have proved the following. If $f\colon M^{2n} \to \mathbb{C}Q^m_c$, $n \geq 2$, is a local isometric immersion of a Kähler manifold into a complex space form of constant holomorphic curvature $c \neq 0$ such that at one point the sectional curvature of M satisfies $K_M \leq c/4$ and $m < (3/2)\,n$, then f is holomorphic.

From Agaoka[1] we know that if $\mathbb{C}P^n$ can be local isometrically immersed into Euclidean space $\mathbb{R}^{2n+k}$, then $k \geq \frac{1}{5}(6\,n - 4)$. This result uses only the Gauss equations.

Let $\mathbb{C}P^n$ and $\mathbb{C}P^m$ have different maximal sectional curvature. Since the Gauss equations for holomorphic isometric immersions of $\mathbb{C}P^n$ into $\mathbb{C}P^m$ with different maximal sectional curvature are equivalent to the the Gauss equations for isometric immersions of $\mathbb{C}P^n$ into Euclidean space $\mathbb{R}^{2n+k}$, the consequence of these results of Dajczer, Rodriguez, and Agaoka is:

If $\alpha_M \leq (1/4)\,\alpha_N$, $n > (2/3)\,m$ and $n \geq 2$, then there are no local real isometric immersions of M into N.

My thesis “On Isometric and Conformal immersions into Riemannian Manifolds” [9] contains the results of this article. It also contains similar results for local conformal immersions and isometric immersions of homogeneous spheres into complex and quaternionic projective space. The hyperbolic cases are also considered.

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