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A CLASS OF METRICS ON TANGENT BUNDLES OF PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We provide the tangent bundle TM of pseudo-Riemannian manifold (M,g) with the Sasaki metric g^s and the neutral metric g^n. First we show that the holonomy group H^s of (TM,g^s) contains the one of (M,g). What allows us to show that if (TM,g^s) is indecomposable reducible, then the basis manifold (M,g) is also indecomposable-reducible. We determine completely the holonomy group of (TM,g^n) according to the one of (M,g). Secondly we found conditions on the base manifold under which (TM,g^s) (respectively (TM,g^n)) is Kählerian, locally symmetric or Einstein manifolds. (TM,g^n) is always reducible. We show that it is indecomposable if (M,g) is irreducible.

1. Introduction

Let (M,g) be a Riemannian manifold. This gives rise to Sasaki metric g^s on the tangent bundle TM. g^s is very rigid in the following sense. When we impose to (TM,g^s) to be locally symmetric (respectively Kählerian or Einstein) manifold, the basis manifold (M,g) must be flat (see [22, 16]). In this paper we study the general case when (M,g) is a pseudo-Riemannian manifold. If (r,s) is the signature of g, the one of g^s is $(2r,2s)$. We prove that g^s is not always rigid when (M,g) is not Riemanniann or Lorentzian manifold. But some very strong conditions are imposed on (M,g). For example if we impose to (TM,g^s) to be locally symmetric, (M,g) must be reducible and its holonomy algebra hol verifies $hol^2=\{0\}$. If we impose to (TM,g^s) to be an Einstein manifold, (M,g) must be reducible, Ricci-flat and $\mathrm{tr}(A^2)=0,\ \forall\, A\in hol$.
We can provide the tangent bundle with another natural metric g^n of neutral signature (see § 4). We determine completely the holonomy algebra of (TM,g^n) according to the one of the basis manifold. the holonomy group of (TM,g^n) leaves invariant the vertical direction witch is totaly isotrope. Hence it is always reducible. (TM,g^n) is not rigid. We prove that it is locally symmetric if and only if (M,g) is locally symmetric and $hol^2=0$. (TM,g^n) is an Einstein manifold, if and only if it is Ricci flat if and only if (M,g) is Ricci flat. Further if (M,g) is a Kählerian pseudo-Riemannian manifold then (TM,g^n) is also a Kählerian pseudo-Riemannian manifold.

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The classification of indecomposable reducible pseudo-Riemannian manifolds remain again an open problem, called holonomy problem. The study of (TM, g^s) and (TM, g^n) permits to construct examples of indecomposable reducible pseudo-Riemannian manifolds. Hence this paper is a contribution to the resolution of the holonomy problem. We recall that this problem is only solved in the Lorentzian case ([5, 20, 8, 9, 17, 19, 24, 25]). The case of neutral signature has been studied ([6, 23]. Even the indecomposable reducible locally symmetric spaces are not yet classified, with the exception of the case of index ≤ 2 ([13, 14, 12]).

2. Preliminaries

2.1. **Results on the Classification of pseudo-Riemannian manifolds.** Let (M, g) be a connected simply connected pseudo-Riemannian manifold of signature (r, s) $(m = r + s)$. We denote by H its holonomy group at a point p.

Definition 1. (M, g) is called *irreducible* if its holonomy group $H \subset O(T_pM, g_p)$ do not leave any proper subspace of T_pM. (M, g) is called *indecomposable* if H do not leave any non-degenerate proper subspace of T_pM.

De Rham-Wu's splitting theorem reduces the study of complete simply connected pseudo-Riemannian manifolds to indecomposables ones.

Theorem 1 ([26, 15])**.** *Let (M, g) be a simply connected complete pseudo-Riemannian manifold of signature (r, s). Then (M, g) is isometric to a product eventually of flat pseudo-Riemannian manifold and of complete simply connected indecomposable pseudo-Riemannian manifolds.*

The irreducible pseudo-Riemannian symmetric spaces were classified by M. Berger in [4]. The list of possible holonomy groups of irreducible non locally symmetric pseudo-Riemannian is given by M. Berger and R. L. Bryant in following theorem

Theorem 2 ([3, 11])**.** *Let (M, g) be a simply connected irreducible non locally symmetric pseudo-Riemannian manifold of signature (r, s). Then its holonomy group is (up to conjugacy in $O(r, s)$) one of the following groups:*
$SO(r, s)$, $U(r, s)$, $SU(r, s)$, $\mathrm{Sp}(r, s)$, $\mathrm{Sp}(r, s) \cdot \mathrm{Sp}(1)$, $SO(r, \mathbb{C})$, $\mathrm{Sp}(p) \cdot SL(2, R)$, $\mathrm{Sp}(p, \mathbb{C}) \cdot SL(2, \mathbb{C})$, $\mathrm{Spin}(7)$, $\mathrm{Spin}(4, 3)$, $\mathrm{Spin}(7)^{\mathbb{C}}$, G_2, $G^*_{2(2)}$, $G_2^{\mathbb{C}}$.

The complete classification of the indecomposable reducible subalgebras $\mathfrak{h}$ of $so(1, 1 + n)$ is given by the following theorem.
We consider on $\mathbb{R}^m$ $(m = n + 2)$ the following Lorentzian scalar product defined by

$$\langle (x_0, x_1, \ldots, x_{n+1}), (y_0, y_1, \ldots, y_{n+1}) \rangle = x_0 y_{n+1} + x_0 y_{n+1} - \sum_{i=1}^{i=n} x_i y_i \,,$$

Theorem 3 ([5])**.** *Let $\mathfrak{h}$ be an indecomposable subalgebra of $so(\langle , \rangle)$ which leaves invariant the light-like direction $\mathbb{R}e_0$. Then*

A) *$\mathfrak{h}$ is a subalgebra of the following algebra*

$$\big(\mathbb{R}\oplus so(n)\big)\ltimes\mathbb{R}^n=\left\{\begin{pmatrix}a & X & 0\\ 0 & A & -{}^tX\\ 0 & 0 & -a\end{pmatrix}\mid a\in\mathbb{R},\ X\in\mathbb{R}^n,\ A\in so(n)\right\}$$

and

• *either $\mathfrak{h}$ contains $\mathcal{N}\cong\mathbb{R}^n$,*

• *or, there exist a a nontrivial decomposition $n=p+q$ and $\mathbb{R}^n=\mathbb{R}^p\oplus\mathbb{R}^q$, a nontrivial abelian subalgebra $\mathcal{C}$ of $so(p)$ (eventually 0), a semisimple subalgebra $\mathcal{D}$ of $so(p)$, commuting with $\mathcal{C}$ and a surjective linear application $\varphi\colon\mathcal{C}\to\mathbb{R}^q$ such that, up to conjugacy in $(\mathbb{R}\oplus so(n))\ltimes\mathbb{R}^n$, $\mathfrak{h}$ is the subalgebra of $(\mathbb{R}\oplus so(n))\ltimes\mathbb{R}^n$, of the following "block" matrixes*

$$\left\{\begin{pmatrix}0 & X & \varphi(A) & 0\\ 0 & A+B & 0 & -{}^tX\\ 0 & 0 & 0 & -{}^t\varphi(A)\\ 0 & 0 & 0 & 0\end{pmatrix}\mid A\in\mathcal{C},B\in\mathcal{D},X\in\mathbb{R}^p\right\}.$$

B) *If we denote by $\mathcal{G}$ the projection of $\mathfrak{h}$ on $so(n)$ with respect to $\mathbb{R}\oplus\mathcal{N}$, the representation of $\mathfrak{h}$ in $\mathbb{R}^n$ is the exterior direct some representation of a trivial representation(eventually) and r irreducible representation $\mathcal{G}_i$.*

The algebras classified in Theorem 3 were all achieved like holonomy algebra of Lorentzian metrics ([5, 20, 8, 17, 19, 24, 25]).

2.2. **The tangent bundle TM.** Let (M,g) be a pseudo-Riemannian manifold and D its Levi-Civita connection. We denote by $\pi\colon TM\to M$ the tangent bundle. The subspace $\mathcal{V}_{(p,u)}=\mathrm{Ker}(d\pi_{|(p,u)})$ is called the vertical subspace of $T_{(p,u)}TM$ at (p,u). The connection application is the application $K_{(p,u)}\colon T_{(p,u)}TM\to T_pM$ defined by

$$K_{(p,u)}(dZ_p(X_p))=(D_XZ)_p\,,$$

where $Z\in\mathfrak{X}(M)$ and $X_p\in T_pM$. The horizontal space $\mathcal{H}_{(p,u)}$ at (p,u) is defined by

$$\mathcal{H}_{(p,u)}=\mathrm{Ker}(K_{(p,u)})\,.$$

The tangent space $T_{(p,u)}TM$ of tangent bundle TM at (p,u) is the direct some of its horizontal space and its vertical space:

$$T_{(p,u)}TM=\mathcal{H}_{(p,u)}\oplus\mathcal{V}_{(p,u)}\,.$$

If $X\in\mathfrak{X}(M)$, we denote by X^h (and X^v, respectively) the horizontal lift (and the vertical lift, respectively) of X to TM. A curve $\widetilde{\gamma}\colon I\to TM,\ t\mapsto(\gamma(t),U(t))$ is a *horizontal curve* if the vector field $U(t)$ is parallel along the curve $\gamma=\pi\circ\widetilde{\gamma}$.

Theorem 4 ([16])**.** *Let (M,g) be a pseudo-Riemannian manifold, D be the Levi-Civita connexion and R be the curvature tensor of D. Then the Lie bracket on the tangent bundle TM of M satisfies the following:*

i) $[X^v,Y^v]=0$,

ii) $[X^h,Y^v]=(D_XY)^v$,

iii) $[X^h, Y^h] = ([X, Y])^h - (R(X, Y)u)^v$.
for all X, $Y \in \mathfrak{X}(M)$ *and* $(p, u) \in TM$.

3. Sasaki pseudo-Riemannian metric

Definition 2. Let (M, g) be a pseudo-Riemannian manifold of signature (r, s) $(m = r + s)$. The Sasaki metric g^s on the tangent bundle TM is defined by the following relations

$$\begin{aligned} &g^s_{(p,u)}(X^h, Y^h) = g^s_{(p,u)}(X^v, Y^v) = g_p(X, Y) \\ &g^s_{(p,u)}(X^h, Y^v) = 0\,, \end{aligned}$$

for X, $Y \in \mathfrak{X}(M)$.

We notice that the signature of g^s is $(2r, 2s)$. With the same computations that in the Riemannian case The Levi-Civita connection associated to g^s is given by

Proposition 1 ([22])**.** *If we denote by* D^s *the Levi-Civita connection of* (TM, g^s)*. Then*

$$\begin{aligned} &(D^s_{X^h}Y^h)_{(p,u)} = (D_XY)^h_{(p,u)} - \frac{1}{2}(R_p(X, Y)u)^v \\ &(D^s_{X^h}Y^v)_{(p,u)} = (D_XY)^v_{(p,u)} + \frac{1}{2}(R_p(u, Y)X)^h \\ &(D^s_{X^v}Y^h)_{(p,u)} = \frac{1}{2}(R_p(u, X)Y)^h \\ &(D^s_{X^v}Y^v)_{(p,u)} = 0 \end{aligned}$$

Proposition 2 ([22])**.** *The curvature* R^s *of* (TM, g^s) *is given by the following formulas*

1) $R^s_{(p,u)}(X^v, Y^v)Z^v = 0$

2) $R^s_{(p,u)}(X^v, Y^v)Z^h = \Big((R(X, Y)Z + \frac{1}{4}R(u, X)(R(u, Y)Z) - \frac{1}{4}R(u, Y)(R(u, X)Z)\Big)^h$

3) $R^s_{(p,u)}(X^h, Y^v)Z^v = -\Big(\frac{1}{2}R(Y, Z)X + \frac{1}{4}R(u, Y)(R(u, Z)X)\Big)^h$

4) $R^s_{(p,u)}(X^h, Y^v)Z^h = \Big(\frac{1}{4}R(R(u, Y)Z, X)u + \frac{1}{2}R(X, Z)Y\Big)^v + \frac{1}{2}((D_XR)(u, Y)Z\Big)^h$

5) $$\begin{aligned} R^s_{(p,u)}(X^h, Y^h)Z^v &= \Big(R(X, Y)Z + \frac{1}{4}R(R(u, Z)Y, X)u - \frac{1}{4}R(R(u, Z)X, Y)u\Big)^v \\ &+ \frac{1}{2}\left((D_XR)(u, Z)Y - (D_YR)(u, Z)X\right)^h \end{aligned}$$

6) $$\begin{aligned} R^s_{(p,u)}(X^h, Y^h)Z^h &= \frac{1}{2}\left((D_ZR)(X, Y)u\right)^v + \Big(R(X, Y)Z + \frac{1}{4}R(u, R(Z, Y)u)X \\ &+ \frac{1}{4}R(u, R(X, Z)u)Y + \frac{1}{2}R(u, R(X, Y)u)Z\Big)^h, \end{aligned}$$

for X, Y, $Z \in \mathfrak{X}(M)$.

3.1. **Holonomy group of** (TM, g^s)**.** Let (M, g) be a pseudo-Riemannian manifold and (TM, g^s) its tangent bundle provided with the Sasaki metric. Let γ be a C^1-piecewise path starting from p in M, its horizontal lift at $(p, 0)$ is $\Gamma\colon t \to (\gamma(t), 0)$. According to Proposition 1, we obtain

$$D^s_{\dot\Gamma(t)} X^h = (D_{\dot\gamma(t)} X)^h$$
$$D^s_{\dot\Gamma(t)} X^v = (D_{\dot\gamma(t)} X)^v$$

for X vector field along γ. Hence, the parallel transport along Γ satisfies

$$\begin{aligned} \tau^s_\Gamma(X^h) &= \big(\tau_\gamma(X)\big)^h \\ \tau^s_\Gamma(X^v) &= \big(\tau_\gamma(X)\big)^v \end{aligned} \tag{1}$$

Then the holonomy group H^s of (TM, g^s) at $(p, 0)$ contains the subgroup $\left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A \in H \right\}$, where H is the holonomy group of (M, g) at p.

Theorem 5. *Let (M, g) be a pseudo-Riemannian manifold and (TM, g^s) its tangent bundle provided with the Sasaki metric. Let H^s (respectively H) the holonomy group of (TM, g^s) at $(p, 0)$ (respectively of (M, g) at p). Then*
1) *H^s contains the subgroup:*

$$H \times H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; A, B \in H \right\}.$$

2) *The holonomy algebra hol^s of (TM, g^s) at $(p, 0)$ contains the set*

$$\left\{ \bar{R}^s_\gamma(X, Y) := \begin{pmatrix} 0 & -\bar{R}_\gamma(Y, X) \\ \bar{R}_\gamma(X, Y) & 0 \end{pmatrix}; X, Y \in T_pM \text{ and } \gamma \in \mathcal{C}_p \right\},$$

where

$$\bar{R}_\gamma(X, Y)(Z) = \tau_\gamma^{-1}\big(R(\tau_\gamma(X), \tau_\gamma(Z))(\tau_\gamma(Y))\big)$$

and $\mathcal{C}_p$ the set of the C^1-piecewise paths starting from p.

Proof. According to the decomposition $T_{(p,0)}TM = \mathcal{H}_{(p,0)} \oplus \mathcal{V}_{(p,0)}$ and from Proposition 2 we have:

$$R^s(X^v, Y^v) = \begin{pmatrix} R(X, Y) & 0 \\ 0 & 0 \end{pmatrix} \quad R^s(X^h, Y^h) = \begin{pmatrix} R(X, Y) & 0 \\ 0 & R(X, Y) \end{pmatrix}$$

and $R^s(X^h, Y^v) = \dfrac{1}{2} \begin{pmatrix} 0 & -\overline{R}(Y, X) \\ \overline{R}(X, Y) & 0 \end{pmatrix}$, with $\bar{R}(X, Y)(Z) = R(X, Z)(Y)$.

(1) implies that

$$\begin{aligned} \tau_\Gamma^{-1}\left(R^s(\tau_\Gamma(X^v), \tau_\Gamma(Y^v))(\tau_\Gamma(Z^h))\right) &= \tau_\Gamma^{-1}\left(R^s((\tau_\gamma(X))^v, (\tau_\gamma(Y))^v((\tau_\gamma(Z))^h)\right) \\ &= \tau_\Gamma^{-1}\left(R^s(\tau_\gamma(X), \tau_\gamma(Y)(\tau_\gamma(Z))\right)^h \\ &= \left(\tau_\gamma^{-1}(R(\tau_\gamma(X), \tau_\gamma(Y))(\tau_\gamma)(Z))\right)^h. \end{aligned}$$

By Ambrose-Singer Theorem ([2]), we deduces 1).
In the same way, according to (1) and Proposition 2, we have

$$\tau_\Gamma^{-1}\big(R^s(\tau_\Gamma(X^h),\tau_\Gamma(Y^v))(\tau_\Gamma(Z^h))\big)=\frac{1}{2}\big(\tau_\gamma^{-1}(R(\tau_\gamma(X),\tau_\gamma(Y))(\tau_\gamma)(Z))\big)^v$$

and

$$\tau_\Gamma^{-1}\big(R^s(\tau_\Gamma(X^h),\tau_\Gamma(Y^v))(\tau_\Gamma(Z^v))\big)=-\frac{1}{2}\left(\tau_\gamma^{-1}(R(\tau_\gamma(Y),\tau_\gamma(X))(\tau_\gamma)(Z))\right)^h$$

Hence we obtain 2). □

Corollary 1. *(TM,g^s) is flat if and only if (M,g) is flat.*

Proof.
a) It is easy to see that the curvature $R^s=0$ if $R=0$. Conversely, if $hol^s=\{0\}$, according to Theorem 5, we get $hol=\{0\}$. □

Theorem 6. *Let (M,g) be a connected, simply connected pseudo-Riemannian manifold.*
1) If (M,g) is decomposable then (TM,g^s) is decomposable.
2) If (TM,g^s) is reducible then (M,g) is reducible.
In particular, if (M,g) is a Riemannian manifold, then (TM,g^s) is irreducible if and only if (M,g) is irreducible.

Proof.
1) If (M,g) is decomposable, i.e. $(M,g)=(M_1,g_1)\times(M_2,g_2)$, then

$$(TM,g^s)=(TM_1,g_1^s)\times(TM_2,g_2^s)\,.$$

2) If (TM,g^s) is reducible, then its holonomy group H^s at $(p,0)$ leaves invariant a proper subspace E_1 of $T_{(p,0)}TM$ and its orthogonal $E_2=E_1^\perp$, i.e. $T_{(p,0)}TM=E_1\oplus E_2$. We suppose that $\dim E_1\geq m$ and $\dim E_2\leq m$. We denote by $\mathcal{V}\equiv\mathcal{V}_{(p,0)}$ and $\mathcal{H}\equiv\mathcal{H}_{(p,0)}$. We will distinguish three cases
• if $\{0\}\subsetneq E_1\cap\mathcal{H}\subsetneq\mathcal{H}$, according to Theorem 5, we have

$$\begin{pmatrix}A&0\\0&0\end{pmatrix}(E_1\cap\mathcal{H})\subset E_1\cap\mathcal{H}$$

for all $A\in hol$. Consequently $E_1\cap\mathcal{H}$ is hol-invariant. Then (M,g) is reducible.
• If $\{0\}=E_1\cap\mathcal{H}$, hence $T_{(p,0)}TM=E_1\oplus\mathcal{H}$. According to Theorem 5, we have for $A\in hol$ that

$$\begin{pmatrix}0&0\\0&A\end{pmatrix}\mathcal{H}=0$$

$$\begin{pmatrix}0&0\\0&A\end{pmatrix}E_1\subset E_1\cap\mathcal{V}$$

then $hol(T_{(p,0)}TM)\subset E_1\cap\mathcal{V}$. We distinguish two cases
⋆ if $E_1\cap\mathcal{V}=0$, then $hol=0$ and (M,g) is reducible.
⋆ If $E_1\cap\mathcal{V}\neq\{0\}$

$$\begin{pmatrix}0&0\\0&A\end{pmatrix}(E_1\cap\mathcal{V})\subset E_1\cap\mathcal{V}$$

for all $A \in hol$. Hence $E_1 \cap \mathcal{V}$ is hol-invariant. Then (M, g) is reducible.
• If $E_1 \cap \mathcal{H} = \mathcal{H}$, then $\mathcal{H} \subset E_1$ and

$$R^s(X^h, Y^v)(\mathcal{H}) \subset E_1 \cap \mathcal{V}\,.$$

$*$ If $E_1 \cap \mathcal{V} = 0$, then $R^s(X^h, Y^v)\mathcal{H} \subset \mathcal{V} \cap E_1 = \{0\}$. Hence $R = 0$ and (M, g) is reducible.
$*$ If $E_1 \cap \mathcal{V} \neq 0$, its is stable by hol, then (M, g) is reducible. □

3.2. **Geometric structure on TM.** In this section, we found conditions on the base manifold (M, g) under which (TM, g^s) is locally symmetric, Einstein or Kählerian manifold.

3.2.1. *Symmetry on TM.*

Proposition 3. *Let (M, g) be a pseudo-Riemannian manifold. Then (TM, g^s) is locally symmetric if and only if (M, g) is locally symmetric and $hol \circ hol = 0$, where hol is the holonomy algebra of (M, g).*

Proof. According to the holonomy principle ([7, Ch. 10]), (TM, g^s) is locally symmetric if and only if its holonomy group H^s preserves the curvature R^s:

$$A \circ R^s(X^*, Y^*) = R^s(AX^*, AY^*) \circ A\,, \quad \forall\, A \in H^s\,, \quad \text{and} \quad \forall\, X^*, Y^* \in T_{(p,u)}TM\,.$$

In term of holonomy algebra, it is equivalent to: $\forall\ \overline{A} \in hol^s$, and $\forall\ X^*, Y^* \in T_{(p,u)}TM$

$$[\overline{A}, R(X^*, Y^*)] = R(\overline{A}X^*, Y^*) + R(X^*, \overline{A}Y^*)\,, \tag{2}$$

For $\overline{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with A, $B \in hol$ and $R(X^*, Y^*) = R(X^v, Y^v)$, (2) implies

$$\begin{cases} [B, R(X, Y)] = 0\,, \\ [A, R(X, Y)] = R(AX, Y) + R(X, AY)\,. \end{cases}$$

Then hol is commutative and (M, g) is locally symmetric. For $\overline{A} = R^s(Z^h, T^v)$ and $R(X^*, Y^*) = R(X^h, Y^h)$, (2) implies

$$BC - DA = R(AX, Y) + R(X, BY) = 0\,, \tag{3}$$

$$CB - AD = R(BX, Y)\,, \tag{4}$$

where $A = \overline{R}(T, Z)$, $B = -\overline{R}(Z, T)$, $C = \overline{R}(X, Y)$ and $D = -\overline{R}(Y, X)$.
If we replace in (4), X by Y and Z by T, we obtain $BC - DA = R(AX, Y)$. Then (3) implies

$$R(X, BY) = R(X, R(T, Y)Z) = 0\,, \quad \forall\, X, Y, Z, T \in T_pM\,.$$

Hence

$$R(X, Y) \circ R(Z, T) = 0\,, \quad \forall\, X, Y, Z, T \in T_pM\,. \tag{5}$$

Because the holonomy algebra of locally symmetric space is only generated by the curvature, (5) is equivalent to $hol \circ hol = 0$. Conversely, if we have (5), and (M, g) is locally symmetric, by a direct computation we get (2). □

Corollary 2. *Let (M,g) be a non-flat pseudo-Riemannian locally symmetric space of dimension $m \geq 2$ satisfying $hol \circ hol = 0$. Then*
a) *(M,g) is reducible.*
b) *The index of g is ≥ 2.*

Proof.
a) If (M,g) is supposed irreducible, the condition $hol \circ hol = 0$ implies $hol = 0$.
b) If (M,g) is Riemannian, according to De Rham-Wu's Theorem , we can suppose that it is irreducible. Then by a) we deduce a contradiction.
Now if (M,g) is Lorentzian, according to a) we can suppose that hol leaves invariant a light-like line. Then

$$hol \subset (\mathbb{R} \oplus so(m-2)) \ltimes \mathbb{R}^{m-2} = \left\{ \begin{pmatrix} a & {}^tX & 0 \\ 0 & A & X \\ 0 & 0 & -a \end{pmatrix} ; a \in \mathbb{R},\ X \in \mathbb{R}^{m-2}, \right.$$

$$\left. A \in so(m-2) \right\}.$$

However if the square of such an element of hol is null, it is necessarily null. Impossible. $\square$

As concerns explicit examples for Corollary 2, see more details in Example 1 at the end of Subsection 3.2.2.

3.2.2. *Einstein structure on TM.* Let (M,g) be a pseudo-Riemannian manifold and $\{e,\dots,e_m\}$ an orthonormal basis of T_pM, then the family $\{e_1^h,\dots,e_m^h,e_1^v,\dots,e_m^v\}$ is an orthonormal basis of $T_{(p,u)}TM$. And hence the Ricci curvature of (TM,g^s) is given by the following formula

$$\mathrm{Ric}^s_{(p,u)}(X^*,Y^*) = \sum_{i=1}^{i=m} \varepsilon_i\, g^s(R^s(X^*,e_i^h)Y^*,e_i^h) + \sum_{i=1}^{i=m} \varepsilon_i\, g^s(R^s(X^*,e_i^v)Y^*,e_i^v)$$

where

$$\varepsilon_i = g^s(e_i^h,e_i^h) = g^s(e_i^v,e_i^v) = g^s(e_i,e_i) = \pm 1\,.$$

According to Proposition 2, we have

$$\begin{aligned} \mathrm{Ric}^s_{(p,u)}(X^h,Y^h) &= \sum_{i=1}^{i=m} \varepsilon_i\, g^s(R^s(X^h,e_i^h)Y^h,e_i^h) + \sum_{i=1}^{i=m} \varepsilon_i g^s(R^s(X^h,e_i^v)Y^h,e_i^v) \\ &= \mathrm{Ric}(X,Y) + \frac{3}{4}\sum_{i=1}^{i=m} \varepsilon_i\, g(R(X,e_i)u,R(Y,e_i)u)\,. \end{aligned} \tag{6}$$

$$\begin{aligned}
\mathrm{Ric}^s_{(p,u)}(X^h,Y^v) &= \sum_{i=1}^{i=m} \varepsilon_i g^s(R^s(X^h,e_i^h)Y^v,e_i^h) + \sum_{i=1}^{i=m} \varepsilon_i\; g^s(R^s(X^h,e_i^v)Y^v,e_i^v) \\
&= \frac{1}{2}\sum_{i=1}^{i=m} \varepsilon_i g((D_X R)(u,Y)e_i,e_i) - \sum_{i=1}^{i=m} \varepsilon_i\; g((D_{e_i}R)(u,Y)X,e_i) \\
&= \frac{1}{2}\sum_{i=1}^{i=m} \varepsilon_i g((D_X R)(e_i,e_i)u,Y) - \delta R(u,Y)X = -\delta R(u,Y)X\,. \qquad (7)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ric}^s_{(p,u)}(X^v,Y^v) &= \sum_{i=1}^{i=m} \varepsilon_i g^s(R^s(X^v,e_i^h)Y^v,e_i^h) + \sum_{i=1}^{i=m} \varepsilon_i\; g^s(R^s(X^v,e_i^v)Y^v,e_i^v) \\
&= \frac{1}{2}\sum_{i=1}^{i=m} \varepsilon_i g(R(X,Y)e_i,e_i) + \frac{1}{4}\sum_{i=1}^{i=m} \varepsilon_i\; g(R(u,X)R(u,Y)e_i,e_i) \\
&= \frac{1}{2}\sum_{i=1}^{i=m} \varepsilon_i g(R(e_i,e_i)X,Y) + \frac{1}{4}\,\mathrm{trace}\,\big(R(u,X)R(u,Y)\big) \\
&= \frac{1}{4}\,\mathrm{trace}\,\big(R(u,X)R(u,Y)\big)\,. \qquad (8)
\end{aligned}$$

Proposition 4. *Let (M,g) be a pseudo-Riemannian manifold. If (TM,g^s) is Einstein, then it is Ricci-flat. And (TM,g^s) is Ricci-flat if and only if (M,g) satisfies the following conditions:*
a) *(M,g) is Ricci-flat,*
b) $\mathrm{trace}\,A^2=0$, *for all $A\in hol$,*
c) *(M,g) admits a harmonic curvature:*

$$\delta R(X,Y)Z = \sum_{i=1}^{i=m} \varepsilon_i\; g(D_{e_i}R(X,Y)Z,e_i) = 0\,, \quad \forall\, X,Y,Z\in\mathfrak{X}(M)\,,$$

d) $\sum_{i=1}^{i=m} \varepsilon_i\; g(R(X,e_i)Z,R(Y,e_i)Z)=0$, *$\forall\, X,Y,Z\in\mathfrak{X}(M)$.*

Proof. Let us suppose that the metric g^s is λ-Einstein then

$$\mathrm{Ric}_{(p,u)}(X^*,Y^*) = \lambda\, g^s_{(p,u)}(X^*,Y^*),\ \forall\, X^*,Y^*\in\mathfrak{X}(M), \quad \text{and} \quad \forall\,(p,u)\in TM\,.$$

If we take $u=0$ in (6) and then in (4), we obtain $\lambda=0$. Then (M,g) is Ricci-flat. Consequently (TM,g^s) is Ricci-flat.
Hence from (4)–(6), we obtain the conditions a)-d). Conversely, if we have the conditions a)-d), it is easy to see that (TM,g^s) i Ricci-flat. □

Corollary 3. *Let (M,g) be a non flat pseudo-Riemannian manifold such that (TM,g^s) is Einstein. Then*
i) *(M,g) is reducible or locally symmetric.*
ii) *The index of g is ≥ 2.*

Proof.
i) If (M,g) is irreducible and non locally symmetric, then its holonomy algebra is one of algebras of Berger's list (Theorem 2). But no algebra of this list verifies the condition b) of Proposition 4.
ii) Now, if (M,g) is Lorentzian. According to De Rham-Wu's Theorem, we can suppose that it is indecomposable.
If it is irreductible, it is well known that $hol = so(1,n+1)$, where $m = n+2$. But according to the condition b) of Proposition 4 it is impossible.
If (M,g) is indecomposable-reducible, we use the following lemma. □

Lemma 1 ([18])**.** *Let (M,g) be a Lorentzian indecomposable reducible non Ricci-flat manifold of signature $(1,1+n)$. Then*
(α) either $hol = (\mathbb{R}\oplus\mathcal{G})\ltimes\mathbb{R}^m$, where $\mathcal{G}\subset so(n)$ is a holonomy algebra of a Riemanian metric and in the decomposition of $\mathcal{G}\subset so(n)$ at least one subalgebra $\mathcal{G}_i\subset so(n_i)$ coincide with one of algebras $so(n_i), u(n_i), sp(\frac{n_i}{4})\oplus sp(1)$ or with a symmetric Berger algebra.
(β) or $hol = \mathcal{G}\ltimes\mathbb{R}^m$ and in the decomposition of $\mathcal{G}\subset so(n)$ each algebra $\mathcal{G}_i\subset so(n_i)$ coincide with one of algebras $so(n_i), \mathrm{su}(n_i), sp(\frac{n_i}{4}), G_2\subset so(7), spin(7)\subset so(8)$.

The condition b) of Proposition 4, impose that *hol* cannot be of type (α) of Lemma 1. Now, if *hol* is of type (β), the same condition b) implies that $\mathcal{G}=0$. Impossible.

Example 1. Let (M,g) be a simply connected pseudo-Riemannian locally symmetric space of signature $(2,2)$ with holonomy group

$$\mathbb{A} = \left\{ \begin{pmatrix} I_2 & aJ \\ 0 & I_2 \end{pmatrix} ,\ a\in\mathbb{R}\right\}, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Its satisfies the conditions of Propositions 3 and 4. Then (TM, g^s) is an Einstein locally symmetric space of signature $(4,4)$. The simply connected pseudo-Riemannian locally symmetric spaces of signature $(2,2)$ with holonomy group $\mathbb{A}$ are given in ([6]).

3.2.3. *Kählerian structure on TM.* Let (M,g) be a pseudo-Riemannian manifold. Let J be the natural almost complex structure definite on TM by

$$\bar{J}(X^h) = X^v \textit{ and } \bar{J}(X^v) = -X^h .$$

It is easy to see that $(TM, g^s, \bar{J})$ is almost Hermitian:

$$g^s(\bar{J}X^*, \bar{J}Y^*) = g^s(X^*, Y^*)\,, \quad \forall\, X^*, Y^* \in \mathfrak{X}(M)$$

Proposition 5. *If $(TM, g^s, \bar{J})$ is Kählerian, then it is flat.*

Proof. We suppose (TM, g^s, J) is Kählerian. According to the holonomy principle, the tensor $\bar{J}$ at $(p,0)$ commute with the curvature and in particular, we have:

$$\bar{J}\circ R^s(X^h, Y^v) = R^s(X^h, Y^v)\circ\bar{J}\,, \quad \forall\, X, Y\in T_pM)\,.$$

This implies $R(X,Z)Y = R(Y,Z)X$, $\forall\, X,Y,Z\in T_pM)$. Hence $R=0$. □

Now, we suppose that (M,g,J) is a Kählerian pseudo-Riemannian manifold and we consider the almost complex structure $\tilde{J}$ defined on TM by

$$\tilde{J}(X^h) = (JX)^v\,, \quad \tilde{J}(X^v) = (JX)^h\,.$$

$(TM, g^s, \tilde{J})$ is an almost Hermitian manifold.

Proposition 6. *If $(TM, g^s, \tilde{J})$ is Kählerian, then it is flat.*

Proof. We suppose $(TM, g^s, \tilde{J})$ is Kählerian. According to the holonomy principle, the tensor $\tilde{J}$ at $(p,0)$ commute with the curvature and in particular, we have:

$$\tilde{J} \circ R^s(X^h, Y^v) = R^s(X^h, Y^v) \circ \tilde{J}\,, \quad \forall\, X, Y \in T_pM\,.$$

This implies $J \circ R(X,Y) = -R(Y,X) \circ J$, $\forall\, X, Y \in T_pM$. Then,
$R(X,JX)X = 0$, $\forall\, X \in T_pM$. Hence, according to ([21, p. 166]), we get $R = 0$. □

4. Neutral metric

Definition 3. Let (M,g) be a pseudo-Riemannian manifold of dimension m with signature (r,s). The neutral metric g^n of g on TM is defined by

$$\begin{aligned} g^n_{(p,u)}(X^h, Y^h) &= g^n_{(p,u)}(X^v, Y^v) = 0 \\ g^n_{(p,u)}(X^v, Y^h) &= g_p(X,Y)\,, \end{aligned}$$

for $X, Y \in \mathfrak{X}(M)$.

g^n is of neutral signature (m,m).
By a simple computation, we obtain

Proposition 7. *If we denote by D^n the Levi-Civita connection of (TM, g^n) then*

$$\begin{aligned} (D^n_{X^h} Y^h)_{(p,u)} &= (D_X Y)^h_{(p,u)} + (R_p(u,X)Y)^v \\ (D^n_{X^h} Y^v)_{(p,u)} &= (D_X Y)^v_{(p,u)} \\ (D^n_{X^v} Y^h)_{(p,u)} &= 0 \\ (D^n_{X^v} Y^v)_{(p,u)} &= 0 \end{aligned}$$

Proposition 8. *If we denote by R^n the tensorial curvature of (TM, g^n). Then we have the following formulas:*

$$\begin{aligned} &R^n_{(p,u)}(X^v, Y^v)Z^v = 0 \\ &R^n_{(p,u)}(X^v, Y^v)Z^h = 0 \\ &R^n_{(p,u)}(X^h, Y^v)Z^v = 0 \\ &R^n_{(p,u)}(X^h, Y^v)Z^h = (R(X,Y)Z)^v \\ &R^n_{(p,u)}(X^h, Y^h)Z^v = (R(X,Y)Z)^v \\ &R^n_{(p,u)}(X^h, Y^h)Z^h = (R_x(X,Y)Z)^h + ((D_X R)_p(u,Y)Z - (D_Y R)_p(u,X)Z)^v \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$.

4.1. **Holonomy group.**

Proposition 9. a) *The holonomy group H of (M,g) is a subgroup of the holonomy group H^n of (TM,g^n):*

$$H \equiv \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}; \; A \in H \right\} \subset H^n .$$

b) *According to the decomposition $\mathbb{R}^{2n} = T_{(p,0}TM = \mathcal{V}_{(p,0} \oplus \mathcal{H}_{(p,0}$, the holonomy algebra hol^n of (TM,g^n) is exactly the algebra*

$$\left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}; \; A, B \in hol \right\},$$

where hol is the holonomy algebra of (M,g).

Proof. Let γ be a C^1-piecewise path starting from p in M, its horizontal lift at $(p,0)$ is $\Gamma\colon t \to (\gamma(t),0)$. According to Proposition 7, we have

$$D^n_{\dot\Gamma(t)} X^h = (D_{\dot\gamma(t)} X)^h$$
$$D^n_{\dot\Gamma(t)} X^v = (D_{\dot\gamma(t)} X)^v$$

for X vector field along γ. Consequently, if γ is a loop at p, the parallel transport along Γ is given by:

$$\tau^s_\Gamma(X^h) = (\tau_\gamma(X))^h \quad \text{and} \quad \tau^s_\Gamma(X^v) = (\tau_\gamma(X))^v .$$

Hence we have a). Moreover, According to Proposition 8, we have

$$R^s(X^h, Y^v)Z^h = (R(X,Y)Z)^v \quad \text{and} \quad R^s(X^h,Y^v)Z^v = 0 .$$

Then

$$\begin{aligned} \tau_\Gamma^{-1}\left(R^n(\tau_\Gamma(X^h), \tau_\Gamma(Y^v))(\tau_\Gamma(Z^h))\right) &= \tau_\Gamma^{-1}\left(R^n((\tau_\gamma(X))^h, (\tau_\gamma(Y))^v((\tau_\gamma(Z))^h)\right) \\ &= \tau_\Gamma^{-1}\left(R^n(\tau_\gamma(X), \tau_\gamma(Y)(\tau_\gamma(Z))\right)^v \\ &= \left(\tau_\gamma^{-1}(R(\tau_\gamma(X), \tau_\gamma(Y))(\tau_\gamma(Z))\right)^v . \end{aligned}$$

In the same way, we have

$$\tau_\Gamma^{-1}(R^n(\tau_\Gamma(X^h), \tau_\Gamma(Y^v))(\tau_\Gamma(Z^v))) = 0 .$$

Then we get

$$\left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}; \;\; A, B \in hol \right\} \subset hol^n .$$

However the definition of g^n from g and the Proposition 7 imply b). □

Proposition 10. *If (M,g) is irreducible then (TM,g^n) is indecomposable. The reciprocal is true if g is a Riemannian metric.*

Proof. First we notice that if *hol* is irreducible $E := \{AX, A \in hol, X \in \mathbb{R}^m\} = \mathbb{R}^m$. Indeed, E is *hol*-invariant, then $E = 0$ or $E = \mathbb{R}^{2m}$. But *hol* is non trivial, hence $E = \mathbb{R}^{2m}$. Now, let F be a non-degenerate proper subspace of $\mathbb{R}^{2m}$ hol^n-invariant, then its projections F_i, $(i = 1, 2)$ on $\mathbb{R}^m$ are *hol*-invariant. Since *hol* is irreducible $F_i = 0$ or $F_i = \mathbb{R}^m$. F is non-degenerate then $F_i = \mathbb{R}^m$. hol^n contains $\left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \ A \in H \right\}$, then F contains the subspace

$$\big\{(AX, 0);\ A \in hol,\ X \in \mathbb{R}^m\big\} = \mathbb{R}^m \times \{0\}.$$

Hence $F = \mathbb{R}^{2m}$. Consequently hol^n is indecomposable. □

Remark 1. According to Proposition 9, the vertical direction $\mathcal{V}_{(p,0)}$ is $hol^n -$ invariant witch is totaly isotrope. Consequently, we get a class of indecomposable-reducible manifolds (TM, g^n) once the base manifold (M, g) is irreducible.

4.2. **Geometric consequences.**

4.2.1. *Symmetry on (TM, g^n).*

Proposition 11. *(TM, g^n) is locally symmetric if and only if (M, g) is locally symmetric and $hol \circ hol = 0$.*

Proof. For the proof we need the following lemma.

Lemma 2. *Let (M, g) be a pseudo-Riemannian manifold. the covariant derivatives of the tensor curvature R^n are given by the following formulas*

1) $$\begin{aligned}(D^n_{W^h} R^n_{(p,u)})(X^h, Y^h)Z^h = {} & ((D_W R)_p(X, Y)Z)^h + ((D_W D_X R)_p(u, Y)Z \\ & - (D_W D_Y R)_p(u, X)Z)^v - ((D_u D_W R)_p(Y, X)Z \\ & + (D_W D_u R)_p(Y, X)Z + (D_{[u,W]} R)_p(Y, X)Z \\ & + 2R_p(Y, R(u, W)X)Z)^v\end{aligned}$$
2) $(D^n_{W^v} R^n_{(p,u)})(X^h, Y^h)Z^h = ((D_W R)_p(X, Y)Z)^v$
3) $(D^n_{W^h} R^n_{(p,u)})(X^h, Y^h)Z^v = ((D_W R)_p(X, Y)Z)^v$
4) $(D^n_{W^v} R^n_{(p,u)})(X^h, Y^h)Z^v = 0$
5) $(D^n_{W^h} R^n_{(p,u)})(X^h, Y^v)Z^h = ((D_W R)_p(X, Y)Z)^v$
6) $(D^n_{W^v} R^n_{(p,u)})(X^h, Y^v)Z^h = 0$
7) $(D^n_{W^h} R^n_{(p,u)})(X^h, Y^v)Z^v = 0$
8) $(D^n_{W^v} R^n_{(p,u)})(X^h, Y^v)Z^v = 0$
9) $(D^n_{W^h} R^n_{(p,u)})(X^v, Y^v)Z^h = 0$
10) $(D^n_{W^v} R^n_{(p,u)})(X^v, Y^v)Z^h = 0$
11) $(D^n_{W^h} R^n_{(p,u)})(X^v, Y^v)Z^v = 0$
12) $(D^n_{W^v} R^n_{(p,u)})(X^v, Y^v)Z^v = 0\,.$

We suppose that (TM, g^n) is locally symmetric. According to 2) of Lemma 2, (M, g) is locally symmetric.
By 1) of Lemma 2, we get $g(R(Y, R(u, W)X)Z, V) = 0$. It is equivalent to

$$g(R(Z, V)R(u, W)X, Y) = 0\,,$$

then

$$R(X,Y)\circ R(Z,V)=0\,,\quad \forall\, X,Y,Z,V\in\chi(M)\,.$$

and since (M,g) is locally symmetric, we have

$$A\circ B=0\,,\quad \forall\, A,B\in hol\,. \tag{9}$$

Conversly, according to Lemma 2, if we have (9) and (M,g) is locally symmetric, we get (TM,g^n) is locally symmetric. □

4.2.2. *Einstein structure on TM.* Let $\{e_1,\dots,e_m\}$ be an orthonormal basis of T_pM, then the Ricci curvature at (p,u) is

$$\mathrm{Ric}^s_{(p,u)}(X^*,Y^*)=\sum_{i=1}^{i=m}\varepsilon_i g^n(R^n(X^*,e_i^h)Y^*,e_i^v)+\sum_{i=1}^{i=m}\varepsilon_i g^n(R^n(X^*,e_i^v)Y^*,e_i^h)$$

where

$$\varepsilon_i=g^n(e_i^h,e_i^v)=g(e_i,e_i)=\pm 1\,.$$

Let's compute Ric^n. We have

$$\begin{aligned}\mathrm{Ric}^n(X^h,Y^h)&=\sum_{i=1}^{i=m}\varepsilon_i g^n(R^n(X^h,e_i^h)Y^h,e_i^v)+\sum_{i=1}^{m}\varepsilon_i g^n(R^n(X^h,e_i^v)Y^h,e_i^h)\\&=2\sum_{i=1}^{m}\varepsilon_i g(R(X,e_i)Y,e_i)=2\,\mathrm{Ric}(X,Y)\,.\end{aligned}\tag{10}$$

$$\begin{aligned}\mathrm{Ric}^n(X^v,Y^v)&=\sum_{i=1}^{m}\varepsilon_i g^n(R^n(X^v,e_i^h)Y^v,e_i^v)\\&\quad+\sum_{i=1}^{m}\varepsilon_i g^n(R^n(X^v,e_i^v)Y^v,e_i^h)=0\,.\end{aligned}\tag{11}$$

$$\begin{aligned}\mathrm{Ric}^n(X^v,Y^h)&=\sum_{i=1}^{m}\varepsilon_i g^n(R^n(X^v,e_i^h)Y^h,e_i^v)\\&\quad+\sum_{i=1}^{m}\varepsilon_i g^n(R^n(X^v,e_i^v)Y^h,e_i^h)=0\,.\end{aligned}\tag{12}$$

Proposition 12. *If (TM,g^n) is λ-Einstein, then it is Ricci-flat. Therefore (TM,g^n) is Ricci-flat if and only if (M,g) is Ricci-flat.*

Proof. According to (10), if (TM,g^n) is Einstein, it is Ricci-flat. According to (8), we deduce the proposition. □

4.2.3. *Kählerian structure on TM.* Let (M, g, J) be a Kählerian pseudo-Riemannian manifold. Let J^n be the natural almost complex structure definite on TM by

$$J^n(X^h) = (JX)^v \textit{ and } J^n(X^h) = (JX)^h\,.$$

It is easy to see that (TM, g^n, J^n) is an almost Hermitian pseudo-Riemannian manifold.

Proposition 13. *(TM, g^n, J^n) is a Kählerian pseudo-Riemannian manifold.*

Proof. According to the decomposition $\mathbb{R}^{2n} = T_{(p,0}TM = \mathcal{V}_{(p,0} \oplus \mathcal{H}_{(p,0}$ the tensor $J^n = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ at $(p, 0)$ commute with hol^n since J commute with hol at p. Then the holonomy principle implies the proposition. $\square$

Remark 2. According to the previous propositions, the tangent bundle can support some reducible-imdecomposable metrics of neutral signature. Notably Einstein, Kählerian or Ricci-flat metrics. For example, if $Hol(M, g) = U(r, s)$, (TM, g^n) is a Kählerian pseudo-Riemannian manifold. If $Hol(M, g) = SU(r, s)$, (TM, g^n) is an Einstein Kählerian pseudo-Riemannian manifold.

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