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ON THE EXISTENCE OF GENERALIZED QUASI-EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to study a type of Riemannian manifold called generalized quasi-Einstein manifold. The existence of a generalized quasi-Einstein manifold have been proved by non-trivial examples.

1. Introduction

A Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([2], p.432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([2], p.432–433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y)\,, \tag{1.2}$$

where a, b are certain scalars and A is a non-zero 1-form such that

$$g(X,U) = A(X), \tag{1.3}$$

for all vector fields X. Moreover, different structures on Einstein manifolds have been studied by several authors. In 1993, Tamassay and Binh [?] studied weakly symmetric structures on Einstein manifolds.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition (1.2).

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Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime are quasi-Einstein manifolds [14]. Also, quasi-Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [13]. So quasi-Einstein manifolds have some importance in the general theory of relativity.

Quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized quasi-Einstein manifolds[4], super quasi-Einstein manifolds[5], nearly quasi-Einstein manifolds [11], generalized Einstein manifolds [1], generalized p-quasi-Einstein manifolds [10], pseudo quasi-Einstein manifolds [18] and many others.

In a recent paper De and Ghosh [12] introduced the notion of generalized quasi-Einstein manifolds in another way. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + dB(X)B(Y)\,, \tag{1.4}$$

where a, b, d are certain non-zero scalars and A, B are two non-zero 1-forms such that

$$g(A,B) = 0, \;\; \|A\| = \|B\| = 1\,.$$

The unit vector fields U and V corresponding to the 1-forms A and B respectively, defined by

$$g(X,U) = A(X)\,, \quad g(X,V) = B(X)\,,$$

for every vector field X are orthogonal, that is, $g(U, V) = 0$. Such a manifold is denoted by $G(QE)_n$. If $d = 0$, then the manifold reduces to a quasi-Einstein manifold [6].

A Riemannian manifold of quasi-constant curvature was given by B. Y. Chen and K. Yano[9] as a conformally flat manifold with the curvature tensor $\acute{R}$ of type $(0, 4)$ which satisfies the condition

$$\begin{aligned}\acute{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q[g(Y,Z)T(X)T(W)\\ &- g(X,Z)T(Y)T(W) + g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z)]\,,\end{aligned} \tag{1.5}$$

where $\acute{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type $(1, 3)$, p, q are scalars, T is a non-zero 1-form defined by

$$g(X,\tilde{\rho}) = T(X)\,, \tag{1.6}$$

and $\tilde{\rho}$ is a unit vector field.

It can be easily seen that if the curvature tensor $\acute{R}$ is of the form (1.5), then the manifold is conformally flat. On the other hand, Gh. Vranceanu [20] defined the notion of almost constant curvature. Later A. L. Mocanu [15] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Gh. Vranceanu are same. If $q = 0$, then it reduces to a manifold of constant curvature.

In [12] the authors generalize the notion of quasi-constant curvature and prove the existence of such a manifold. A Riemannian manifold is said to be a manifold of generalized quasi-constant curvature, if the curvature tensor $\acute{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\acute{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + q[g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W) \\ &\quad + g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z)] \\ &\quad\ s[g(Y,Z)D(X)D(W) - g(X,Z)D(Y)D(W) \\ &\quad + g(X,W)D(Y)D(Z) - g(Y,W)D(X)D(Z)]\,, \end{aligned} \tag{1.7}$$

where p, q, s are scalars, T and D are non-zero 1-forms.
We assume that the unit vector fields ρ and $\tilde{\rho}$ defined by

$$g(X,\rho) = T(X)\,, \quad g(X,\tilde{\rho}) = D(X)$$

are orthogonal, i.e., $g(\rho,\tilde{\rho}) = 0$.
In 2008, De and Gazi [11] introduced the notion of nearly quasi-Einstein manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a nearly quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bE(X,Y)\,, \tag{1.8}$$

where a and b are non-zero scalars and E is a non-zero symmetric tensor of type $(0,2)$. An n-dimensional nearly quasi-Einstein manifold was denoted by $N(QE)_n$.
Let M be a smooth manifold with odd dimension $2n+1$. An almost contact structure on M is a triple (ϕ,ξ,η) where ξ is a vector field, η is a 1-form and ϕ is a $(1,1)$ tensor field satisfying the relation:

$$\phi^2 = -\operatorname{Id} + \eta\circ\xi\,, \tag{1.9}$$

$$\eta(\xi) = 1\,, \tag{1.10}$$

where Id is the identity endomorphism on TM. Then we have $\phi\xi = 0$ and $\eta\circ\phi = 0$. Moreover, if g is a Riemannian metric associated on M, i.e., a metric which satisfies for any X and Y on $\Gamma(TM)$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)\,, \tag{1.11}$$

then we say that (ϕ,ξ,η,g) is an almost contact metric structure. A manifold equipped with such a structure is called an almost contact metric manifold. The fundamental 2-form Φ on M is given by

$$\Phi(X,Y) = g(X,\phi Y)\,,$$

for any X and Y on $\Gamma(TM)$.
An almost contact metric structure (ϕ,ξ,η,g) is normal if the Nijenhuis tensor N_ϕ satisfies[3]:

$$N_\phi + 2d\eta\otimes\xi = 0\,.$$

A contact manifold is a smooth manifold M together with a 1-form η such that $\eta\wedge d\eta \neq 0$. We say that $M(\phi,\xi,\eta,g)$ is a Sasakian manifold if it is a normal contact metric manifold such that $\Phi = d\eta$. An almost contact metric structure (ϕ, ξ, η, g) is Sasakian[3] if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X\,, \tag{1.12}$$

where ∇ is the Riemannian connection of M.
>From (1.12), for a Sasakian manifold we have

$$\nabla_X \xi = -\phi X\,. \tag{1.13}$$

Also a contact metric manifold is a Sasakian manifold if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y\,. \tag{1.14}$$

A plane section in the tangent space $T_x\tilde{M}$ at $x \in \tilde{M}$ is called a ϕ-section if it is spanned by a vector X orthogonal to ξ and ϕX. The sectional curvature $K(X, \phi X)$ with respect to a ϕ-section determined by a vector X is called a ϕ-sectional curvature. A Sasakian manifold with constant ϕ-sectional curvature c is a Sasakian space form and is denoted by $\tilde{M}(c)$. The curvature tensor of a Sasakian space form $\tilde{M}(c)$ is given by

$$\begin{aligned}\tilde{R}(X, Y)Z = \frac{1}{4}(c+3)[g(Y, Z)X - g(X, Z)Y] - \frac{1}{4}(c-1)[\eta(Y)\eta(Z)X \\ - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ - g(\phi Y, Z)\phi X\, g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z]\,.\end{aligned} \tag{1.15}$$

In [12] the authors prove the existence of a $G(QE)_n$ and study some geometric properties. The present paper is organised as follows:
After preliminaries in Section 3, we prove that a quasi-conformally flat $G(QE)_n$ is a manifold of generalized quasi-constant curvature. In the next section it is shown that a Ricci semi-symmetric $G(QE)_n$ is a $N(QE)_n$ and in Section 5, we prove that a Ricci-recurrent $G(QE)_n$ is a $N(QE)_n$. Finally, non-trivial examples of $G(QE)_n$ have been constructed.

2. Preliminaries

We consider a $G(QE)_n$ with associated scalars a, b, d and associated 1-forms A, B. From (1.4) we get

$$r = na + b + d\,, \tag{2.1}$$

where r denotes the scalar curvature of the manifold. Since U and V are orthogonal unit vector fields,

$$g(U, U) = 1\,, \quad g(V, V) = 1\,, \quad g(U, V) = 0\,.$$

Putting X=Y=U in (1.4) we get

$$S(U, U) = a + b\,. \tag{2.2}$$

Again putting X=Y=V in (1.4) we obtain

$$S(V, V) = a + d\,. \tag{2.3}$$

Let L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then $g(LX, Y) = S(X, Y),\ \forall X, Y$.

In an n-dimensional ($n > 2$) Riemannian manifold the quasi-conformal curvature tensor is defined as [21]

$$\begin{aligned}\acute{C}^{\star}(X,Y,Z,W) &= a_1\acute{R}(X,Y,Z,W) + b_1[S(Y,Z)g(X,W)\\ &\quad - S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)]\\ &\quad - \frac{r}{n}[\frac{a_1}{n-1} + 2b_1][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\,,\end{aligned} \tag{2.4}$$

where $g(C^{\star}(X,Y)Z,W) = \acute{C}^{\star}(X,Y,Z,W)$.
If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (2.4) takes the form of conformal curvature tensor C, where

$$\begin{aligned}C(X,Y,Z,W) &= \acute{R}(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W)\\ &\quad - S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)]\\ &\quad + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\,.\end{aligned} \tag{2.5}$$

3. Quasi conformally flat $G(QE)_n$

A manifold of generalized quasi-constant curvature is a $G(QE)_n$. But the converse is not true, in general. In this section, we enquire under what conditions the converse will be true.
Let us suppose that the manifold under consideration is quasi-conformally flat. Then quasi-conformal curvature tensor vanishes identically.
So from (2.4) we obtain

$$\begin{aligned}\acute{R}(X,Y,Z,W) &= \frac{b_1}{a_1}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)\\ &\quad + g(X,Z)g(QY,W) - g(Y,Z)g(QX,W)]\\ &\quad + \frac{r}{n}\left[\frac{a_1}{n-1} + 2b_1\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\,.\end{aligned} \tag{3.1}$$

Using (1.4), (2.1) in (3.1) we obtain

$$\begin{aligned}\acute{R}(X,Y,Z,W) &= \alpha[g(Y,Z)g(X,W) - g(Y,Z)g(Y,W)]\\ &\quad + \beta[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)\\ &\quad + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)]\\ &\quad \gamma[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)\\ &\quad + g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)]\,,\end{aligned} \tag{3.2}$$

where $\alpha = \frac{(na+b+d)a_1}{n(n-1)} + \frac{2b_1(na+b+d)}{n} - \frac{2ab_1}{a_1}$, $\beta = -\frac{bb_1}{a_1}$, $\gamma = -\frac{db_1}{a_1}$.
This shows that the manifold is one of generalized quasi-constant curvature.
This leads to the following theorem:

Theorem 3.1. *A quasi-conformally flat $G(QE)_n$ is one of generalized quasi-constant curvature.*

4. Ricci semi-symmetric $G(QE)_n$

Let us suppose that the manifold under consideration is Ricci semi-symmetric. This means that $R(X,Y)\cdot S = 0$, $\forall X$, Y, where $R(X,Y)$ denotes the curvature operator and the dot means that $R(X,Y)$ acts as derivation on the tensor algebra. Now $(R(X,Y)\cdot S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W)$.
Then,

$$\begin{aligned} ag\big(R(X,Y)Z,W\big) + bA\big(R(X,Y)Z\big)A(W) + dB\big(R(X,Y)Z\big)B(W) \\ + ag\big(Z,R(X,Y)W\big) + bA(Z)A\big(R(X,Y)W\big) \\ + dB(Z)B\big(R(X,Y)W\big) = 0\,. \end{aligned} \tag{4.1}$$

In (4.1) putting $W = U$ and $Z = V$ we obtain

$$bA\big(R(X,Y)V\big) + dB\big(R(X,Y)U\big) = 0\,,$$

$$\text{i.e.,}\quad (d-b)\acute{R}(X,Y,U,V) = 0\,, \quad \text{where} \quad \acute{R}(X,Y,U,V) = g\big(R(X,Y)U,V\big)\,.$$

Since $\acute{R} \neq 0$, we obtain $b = d$.
Using $b = d$ in (1.4), we obtain

$$S(X,Y) = ag(X,Y) + b[A(X)A(Y) + B(X)B(Y)]\,,$$

$$\text{i.e.,}\qquad S(X,Y) = ag(X,Y) + bE(X,Y)\,,$$

where $E(X,Y) = A(X)A(Y) + B(X)B(Y)$.
This leads to the following theorem:

Theorem 4.1. *Every Ricci semi-symmetric $G(QE)_n$ is a $N(QE)_n$.*

5. Ricci-recurrent $G(QE)_n$

A Riemannian manifold M^{2n+1} is said to be Ricci-recurrent [17] if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z)\,, \tag{5.1}$$

where α is non-zero 1-form. We now define a function f on M^{2n+1} by

$$f^2 = g(Q,Q)\,,$$

where $g(QX,Y) = S(X,Y)$ and the Riemannian metric g is extended to the inner product between the tensor fields in the standard fashion. Then we obtain

$$f(Yf) = f^2\alpha(Y)\,.$$

So from this we have

$$Yf = f\alpha(Y)\,. \tag{5.2}$$

>From (5.2) we have

$$X(Yf) - Y(Xf) = [X\alpha(Y) - Y\alpha(X)]f\,. \tag{5.3}$$

Therefore, we get

$$[\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}]f = [X\alpha(Y) - Y\alpha(X) - \alpha[X,Y]]f\,. \tag{5.4}$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M^{2n+1} by our assumption, we obtain

$$d\alpha(X,Y) = 0\,, \tag{5.5}$$

that is, the 1-form α is closed. Now from $(\nabla_Y S)(U,V) = \alpha(Y)S(U,V)$, we get

$$(\nabla_X \nabla_Y S)(U,V) = [X\alpha(Y) + \alpha(X)\alpha(Y)]S(U,V)\,.$$

Hence from (5.5) we get

$$(R(X,Y)\cdot S)(U,V) = 2d\alpha(X,Y)S(U,V)\,.$$

That is, our manifold satisfies $R(X,Y)\cdot S = 0$.
Thus a Ricci-recurrent manifold is Ricci semi-symmetric.
Hence from Theorem 4.1, we can state the following:

Theorem 5.1. *A Ricci-recurrent $G(QE)_n$ is a $N(QE)_n$.*

6. Examples of a generalized quasi-Einstein manifold

Example 6.1. A 2-quasi-umbilical hypersurface of a space of constant curvature is a $G(QE)_n$.

Let $(M^{n-1},\tilde{g})$ be a hypersurface of (M^n,g). If A is the $(1,1)$ tensor corresponding to the normal valued second fundamental tensor H, then we have ([7, p.41])

$$\tilde{g}(A_\xi(X),Y) = g(H(X,Y),\xi)\,, \tag{6.1}$$

where ξ is the unit normal vector field and X, Y are tangent vector fields.
Let H_ξ be the symmetric $(0,2)$ tensor associated with A_ξ in the hypersurface defined by

$$\tilde{g}(A_\xi(X),Y) = H_\xi(X,Y)\,. \tag{6.2}$$

A hypersurface of a Riemannian manifold (M^n,g) is called quasi-umbilical ([7, p.147]) if its second fundamental tensor has the form

$$H_\xi(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y)\,, \tag{6.3}$$

where ω is a 1-form, the vector field corresponding to the 1-form ω is a unit vector field, and α,β are scalars. If $\alpha = 0$ (resp. $\beta = 0$ or $\alpha = \beta = 0$) holds, then it is called cylindrical (resp. umbilical or geodesic).
Now from (6.1), (6.2) and (6.3) we obtain

$$g(H(X,Y),\xi) = \alpha g(X,Y)g(\xi,\xi) + \beta\omega(X)\omega(Y)g(\xi,\xi)\,,$$

which implies that

$$H(X,Y) = \alpha g(X,Y)\xi + \beta\omega(X)\omega(Y)\xi\,, \tag{6.4}$$

since ξ is the only unit normal vector field.
A hypersurface $(M^{n-1},\tilde{g})$ immersed isometrically in a Riemannian manifold (M^n,g) is said to be 2-quasi-umbilical [16] if the second fundamental tensor H satisfies the equality

$$H(X,Y) = \alpha g(X,Y)\xi + \beta\omega(X)\omega(Y)\xi + \gamma\eta(X)\eta(Y)\xi\,, \tag{6.5}$$

where ξ is the only unit normal vector field, α, β, γ are scalars and ω, η are 1-forms, $\tilde{U}$, $\tilde{V}$ are unit vectors such that

$$g(\tilde{U},\tilde{V}) = 0\,,\quad \omega(X) = g(X,\tilde{U})\,,\quad \eta(X) = g(X,\tilde{V})\,,$$

for every vector field X. Evidently, 2-quasi-umbilical hypersurfaces form a natural extension of the class of quasi-umbilical hypersurfaces. The above definition of 2-quasi-umbilical hypersurface $(M^{n-1},\tilde{g})$ at a point X is equivalent to the following: The hypersurface $(M^{n-1},\tilde{g})$, $n \geq 5$ immersed isometrically in a Riemannian manifold (M^n, g) is said to be 2-quasi-umbilical at a point $x \in M^{n-1}$, when it has a principal curvature with *multiplicity* $\geq n-3$.
We have the following equation of Gauss ([7, p.45]) for any vector field X, Y, Z, W tangent to the hypersurface

$$\begin{aligned}\tilde{g}\big(\tilde{R}(X,Y)Z,W\big) &= g\big(R(X,Y)Z,W\big) + g\big(H(X,W),H(Y,Z)\big)\\ &\quad - g\big(H(Y,W),H(X,Z)\big)\,,\end{aligned} \tag{6.6}$$

where $\tilde{R}$ is the curvature tensor of the hypersurface.
Now we consider 2-quasi-umbilical hypersurface of a space of constant curvature. Then we have

$$R(X,Y,Z,W) = \lambda[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\,,$$

where λ is a non-zero constant.

Let us suppose that the hypersurface is 2-quasi-umbilical. Then from (6.5) and (6.6), it follows that

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Z,W) &= \lambda[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &\quad + \alpha^2[g(X,W)g(Y,Z) - g(Y,W)g(X,Z)]\\ &\quad + \alpha\beta[g(X,W)\omega(Y)\omega(Z) + g(Y,Z)\omega(X)\omega(W)\\ &\quad - g(Y,W)\omega(X)\omega(Z) - g(X,Z)\omega(Y)\omega(W)]\\ &\quad + \alpha\gamma[g(X,W)\eta(Y)\eta(Z) + g(Y,Z)\eta(X)\eta(W)\\ &\quad - g(Y,W)\eta(X)\eta(Z) - g(X,Z)\eta(Y)\eta(W)]\\ &\quad + \beta\gamma[\omega(X)\eta(Y)\eta(Z)\omega(W) + \eta(X)\eta(W)\omega(Y)\omega(Z)\\ &\quad - \eta(Y)\eta(W)\omega(X)\omega(Z) - \omega(Y)\omega(W)\eta(X)\eta(Z)]\,.\end{aligned} \tag{6.7}$$

Now from (6.7) we get on contraction

$$\begin{aligned}S(Y,Z) &= [(\lambda+\alpha^2)(n-2) + \alpha\beta + \alpha\gamma]g(Y,Z)\\ &\quad + [\alpha\beta(n-3) + \beta\gamma]\omega(Y)\omega(Z)\\ &\quad + [\alpha\gamma(n-3) + \beta\gamma]\eta(Y)\eta(Z)\,,\end{aligned} \tag{6.8}$$

which shows that a 2-quasi-umbilical hypersurface of a space of constant curvature is a $G(QE)_n$, which is not a quasi-Einstein manifold. However a quasi-umbilical hypersurface of a space of constant curvature is quasi-Einstein.

Example 6.2. A quasi-umbilical hypersurface of a Sasakian space form is a $G(QE)_n$.

An n-dimensional hypersurface M, $n \geq 3$, in a Riemannian manifold M is said to be quasi-umbilical [8] at a point $x \in M$ if at the point x its second fundamental tensor H satisfies the equality

$$H(X,Y) = \alpha g(X,Y)\xi + \beta\omega(X)\omega(Y)\xi\,, \tag{6.9}$$

where ω is a 1-form and α and β are some functions on M. If (6.9) is fulfilled at every point of M, then it is called a quasi-umbilical hypersurface.
Using (6.6) and (6.9) we get

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) - g(\alpha g(X,W)\xi \\ &\quad + \beta\omega(X)\omega(W)\xi, \alpha g(Y,Z)\xi + \beta\omega(Y)\omega(Z)\xi) \\ &\quad + g(\alpha g(Y,W)\xi + \beta\omega(Y)\omega(W)\xi, \alpha g(X,Z)\xi + \beta\omega(X)\omega(Z)\xi)\,.\end{aligned} \tag{6.10}$$

Since $M(c)$ is a Sasakian space form, by the use of (1.15) in (6.10) we obtain

$$\begin{aligned}&\frac{1}{4}(c+3)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] - \frac{1}{4}(c-1)[\eta(Y)\eta(Z)g(X,W) \\ &\quad - \eta(X)\eta(Z)g(Y,W) + g(Y,Z)\eta(X)g(\xi,W) - g(X,Z)\eta(Y)g(\xi,W) \\ &\quad - g(\phi Y,Z)g(\phi X,W) + g(\phi X,Z)g(\phi Y,W) + 2g(\phi X,Y)g(\phi Z,W)] \\ &= R(X,Y,Z,W) - \alpha^2 g(X,W)g(Y,Z) - \alpha\beta g(X,W)\omega(Y)\omega(Z) \\ &\quad - \alpha\beta g(Y,Z)\omega(X)\omega(W) + \alpha^2 g(Y,W)g(X,Z) \\ &\quad + \alpha\beta g(Y,W)\omega(X)\omega(Z) + \alpha\beta g(X,Z)\omega(Y)\omega(W)\,.\end{aligned} \tag{6.11}$$

Contracting the above equation over X and W and using (1.11) and $g(\phi X,\xi) = 0$, we obtain

$$\begin{aligned}S(Y,Z) &= [\frac{c+3}{2}(2n-1) + \frac{c-1}{4} + (2n-1)\alpha^2 - \alpha\beta]g(Y,Z) \\ &\quad -\frac{c-1}{4}(2n+1)\eta(Y)\eta(Z) + 2n\alpha\beta\omega(Y)\omega(Z),\end{aligned} \tag{6.12}$$

which shows that a quasi-umbilical hypersurface of Sasakian space form is a $G(QE)_n$, which is not a quasi-Einstein manifold. However a totally umbilical hypersurface of a Sasakian space form is a quasi-Einstein manifold.

Example 6.3. Let us consider a Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{4/3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2\,. \tag{6.13}$$

Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{1/3},$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}\,,$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{2/3}\,,$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensor are:

$$R_{11} = R_{22} = R_{33} = \frac{2}{3(x^4)^{2/3}}\,, \quad R_{44} = -\frac{2}{3(x^4)^2}\,.$$

It can be easily shown that the scalar curvature of the resulting manifold $(\mathbb{R}^4, g)$ is $\frac{4}{3(x^4)^2}$ which is non-vanishing and non-constant.
Let us now consider the associated scalars as follows:

$$a = -\frac{2}{3(x^4)^2}\,, \quad b = -\frac{1}{9(x^4)^1}\,, \quad d = \frac{13}{9}\,. \tag{6.14}$$

We also choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} (x^4)^{1/6}\,, & i = 1, 2, 3 \\ 0\,, & i = 4\,, \end{cases}$$

$$B_i(x) = \begin{cases} \frac{1}{(x^4)^{1/3}}\,, & i = 1, 2, 3 \\ 0\,, & i = 4\,, \end{cases}$$

at any point $x \in \mathbb{R}^4$.
In terms of local coordinates, the defining condition of a $G(QE)_n$ can be written as

$$R_{ij} = a g_{ij} + b A_i A_j + d B_i B_j, \tag{6.15}$$

for i, $j = 1, 2, 3, 4$.
By virtue of (6.14) and choice of the 1-forms, it can be easily shown that (6.15) holds for $i = 1, 2, 3, 4$. Therefore, (M^4, g) is a (special) generalized quasi-Einstein manifold, which is not a quasi-Einstein manifold.

Example 6.4. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on M given by

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\,, \quad e_2 = \frac{\partial}{\partial y}\,, \quad e_3 = \frac{\partial}{\partial z}\,.$$

Let g be the Riemannian metric defined by $g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$, $g(e_1, e_1) = g(e_2, e_2) = 1$, $g(e_3, e_3) = 1$.
Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = e_3\,, \quad [e_1, e_3] = 0\,, \quad [e_2, e_3] = 0\,.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\ &\quad - g(Y,[X,Z]) + g(Z,[X,Y])\,,\end{aligned} \tag{6.16}$$

which is known as Koszul's formula.
Koszul's formula yields

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0\,, && \nabla_{e_1} e_2 = \frac{1}{2} e_3\,, && \nabla_{e_1} e_3 = -\frac{1}{2} e_2\,,\\
&\nabla_{e_2} e_1 = -\frac{1}{2} e_3\,, && \nabla_{e_2} e_2 = 0\,, && \nabla_{e_2} e_3 = \frac{1}{2} e_1\,,\\
&\nabla_{e_3} e_1 = -\frac{1}{2} e_2\,, && \nabla_{e_3} e_2 = \frac{1}{2} e_1\,, && \nabla_{e_3} e_3 = 0\,.
\end{aligned}$$

It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\,. \tag{6.17}$$

With the help of the above results and using (6.17), we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned}
&R(e_2,e_3)e_3 = \frac{1}{4} e_2\,, && R(e_1,e_3)e_3 = \frac{1}{4} e_1\,, && R(e_1,e_2)e_2 = -\frac{3}{4} e_1\,,\\
&R(e_2,e_3)e_2 = -\frac{1}{4} e_3\,, && R(e_1,e_3)e_1 = -\frac{1}{4} e_3\,, && R(e_1,e_2)e_1 = \frac{3}{4} e_2\,,
\end{aligned}$$

and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(e_1,e_1) = -\frac{1}{2}\,, \quad S(e_2,e_2) = -\frac{1}{2}\,, \quad S(e_3,e_3) = \frac{1}{2}\,,$$

and the scalar curvature is $-\frac{1}{2}$. Since $\{e_1, e_2, e_3\}$ is a frame field, any vector field $X, Y \in \chi(M)$ can be written as

$$X = \acute{a}_1 e_1 + \acute{b}_1 e_2 + \acute{c}_1 e_3\,,$$

and

$$Y = \acute{a}_2 e_1 + \acute{b}_2 e_2 + \acute{c}_2 e_3\,,$$

where $\acute{a}_i, \acute{b}_i, \acute{c}_i \in \mathbb{R}^+$ (the set of positive real numbers), $i = 1,2,3$, such that $\acute{a}_1\acute{a}_2 + \acute{b}_1\acute{b}_2 + \acute{c}_1\acute{c}_2 \neq 0$. Hence

$$S(X,Y) = -\frac{1}{2}(\acute{a}_1\acute{a}_2 + \acute{b}_1\acute{b}_2 - \acute{c}_1\acute{c}_2)\,,$$

and

$$g(X,Y) = \acute{a}_1\acute{a}_2 + \acute{b}_1\acute{b}_2 + \acute{c}_1\acute{c}_2\,.$$

Let us now consider the associated scalars as follows:

$$a = 1\,, \quad b = -\frac{3}{2}\,, \quad d = -\frac{1}{2}\,.$$

We also choose the associated 1-forms as follows:

$$A(X) = (\acute{a}_1\acute{a}_2 + \acute{b}_1\acute{b}_2)^{1/2}\,, \qquad \forall\ X\,,$$
$$B(X) = (\acute{c}_1\acute{c}_2)^{1/2}\,, \qquad \forall\ X\,.$$

By virtue of the choices of scalars and 1-forms and using (1.4), we can say that (M^3, g) is a (special) generalized quasi-Einstein manifold, which is not quasi-Einstein.
Thus we can state the following:

Theorem 6.1. *There exists a generalized quasi-Einstein manifold of dimension 3 with non-zero constant scalar curvature which is non quasi-Einstein.*

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