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ELLIPTICITY OF THE SYMPLECTIC TWISTOR COMPLEX

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Abstract. For a Fedosov manifold (symplectic manifold equipped with a symplectic torsion-free affine connection) admitting a metaplectic structure, we shall investigate two sequences of first order differential operators acting on sections of certain infinite rank vector bundles defined over this manifold. The differential operators are symplectic analogues of the twistor operators known from Riemannian or Lorentzian spin geometry. It is known that the mentioned sequences form complexes if the symplectic connection is of Ricci type. In this paper, we prove that certain parts of these complexes are elliptic.

1. Introduction

In this article, we prove the ellipticity of certain parts of the so called symplectic twistor complexes. The symplectic twistor complexes are two sequences of first order differential operators defined over Ricci type Fedosov manifolds admitting a metaplectic structure. The mentioned parts of these complexes will be called truncated symplectic twistor complexes and will be defined later in this text.

Now, let us say a few words about the Fedosov manifolds. Formally speaking, a Fedosov manifold is a triple (M^{2l}, ω, ∇) where (M^{2l}, ω) is a (for definiteness $2l$ dimensional) symplectic manifold and ∇ is a symplectic torsion-free affine connection. Connections satisfying these two properties are usually called *Fedosov connections* in honor of Boris Fedosov who used them to obtain a deformation quantization for symplectic manifolds. (See Fedosov [5].) Let us also mention that in contrary to torsion-free Levi-Civita connections, the Fedosov ones are not unique. We refer an interested reader to Tondeur [18] and Gelfand, Retakh, Shubin [6] for more information.

To formulate the result on the ellipticity of the truncated symplectic twistor complexes, one should know some basic facts on the structure of the curvature tensor field of a Fedosov connection. In Vaisman [19], one can find a proof of a theorem which says that such curvature tensor field splits into two parts if $l \geq 2$, namely into the symplectic Ricci and symplectic Weyl curvature tensor fields. If

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$l = 1$, only the symplectic Ricci curvature tensor field occurs. Fedosov manifolds with zero symplectic Weyl curvature are usually called of Ricci type. (See also Cahen, Schwachhöfer [3] for another but related context.)

After introducing the underlying geometric structure, let us start describing the fields on which the differential operators from the symplectic twistor complexes act. These fields are certain exterior differential forms with values in the so called symplectic spinor bundle which is an associated vector bundle to the *metaplectic bundle*. We shall introduce the metaplectic bundle briefly now. Because the first homotopy group of the symplectic group $Sp(2l, \mathbb{R})$ is isomorphic to $\mathbb{Z}$, there exists a connected two-fold covering of this group. The covering space is called the metaplectic group, and it is usually denoted by $Mp(2l, \mathbb{R})$. Let us fix an element of the isomorphism class of all connected $2:1$ coverings of $Sp(2l, \mathbb{R})$ and denote it by λ. In particular, the mapping $\lambda\colon Mp(2l, \mathbb{R}) \to Sp(2l, \mathbb{R})$ is a Lie group homomorphism, and in this case it is also a Lie group representation. A metaplectic structure on a symplectic manifold (M^{2l}, ω) is a notion parallel to that of a spin structure known from Riemannian geometry. In particular, one of its part is a principal $Mp(2l, \mathbb{R})$-bundle $\mathcal{Q}$ covering twice the bundle of symplectic repères $\mathcal{P}$ on (M, ω). This principal $Mp(2l, \mathbb{R})$-bundle is the mentioned metaplectic bundle and will be denoted by $\mathcal{Q}$ in this paper.

As we have already written, the fields we are interested in are certain exterior differential forms on M^{2l} with values in the *symplectic spinor bundle* which is a vector bundle over M associated to the chosen principal $Mp(2l, \mathbb{R})$-bundle $\mathcal{Q}$ via an 'analytic derivate' of the Segal-Sahle-Weil representation. The Segal-Shale-Weil representation is a faithful unitary representation of the metaplectic group $Mp(2l, \mathbb{R})$ on the vector space $L^2(\mathbb{L})$ of complex valued square Lebesgue integrable functions defined on a Lagrangian subspace $\mathbb{L}$ of the canonical symplectic vector space $(\mathbb{R}^{2l}, \omega_0)$. For technical reasons, we shall use the so called Casselman-Wallach globalization of the underlying Harish-Chandra $(\mathfrak{g}, \tilde{K})$-module of the Segal-Shale-Weil representation. Here, $\mathfrak{g}$ is the Lie algebra of the metaplectic group $\tilde{G}$ and $\tilde{K}$ is a maximal compact subgroup of the group $\tilde{G}$. The vector space carrying this globalization is the Schwartz space $\mathbf{S} := \mathcal{S}(\mathbb{L})$ of smooth functions on $\mathbb{L}$ rapidly decreasing in infinity with its usual Fréchet topology. This Schwartz space is the 'analytic derivate' mentioned above. We shall denote the resulting representation of $Mp(2n, \mathbb{R})$ on $\mathbf{S}$ by L and call it the *metaplectic representation*, i.e., we have $L\colon Mp(2l, \mathbb{R}) \to \mathrm{Aut}(\mathbf{S})$. Let us mention that $\mathbf{S}$ decomposes into two irreducible $Mp(2l, \mathbb{R})$-submodules $\mathbf{S}_+$ and $\mathbf{S}_-$, i.e., $\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-$. The elements of $\mathbf{S}$ are usually called *symplectic spinors*. See Kostant [11] who used them in the context of geometric quantization.

The underlying algebraic structure of the symplectic spinor valued exterior differential forms is the vector space $\mathbf{E} := \bigwedge^\bullet(\mathbb{R}^{2l})^* \otimes \mathbf{S} = \bigoplus_{r=0}^{2l} \bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}$. Obviously, this vector space is equipped with the following tensor product representation ρ of the metaplectic group $Mp(2l, \mathbb{R})$. For $r = 0, \ldots, 2l$, $g \in Mp(2l, \mathbb{R})$ and $\alpha \otimes s \in \bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}$, we set $\rho(g)(\alpha \otimes s) := \lambda(g)^{*\wedge r}\alpha \otimes L(g)s$ and extend this prescription by linearity. With this notation in mind, the symplectic spinor valued exterior differential forms are sections of the vector bundle $\mathcal{E}$ associated to the chosen principal $Mp(2l,\mathbb{R})$-bundle $\mathcal{Q}$ via ρ, i.e., $\mathcal{E} := \mathcal{Q} \times_\rho \mathbf{E}$. Now, we shall restrict our attention to the mentioned *specific* symplectic spinor valued exterior differential forms. For each $r = 0, \ldots, 2l$, there exists a distinguished irreducible submodule of $\bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}_\pm$ which we denote by $\mathbf{E}^r_\pm$. Actually, the submodules $\mathbf{E}^r_\pm$ are the Cartan components of $\bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}_\pm$, i.e., the highest weight of each of them is the largest one of the highest weights of all irreducible constituents of $\bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}_\pm$ wrt. the standard choices. For $r = 0, \ldots, 2l$, we set $\mathbf{E}^r := \mathbf{E}^r_+ \oplus \mathbf{E}^r_-$ and $\mathcal{E}^r := \mathcal{Q} \times_\rho \mathbf{E}^r$. Further, let us denote the corresponding $Mp(2l,\mathbb{R})$-equivariant projection from $\bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S}$ onto $\mathbf{E}^r$ by p^r. We denote the lift of the projection p^r to the associated (or 'geometric') structures by the same symbol, i.e., $p^r : \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^r(\mathbb{R}^{2l})^* \otimes \mathbf{S})) \to \Gamma(M, \mathcal{E}^r)$.

Now, we are in a position to define the main subject of our investigation, namely the symplectic twistor complexes. Let us consider a Fedosov manifold (M, ω, ∇) and suppose that (M, ω) admits a metaplectic structure. Let d^{∇^S} be the exterior covariant derivative associated to ∇. For each $r = 0, \ldots, 2l$, let us restrict the associated exterior covariant derivative d^{∇^S} to $\Gamma(M, \mathcal{E}^r)$ and compose the restriction with the projection p^{r+1}. The resulting operator, denoted by T_r, will be called *symplectic twistor operator*. In this way, we obtain two sequences of differential operators, namely $0 \longrightarrow \Gamma(M, \mathcal{E}^0) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^1) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(M, \mathcal{E}^l) \longrightarrow 0$ and $0 \longrightarrow \Gamma(M, \mathcal{E}^l) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l}) \longrightarrow 0$. It is known, see Krýsl [14], that these sequences form complexes provided the Fedosov manifold (M^{2l}, ω, ∇) is of Ricci type. These two complexes are the mentioned *symplectic twistor complexes*. Let us notice, that we did not choose the full sequence of all symplectic spinor valued exterior differential forms together with the exterior covariant derivative acting between them because for a general or even Ricci type Fedosov manifold, this sequence would not form a complex in general.

As we have mentioned, we shall prove that some parts of these two complexes are elliptic. To obtain these parts, one should remove the last (i.e., the zero) term and the second last term from the first complex and the first term (the zero space again) from the second complex. The complexes obtained in this way will be called truncated symplectic twistor complexes. Let us mention that by an elliptic complex, we mean a complex of differential operators such that its associated symbol sequence is an exact sequence of the sheaves in question. (See, e.g., Wells [21] for details.)

Let us make some remarks on the methods we have used to prove the ellipticity of the truncated symplectic twistor complexes. We decided to use the so called Schur-Weyl-Howe correspondence, which is referred to as the Howe correspondence for simplicity in this text. The Howe correspondence in our case, i.e., for the metaplectic group $Mp(2l,\mathbb{R})$ acting on the space $\mathbf{E}$ of symplectic spinor valued exterior forms, leads to the ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2)$ and a certain representation of this algebra on $\mathbf{E}$. We decided to use the Howe type correspondence mainly because the spaces $\mathbf{E}^r$ (defined above) can be characterized via the mentioned representation of $\mathfrak{osp}(1|2)$ easily and in a way described in this paper. See R. Howe [10] for more information on the Howe type correspondence in general. Let us also mention that besides this duality, the Cartan lemma on exterior differential forms was used. For other examples of elliptic complexes, we refer an interested reader, e.g., to Stein and Weiss [17], Schmid [15], Hotta [9], and Branson [2].

For an application of symplectic spinors in mathematical physics, see, e.g., Shale [16] and Green, Hull [7] and the already mentioned article of Kostant [11]. In the first reference, one can find an application of these spinors in quantizing of Klein-Gordon fields and in the second one in the 10 dimensional super-string theory. The purpose for taking symplectic spinor valued forms might be justified by the intention to describe higher spin boson fields.

In the second section, we recall some known facts on symplectic spinors and the space of symplectic spinor valued exterior forms and its decomposition into irreducible submodules (Theorem 1). In the third chapter, basic information on Fedosov manifolds and their curvature are mentioned and the symplectic twistor complexes are introduced. In the fourth section, the symbol sequence of the symplectic twistor complexes is computed and the ellipticity of the truncated symplectic twistor complexes is proved (Theorem 7).

2. Symplectic spinor valued forms

In this paper the Einstein summation convention is used for finite sums, not mentioning it explicitly unless otherwise is stated. (We will not use this convention in the proof of the Lemma 6 and in the item 3 of the proof of the Theorem 7 only.) The category of representations of Lie groups we shall consider is that one the object of which are finite length admissible representations of a fixed reductive group G on Fréchet vector spaces and the morphisms are continuous G-equivariant maps between the objects. All manifolds, vector bundles and their sections in this text are supposed to be smooth. The only manifolds which are allowed to be of infinite dimension are the total spaces of vector bundles. If this is the case, the bundles are supposed to be Fréchet. The base manifolds are always finite dimensional. The sheaves we will consider are sheaves of smooth sections of vector bundles. If $E \to M$ is a Fréchet vector bundle, we denote the sheaf of sections by Γ, i.e., $\Gamma(U) := \Gamma(U, E)$ for each open set U in M. For $m \in M$, we denote the stalk of Γ at m by Γ_m.

2.1. *Symplectic linear algebra and basic notation.* In order to set the notation, let us start recalling some simple results from symplectic linear algebra. Let $(\mathbb{V}, \omega_0)$ be a real symplectic vector space of dimension $2l$, $l \geq 1$. Let us choose two Lagrangian subspaces $\mathbb{L}$ and $\mathbb{L}'$, such that $\mathbb{V} \simeq \mathbb{L} \oplus \mathbb{L}'$ [1]. It is easy to see that $\dim \mathbb{L} = \dim \mathbb{L}' = l$. Further, let us choose an adapted symplectic basis $\{e_i\}_{i=1}^{2l}$ of $(\mathbb{V} \simeq \mathbb{L} \oplus \mathbb{L}', \omega_0)$, i.e., $\{e_i\}_{i=1}^{2l}$ is a symplectic basis of $(\mathbb{V}, \omega_0)$ and $\{e_i\}_{i=1}^{l} \subseteq \mathbb{L}$ and $\{e_i\}_{i=l+1}^{2l} \subseteq \mathbb{L}'$. The basis dual to the basis $\{e_i\}_{i=1}^{2l}$ will be denoted by $\{\epsilon^i\}_{i=1}^{2l}$, i.e., for $i, j = 1, \ldots, 2l$ we have $\epsilon^j(e_i) = \iota_{e_i}\epsilon^j = \delta_i^j$, where $\iota_v \alpha$ for an element $v \in \mathbb{V}$ and an exterior form $\alpha \in \bigwedge^\bullet \mathbb{V}^*$, denotes the contraction of the form α by the vector v. Further for $i, j = 1, \ldots, 2l$, we set $\omega_{ij} := \omega_0(e_i, e_j)$ and define ω^{ij}, i, $j = 1, \ldots, 2l$, by

[1] Let us recall that by Lagrangian, we mean maximal isotropic wrt. ω_0.

the equation $\omega_{ij}\omega^{kj} = \delta_i^k$ for all i, $k = 1, \ldots, 2l$. Let us remark that not only $\omega_{ij} = -\omega_{ji}$, but also $\omega^{ij} = -\omega^{ji}$ for i, $j = 1, \ldots, 2l$.

As in the Riemannian case, we would like to rise and lower indices of tensor coordinates. In the symplectic case, one should be more careful because of the anti-symmetry of ω_0. For coordinates $K_{ab\ldots c\ldots d}{}^{rs\ldots t\ldots u}$ of a tensor K over $\mathbb{V}$. we denote the expression $\omega^{ic}K_{ab\ldots c\ldots d}{}^{rs\ldots t}$ by $K_{ab\ldots}{}^{i}{}_{\ldots d}{}^{rs\ldots t}$ and $K_{ab\ldots c}{}^{rs\ldots t\ldots u}\omega_{ti}$ by $K_{ab\ldots c}{}^{rs\ldots}{}_{i}{}^{\ldots u}$ and similarly for other types of tensors and also in the geometric setting when we will be considering tensor fields over a symplectic manifold (M^{2l}, ω). Let us remark that $\omega_i{}^j = -\omega^j{}_i = \delta_i^j$, i, $j = 1, \ldots, 2l$. Further, one can also define an isomorphism $\sharp\colon \mathbb{V}^* \to \mathbb{V}, \mathbb{V}^* \ni \alpha \mapsto \alpha^\sharp \in \mathbb{V}$, by the formula

$$\alpha(w) = \omega_0(\alpha^\sharp, w) \quad \text{for each} \quad \alpha \in \mathbb{V}^* \quad \text{and} \quad w \in \mathbb{V}\,.$$

For $\alpha = \alpha_i\epsilon^i$ and $j = 1, \ldots, 2l$, we get $\alpha_j = \alpha(e_j) = \omega_0((\alpha^\sharp)^i e_i, e_j) = \omega_{ij}(\alpha^\sharp)^i = (\alpha^\sharp)_j$ which implies $\alpha^\sharp = (\alpha^\sharp)^i e_i = \alpha^i e_i$. Thus, we see that the rising of indices via the form ω_0 is realized by the isomorphism $\sharp$.

Finally, let us introduce the groups we will be using. Let us denote the symplectic group of $(\mathbb{V}, \omega_0)$ by G, i.e., $G := \mathit{Sp}(\mathbb{V}, \omega_0) \simeq \mathit{Sp}(2l, \mathbb{R})$. Because the fundamental group of $G = \mathit{Sp}(\mathbb{V}, \omega_0)$ is $\mathbb{Z}$, there exists a connected $2\colon 1$, necessarily non-universal, covering of G by the so called metaplectic group $\mathit{Mp}(\mathbb{V}, \omega_0)$ denoted by $\tilde{G}$ in this text. Let us denote the mentioned two-fold covering map by λ, in particular $\lambda\colon \tilde{G} \to G$. (See, e.g., Habermann, Habermann [8].)

2.2. *Segal-Shale-Weil representation and symplectic spinor valued forms.* The Segal-Shale-Weil representation is a distinguished representation of the metaplectic group $\tilde{G} = \mathit{Mp}(\mathbb{V}, \omega_0)$.[2] This representation is unitary, faithful and does not descend to a representation of the symplectic group. Its underlying vector space is the vector space of complex valued square Lebesgue integrable functions $L^2(\mathbb{L})$ defined on the chosen Lagrangian subspace $\mathbb{L}$. Let us set $\mathbf{S} := V^\infty(HC(L^2(\mathbb{L})))$, where V^∞ is the Casselman-Wallach globalization functor and HC denotes the forgetful Harish-Chandra functor from the category of $\tilde{G}$-modules defined above into the category of Harish-Chandra $(\mathfrak{g}, \tilde{K})$-modules[3]. We shall denote the resulting representation by L and call it the *metaplectic representation.* Thus, we have

$$L\colon\ \mathit{Mp}(\mathbb{V}, \omega_0) \to \mathrm{Aut}(\mathbf{S})\,.$$

The elements of $\mathbf{S}$ will be called *symplectic spinors.* It is well known that $\mathbf{S}$ splits into two irreducible $\mathit{Mp}(\mathbb{V}, \omega_0)$-submodules $\mathbf{S}_+$ and $\mathbf{S}_-$. Thus, we have $\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-$. See the foundational paper of A. Weil [20] for more detailed information on the Segal-Shale-Weil representation and Casselman [4] on this type of globalization. Let us mention that choosing this particular globalization seems to be rather technical from the point of view of the aim of our article.

In the proof of the ellipticity of the truncated symplectic twistor complexes, we shall need some facts on the underlying vector space of the metaplectic representation. Let us mention that it is known that $\mathbf{S}$ is isomorphic to the Schwartz

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[2]The names oscillator and metaplectic are also used in the literature. See, e.g., Howe [10].

[3]Here, $\mathfrak{g}$ is the Lie algebra of $\tilde{G}$ and $\tilde{K}$ is the maximal compact Lie subgroup of $\tilde{G}$.

space $\mathcal{S}(\mathbb{L})$ of smooth functions rapidly decreasing in the infinity equipped with the standard (locally convex) Fréchet topology generated by the supremum semi-norms. (See, e.g., Habermann, Habermann [8] or Borel, Wallach [1].) For the convenience of the reader, let us briefly recall the definition of the involved semi-norms. For each a, $b \in \mathbb{N}_0^l$, the semi-norm $q_{a,b}$ is defined by the formula $q_{a,b}(f) := \sup_{x\in\mathbb{L}}|(x^a\partial^b f)(x)|$, $f \in \mathcal{S}(\mathbb{L})$. Let us order the set $(q_{a,b})_{a,b}$ in the standard 'lexicographical' way and denote the resulting sequence of semi-norms by $(q^k)_{k\in\mathbb{N}_0}$. These semi-norms generate a complete metric topology on $\mathcal{S}(\mathbb{L})$. Taking $a = b = 0$, one sees that the convergence with respect to the semi-norms implies the uniform convergence immediately. Further, it is well known that the Schwartz space $\mathcal{S}(\mathbb{L})$ possesses a Schauder basis. For a complex metric (e.g., Fréchet) space F, an ordered countable set $(f_i)_{i\in\mathbb{N}} \subseteq F$ is called a Schauder basis of F if each element $f \in F$ can be uniquely expressed as $f = \sum_{i=1}^{\infty} a_i f_i$ for some $a_i \in \mathbb{C}$. Notice that from the uniqueness of the coefficients a_i immediately follows that $0 = \sum_{i=1}^{\infty} a_i f_i$ implies $a_i = 0$ for all $i \in \mathbb{N}$. From the basic mathematical analysis courses, one knows that in the case of the Schwartz space $\mathcal{S}(\mathbb{L})$, one can take, e.g., the lexicographically ordered sequence of Hermite functions in l variables as the Schauder basis. We denote this basis by $(h_i)_{i\in\mathbb{N}}$.

Now, we may define the so called *symplectic Clifford multiplication* $\cdot\colon \mathbb{V}\times\mathbf{S}\to\mathbf{S}$. For $s\in\mathbf{S}$, $x = x^j e_j \in \mathbb{L}$, $x^j\in\mathbb{R}$ and $i, j = 1,\dots,l$, let us set

$$e_i \cdot s(x) := \imath x^i s(x) \quad \text{and} \quad e_{i+l}\cdot s(x) := \frac{\partial s}{\partial x^i}(x)\,.$$

In physics, this mapping (up to a constant multiple) is usually called the canonical quantization. Let us remark that the definition is correct due to the preceding paragraph. For each v, $w\in\mathbb{V}$ and $s\in\mathbf{S}$, one can easily derive the following commutation relation

$$v\cdot w\cdot s - w\cdot v\cdot s = -\imath\omega_0(v,w)s\,. \tag{1}$$

(See, e.g., Habermann, Habermann [8].) We shall use this relation repeatedly and without mentioning its use. Now, we prove that the symplectic Clifford multiplication by a fixed non-zero vector $v\in\mathbb{V}$ is injective as a mapping from $\mathbf{S}$ into $\mathbf{S}$. We shall use the $\tilde{G}$-equivariance of the symplectic Clifford multiplication, i.e., the fact $L(g)(v\cdot s) = [\lambda(g)v]\cdot L(g)s$ which holds for each $g\in\tilde{G}$, $v\in\mathbb{V}$ and $s\in\mathbf{S}$ (see Habermann, Habermann [8]). Thus, let us suppose that a fixed $s\in\mathbf{S}$ and a fixed $0\neq v\in\mathbb{V}$ are given such that $v\cdot s = 0$. Because the action of the symplectic group G on $\mathbb{V}-\{0\}$ is transitive and λ is a covering, there exists an element $g\in\tilde{G}$ such that $\lambda(g)v = e_1$. Applying $L(g)$ on the equation $v\cdot s = 0$, we get $L(g)(v\cdot s) = 0$. Using the above mentioned equivariance of the symplectic Clifford multiplication, we get $0 = L(g)(v\cdot s) = [\lambda(g)v]\cdot(L(g)s) = e_1\cdot(L(g)s)$. Denoting $L(g)s =: \psi$ and using the definition of the symplectic Clifford multiplication, we obtain $\imath x^1\psi = 0$, which implies $\psi(x) = 0$ for each $x = (x^1,\dots,x^l)\in\mathbb{L}$ such that $x^1\neq 0$. By continuity of $\psi\in\mathbf{S}$, we get $\psi = 0$. Because L is a group representation, we get $s = 0$ from $0 = \psi = L(g)s$, i.e., the injectivity of the symplectic Clifford multiplication.

Having defined the metaplectic representation and the symplectic Clifford multiplication, we shall introduce the underlying algebraic structure of the basic geometric object we are interested in, namely the space $\mathbf{E} := \bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$ of symplectic spinor valued exterior forms. The vector space $\mathbf{E}$ is considered with its canonical (Fréchet) direct sum topology induced by the metric topology on the (finite dimensional) space of exterior forms and the Fréchet topology on $\mathbf{S}$. The metaplectic group $\tilde{G}$ acts on $\mathbf{E}$ by the representation

$$\rho\colon \tilde{G} \to \mathrm{Aut}(\mathbf{E}) \quad \text{defined by the formula}$$

$$\rho(g)(\alpha \otimes s) := (\lambda(g)^*)^{\wedge r}\alpha \otimes L(g)s\,,$$

where $\alpha \in \bigwedge^r \mathbb{V}^*$, $s \in \mathbf{S}$, $r = 0, \dots, 2l$, and it is extended by linearity also for non-homogeneous elements.

For $\psi = \alpha \otimes s \in \mathbf{E}$, $v \in \mathbb{V}$ and $\beta \in \bigwedge^\bullet \mathbb{V}^*$, we set $\iota_v\psi := \iota_v\alpha \otimes s$, $\beta \wedge \psi := \beta \wedge \alpha \otimes s$ and $v \cdot \psi := \alpha \otimes v \cdot s$ and extend these definitions by linearity to non-homogeneous elements. Obviously, the contraction, the exterior multiplication and the Clifford multiplication by a fixed vector or co-vector are continuous on $\mathbf{E}$.

Now, we shall describe the decomposition of the space $\mathbf{E}$ into irreducible $\tilde{G}$-submodules. For $i = 0, \dots, l$, let us set $m_i := i$, and for $i = l+1, \dots 2l$, $m_i := 2l - i$, and define the set Ξ of pairs of non-negative integers

$$\Xi := \left\{(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid i = 0, \dots, 2l,\ j = 0, \dots, m_i\right\}.$$

One can say the set Ξ has a shape of a triangle if visualized in a 2-plane. (See the Figure 1. below.) We use the elements of Ξ for parameterizing the irreducible submodules of $\mathbf{E}$.

In Krýsl [12] for each $(i,j) \in \Xi$, two irreducible $\tilde{G}$-modules $\mathbf{E}^{ij}_\pm$ were uniquely defined via the highest weights of their underlying Harish-Chandra modules and by the fact that they are irreducible submodules of $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}_\pm$. For convenience for each $(i,j) \in \mathbb{Z} \times \mathbb{Z} \setminus \Xi$, we set $\mathbf{E}^{ij}_\pm := 0$, and for each $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, we define $\mathbf{E}^{ij} := \mathbf{E}^{ij}_+ \oplus \mathbf{E}^{ij}_-$.

In the following theorem, the decomposition of $\mathbf{E}$ into irreducible $\tilde{G}$-submodules is described.

Theorem 1. *For $r = 0, \dots, 2l$, the following decomposition into irreducible $\tilde{G}$-modules*

$$\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}_\pm \simeq \bigoplus_{\substack{j \\ (r,j) \in \Xi}} \mathbf{E}^{rj}_\pm \quad \textit{holds}.$$

Proof. See Krýsl [12]. □

The following remark on the multiplicity structure of the module $\mathbf{E}$ is crucial. It follows from the prescriptions for the highest weights of the underlying Harish-Chandra modules of $\mathbf{E}^{ij}_\pm$ (see Krýsl [13]).

Remark. 1. For any $(r,j), (r,k) \in \Xi$ such that $j \neq k$, we have

$$\mathbf{E}^{rj}_\pm \not\simeq \mathbf{E}^{rk}_\pm$$

$$\begin{array}{ccccccc}
\mathbf{E}_{\pm}^{00} & \mathbf{E}_{\pm}^{10} & \mathbf{E}_{\pm}^{20} & \mathbf{E}_{\pm}^{30} & \mathbf{E}_{\pm}^{40} & \mathbf{E}_{\pm}^{50} & \mathbf{E}_{\pm}^{60} \\
 & \mathbf{E}_{\pm}^{11} & \mathbf{E}_{\pm}^{21} & \mathbf{E}_{\pm}^{31} & \mathbf{E}_{\pm}^{41} & \mathbf{E}_{\pm}^{51} & \\
 & & \mathbf{E}_{\pm}^{22} & \mathbf{E}_{\pm}^{32} & \mathbf{E}_{\pm}^{42} & & \\
 & & & \mathbf{E}_{\pm}^{33} & & &
\end{array}$$

FIG. 1: Decomposition of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ for $2l = 6$.

(any combination of $\pm$ at both sides of the preceding relation is allowed). Thus in particular, $\bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$ is multiplicity-free for each $r = 0, \ldots, 2l$.

2. Moreover, it is known that $\mathbf{E}_{\pm}^{rj} \simeq \mathbf{E}_{\mp}^{sj}$ for each $(r, j), (s, j) \in \Xi$. One cannot change the order of $+$ and $-$ at precisely one side of the preceding isomorphism without changing its trueness.
3. From the preceding two items, one gets immediately that there are no submodules of $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}_{\pm}^{i+1,i+1}$ for each $i = 0, \ldots, l-1$.

In the Figure 1, one can see the decomposition structure of $\bigwedge^{\bullet} \mathbb{V}^* \otimes \mathbf{S}_{\pm}$ in the case of $l = 3$. For $i = 0, \ldots, 6$, the i^{th} column constitutes of the irreducible modules in which the $\mathbf{S}_{\pm}$-valued exterior forms of form-degree i decompose.

In the next theorem, the decomposition of $\mathbb{V}^* \otimes \mathbf{E}^{ij}$, $(i, j) \in \Xi$, into irreducible $\tilde{G}$-submodules is described. Let us remind the reader that due to our convention $\mathbf{E}^{ij} = 0$ for $(i, j) \in \mathbb{Z} \times \mathbb{Z} \setminus \Xi$. We will use this theorem in the proofs of Lemma 6 and Theorem 7 on the ellipticity of the truncated symplectic twistor complexes.

Theorem 2. *For $(i, j) \in \Xi$, we have*

$$(\mathbb{V}^* \otimes \mathbf{E}^{ij}) \cap (\bigwedge^{i+1} \mathbb{V}^* \otimes \mathbf{S}) \simeq \mathbf{E}^{i+1,j-1} \oplus \mathbf{E}^{i+1,j} \oplus \mathbf{E}^{i+1,j+1}.$$

Proof. See Krýsl [13]. □

Remark. Roughly speaking, the theorem says that the wedge multiplication sends each irreducible module $\mathbf{E}^{ij}$ into at most three "neighbor" modules in the $(i+1)^{st}$ column. (See the Figure 1.)

2.3. *Operators related to a Howe type correspondence***.** In this section, we will introduce five continuous linear operators acting on the space $\mathbf{E}$ of symplectic spinor valued exterior forms. Let us mention that these operators are related to the so called Howe type correspondence for the metaplectic group $Mp(\mathbb{V}, \omega_0)$ acting on

$\mathbf{E}$ via the representation ρ. For $r = 0, \dots, 2l$ and $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, we set

$$F^+ \colon \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}\,, \qquad F^+(\alpha \otimes s) := \frac{\imath}{2} \sum_{i=1}^{2l} \epsilon^i \wedge \alpha \otimes e_i \cdot s$$

and

$$F^- \colon \bigwedge^r \mathbb{V}^* \otimes \mathbf{S} \to \bigwedge^{r-1} \mathbb{V}^* \otimes \mathbf{S}\,, \qquad F^-(\alpha \otimes s) := \frac{1}{2} \sum_{i=1}^{2l} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s$$

and extend them linearly. Further, we shall introduce the operators H, E^+ and E^- acting also continuously on the space $\mathbf{E} = \bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$. We define

$$H := 2\{F^+, F^-\} \quad \text{and} \quad E^\pm := \pm 2\{F^\pm, F^\pm\}\,,$$

where $\{\,,\}$ denotes the anti-commutator in the associative algebra $\mathrm{End}(\mathbf{E})$. By a direct computation, we get

$$E^-(\alpha \otimes s) = \frac{\imath}{2} \omega^{ij} \iota_{e_i} \iota_{e_j} \alpha \otimes s \tag{2}$$

for any $\alpha \otimes s \in \bigwedge^\bullet \mathbb{V}^* \otimes \mathbf{S}$. Thus, we see that the operator E^- acts on the form-part of a symplectic spinor valued exterior form only. Because of that we will write $E^- \alpha \otimes s$ instead of $E^-(\alpha \otimes s)$ simply.

In the next lemma, we sum-up some known facts and derive some new information on the operators $F^\pm$, $E^\pm$ and H which we shall need in the proof of the ellipticity of the truncated symplectic twistor complexes.

Lemma 3. 1. *The operators $F^\pm$, $E^\pm$ and H are $\tilde{G}$-equivariant.*

2. *For $i = 0, \dots, l$, the operator $F^-_{|\mathbf{E}^{im_i}} = 0$ and for $i = l, \dots, 2l$, the operator $F^+_{|\mathbf{E}^{im_i}} = 0$.*
3. *The associative algebra $End_{\tilde{G}}(\mathbf{E}) := \{A \colon \mathbf{E} \to \mathbf{E} \text{ continuous} \mid A\rho(g) = \rho(g)A \text{ for all } g \in \tilde{G}\}$ is, as an associative algebra, finitely generated by F^+ and F^- and the $\tilde{G}$-equivariant projections $p_\pm \colon \mathbf{S} \to \mathbf{S}_\pm$.*
4. *For $\alpha \otimes s \in \bigwedge^r \mathbb{V}^* \otimes \mathbf{S}$, the following relations hold on $\mathbf{E}$*

$$[E^+, E^-] = H\,, \qquad [E^-, F^+] = -F^-\,, \tag{3}$$

$$H(\alpha \otimes s) = \frac{1}{2}(r - l)\alpha \otimes s\,, \tag{4}$$

$$\{F^+, \iota_v\}(\alpha \otimes s) = \frac{\imath}{2} \alpha \otimes v \cdot s \quad \text{and} \quad [F^-, v\cdot](\alpha \otimes s) = \frac{\imath}{2} \iota_v \alpha \otimes s\,. \tag{5}$$

Proof. See Krýsl [13] for the proof of the items 1 and 2, and Krýsl [12] for a proof of the item 3 and of the relations in the rows (3) and (4). Now, suppose we are given an element $v = v^i e_i \in \mathbb{V}$, $v^i \in \mathbb{R}$, $i = 1, \dots, 2l$, and a homogeneous element $\alpha \otimes s \in \bigwedge^j \mathbb{V}^* \otimes \mathbf{S}$, $j = 0, \dots, 2l$. First, let us prove the first relation in the row (5). Using the definition of F^+, we may write $\{F^+, \iota_v\}(\alpha \otimes s) = F^+(\iota_v \alpha \otimes s) + \frac{\imath}{2} \iota_v(\epsilon^i \wedge \alpha \otimes e_i \cdot s) = \frac{\imath}{2}[\epsilon^i \wedge \iota_v \alpha \otimes e_i \cdot s + v^i \alpha \otimes e_i \cdot s - \epsilon^i \wedge \iota_v \alpha \otimes e_i \cdot s] = \frac{\imath}{2} \alpha \otimes v \cdot s$. Thus, the first relation of (5) follows now by linearity. Now, let us prove the second relation at the row (5). Using the definition of F^- and the commutation relation (1), we get $F^-(\alpha\otimes v\cdot s) = \frac{1}{2}(\omega^{ij}\iota_{e_i}\alpha\otimes e_j\cdot v\cdot s) = \frac{1}{2}\omega^{ij}\iota_{e_i}\alpha\otimes(v\cdot e_j\cdot s - \imath\omega_0(e_j,v)s) = v\cdot F^-(\alpha\otimes s) + \frac{\imath}{2}\omega^{ij}\iota_{e_i}\alpha\otimes v_j s = v\cdot F^-(\alpha\otimes s) + \frac{\imath}{2}\iota_v\alpha\otimes s$. Thus, the second relation at the row (5) is proved. □

Remark. The operators $F^\pm$, $E^\pm$ and H satisfy the commutation and anti-commutation relations identical to that ones which are satisfied by the usual generators of the ortho-symplectic super Lie algebra $\mathfrak{osp}(1|2)$.

3. Symplectic twistor complexes and their elliptic parts

In this section, we define the notion of a Fedosov manifold, recall some information on its curvature, introduce a symplectic analogue of the spin structure (the metaplectic structure) and define the symplectic twistor complexes.

Let (M,ω) be a symplectic manifold. Let us consider an affine torsion-free symplectic connection ∇ on (M,ω) and denote the induced connection on $\Gamma(M,\bigwedge^2 T^*M)$ by ∇ as well. Let us recall that by torsion-free and symplectic, we mean $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$ for all $X,Y\in\mathfrak{X}(M)$ and $\nabla\omega = 0$. Such connections are usually called Fedosov connections, and the triple (M,ω,∇) a Fedosov manifold. See the Introduction and the references therein for more information on these connections. The curvature tensor R^∇ of a Fedosov connection is defined in the classical way, i.e., formally by the same formula as in the Riemannian geometry. It is known, see Vaisman [19], that R^∇ splits into two parts, namely into the extended symplectic Ricci and Weyl curvature tensor fields, here denoted by $\widetilde{\sigma}^\nabla$ and W^∇ respectively. Let us display the definitions of these two curvature parts although we shall not use them explicitly. For a symplectic frame $(U,\{e_i\}_{i=1}^{2l})$, $U\subseteq M$, we have the following local formulas

$$\begin{aligned}\sigma_{ij} &:= R^k{}_{ikj}\,,\\ 2(l+1)\widetilde{\sigma}^\nabla_{ijkn} &:= \omega_{in}\sigma_{jk} - \omega_{ik}\sigma_{jn} + \omega_{jn}\sigma_{ik} - \omega_{jk}\sigma_{in} + 2\sigma_{ij}\omega_{kn} \quad \text{and}\\ W^\nabla &:= R^\nabla - \widetilde{\sigma}^\nabla\,,\end{aligned}$$

where $i,j,k,n = 1,\ldots,2l$. Let us call a Fedosov manifold (M,ω,∇) of *Ricci type* if $W^\nabla = 0$.

Remark. Because the Ricci curvature tensor field σ_{ij} is symmetric (see Vaisman [19]), a possible candidate for the scalar curvature, namely $\sigma^{ij}\omega_{ij}$, is zero.

Example. It is easy to see that each Riemann surface equipped with its volume form as the symplectic form and with the Riemann connection is a Fedosov manifold of Ricci type. Further for any $l\geq 1$, the Fedosov manifold $(\mathbb{CP}^l,\omega_{FS},\nabla)$ is also a Fedosov manifold of Ricci type. Here, ω_{FS} is the Kähler form associated to the Fubini-Study metric and to the complex structure on the complex projective space $\mathbb{CP}^l$, and ∇ is the Riemannian connection associated to the Fubini-Study metric.

Now, let us introduce the metaplectic structure the definition of which we have sketched briefly in the Introduction. For a symplectic manifold (M^{2l},ω) of dimension $2l$, let us denote the bundle of symplectic repères in TM by $\mathcal{P}$ and the foot-point projection from $\mathcal{P}$ onto M by p. Thus $(p\colon \mathcal{P} \to M, G)$, where $G \simeq Sp(2l,\mathbb{R})$, is a principal G-bundle over M. As in the subsection 2.1, let $\lambda\colon \tilde{G} \to G$ be a member of the isomorphism class of the non-trivial two-fold coverings of the symplectic group G. In particular, $\tilde{G} \simeq Mp(2l,\mathbb{R})$. Now, let us consider a principal $\tilde{G}$-bundle $(q\colon \mathcal{Q} \to M, \tilde{G})$ over the chosen symplectic manifold (M,ω). We call the pair $(\mathcal{Q},\Lambda)$ metaplectic structure if $\Lambda\colon \mathcal{Q} \to \mathcal{P}$ is a surjective bundle morphism *compatible* with the actions of G on $\mathcal{P}$ and that of $\tilde{G}$ on $\mathcal{Q}$ and with the covering λ in the same way as in the Riemannian spin geometry. (For a more elaborate definition see, e.g., Habermann, Habermann [8].) Let us remark, that typical examples of symplectic manifolds admitting a metaplectic structure are cotangent bundles of orientable manifolds (phase spaces), Calabi-Yau manifolds and the complex projective spaces $\mathbb{CP}^{2k+1}$, $k \in \mathbb{N}_0$.

Now, let us denote the Fréchet vector bundle associated to the introduced principal $\tilde{G}$-bundle $(q : \mathcal{Q} \to M, \tilde{G})$ via the metaplectic representation L on $\mathbf{S}$ by $\mathcal{S}$. Thus, we have $\mathcal{S} = \mathcal{Q} \times_L \mathbf{S}$. We shall call this associated vector bundle $\mathcal{S} \to M$ the *symplectic spinor bundle*. The sections $\phi \in \Gamma(M,\mathcal{S})$ will be called *symplectic spinor fields*. Let us put $\mathcal{E} := \mathcal{Q} \times_\rho \mathbf{E}$. For $r = 0,\ldots,2l$, we define $\mathcal{E}^r := \mathcal{Q} \times_\rho \mathbf{E}^r$, where $\mathbf{E}^r$ abbreviates $\mathbf{E}^{rm_r}$. The smooth sections $\Gamma(M,\mathcal{E})$ will be called symplectic spinor valued exterior differential forms. Because the operators $E^\pm$, $F^\pm$ and H are $\tilde{G}$-equivariant (see the Lemma 3 item 1), they lift to operators acting on sections of the corresponding associated vector bundles. The same is true about the projections p^{ij}, $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. We shall use the same symbols as for the mentioned operators as for their “lifts” to the associated vector bundle structure.

Now, we shall make a use of the Fedosov connection. The Fedosov connection ∇ determines the induced principal G-bundle connection on the principal bundle $(p\colon \mathcal{P} \to M, G)$. This connection lifts to a principal $\tilde{G}$-bundle connection on the principal bundle $(q\colon \mathcal{Q} \to M, \tilde{G})$ and defines the associated covariant derivative on the symplectic bundle $\mathcal{S}$, which we shall denote by ∇^S, and call it the symplectic spinor covariant derivative. See, e.g., Habermann, Habermann [8] for this classical construction. The symplectic spinor covariant derivative ∇^S induces the exterior covariant derivative d^{∇^S} acting on $\Gamma(M,\mathcal{E})$. For $r = 0,\ldots,2l$, we have $d^{\nabla^S} : \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^r \mathbb{V}^* \otimes \mathbf{S})) \to \Gamma(M, \mathcal{Q} \times_\rho (\bigwedge^{r+1} \mathbb{V}^* \otimes \mathbf{S}))$. Now, we are able to define the symplectic twistor operators. For $r = 0,\ldots,2l$, we set

$$T_r\colon \Gamma(M,\mathcal{E}^r) \to \Gamma(M,\mathcal{E}^{r+1})\,, \qquad T_r := p^{r+1,m_{r+1}} d^{\nabla^S}_{|\Gamma(M,\mathcal{E}^r)}$$

and call these operators *symplectic twistor operators*. Informally, one can say that the operators are going on the lower edges of the triangle at the Figure 1. Let us notice that $F^-(\nabla^S - T_0)$ is, up to a non-zero scalar multiple, the so called symplectic Dirac operator introduced by K. Habermann. See, e.g., Habermann, Habermann [8].

In the next theorem, we state that the sequences consisting of the symplectic twistor operators form complexes. These sequences will be called symplectic twistor sequences or complexes.

Theorem 4. *Let $l \geq 2$ and (M^{2l}, ω, ∇) be a Fedosov manifold of Ricci type admitting a metaplectic structure. Then*

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{00}) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^{11}) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(M, \mathcal{E}^{ll}) \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma(M, \mathcal{E}^{ll}) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1,l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l,2l}) \longrightarrow 0$$

are complexes.

Proof. See Krýsl [14]. □

4. Ellipticity of the symplectic twistor complex

After the preceding summarizing parts, we now tend to the proof the ellipticity of the truncated symplectic twistor complexes. Let us recall that by an elliptic complex of differential operators we mean a complex of differential operators acting on the sections of Fréchet bundles such that the associated complex of symbols of the considered differential operators forms an exact sequence of sheaves. Let us recall that a sequence $(\Gamma(\mathcal{F}^\bullet), \pi^\bullet)$ in the category of complexes of sheaves of sections of Fréchet bundles $\mathcal{F}^\bullet$ is called exact if the stalks $[\mathrm{Ker}(\pi^i)]_m$, $[\mathrm{Im}(\pi^{i-1})]_m$ satisfy the equality $[\mathrm{Ker}(\pi^i)]_m = [\mathrm{Im}(\pi^{i-1})]_m$ for each $i \in \mathbb{Z}$ and each $m \in M$, where always when arriving at a preshaef and not at a sheaf, we consider its sheafification not distinguishing it at the notation level. Let us notice that in the case of symbols, we may speak about fibers and not necessarily about stalks because the symbols are bundle and not only sheaf morphisms. See the classical text-book of Wells [21] for more on ellipticity of complexes of differential operators.

After this introductory paragraph, we start with a simple lemma in which the symbol of the exterior covariant symplectic spinor derivative associated to a Fedosov manifold admitting a metaplectic structure is computed.

Lemma 5. *Let (M, ω, ∇) be a Fedosov manifold admitting a metaplectic structure, $\mathcal{S} \to M$ be the corresponding symplectic spinor bundle and d^{∇^S} denotes the exterior covariant derivative. Then for each $\xi \in \Gamma(M, T^*M)$ and $\alpha \otimes \phi \in \Gamma(M, \mathcal{E})$, the symbol σ^ξ of d^{∇^S} is given by*

$$\sigma^\xi(\alpha \otimes \phi) = \xi \wedge \alpha \otimes \phi\,.$$

Proof. For $f \in \mathcal{C}^\infty(M)$, $\xi \in \Gamma(M, T^*M)$ and $\alpha \otimes s \in \Gamma(M, \mathcal{E})$, let us compute $d^{\nabla^S}(f\alpha \otimes s) - f d^{\nabla^S}(\alpha \otimes s) = df \wedge \alpha \otimes s + f d^{\nabla^S}(\alpha \otimes s) - f d^{\nabla^S}(\alpha \otimes s) = df \wedge \alpha \otimes s$. Using this computation, we get the statement of the lemma. □

From now on, we shall denote the projections p^{im_i} onto $\mathbf{E}^i$ by p^i simply, $i = 0, \ldots, 2l$. (In order not to cause a possible confusion, we will make no use of the projections from $\mathbf{E}$ onto $\bigwedge^i \mathbb{V}^* \otimes \mathbf{S}$ or of their lifts to the associated geometric structures.) Due to the previous lemma and the definition of the symplectic twistor operators, we get easily that for each $i = 0, \ldots, 2l$ and $\xi \in \Gamma(M, T^*M)$, the symbol σ_i^ξ of the symplectic twistor operator T_i is given by the formula

$$\sigma_i^\xi(\alpha \otimes s) := p^{i+1}(\xi \wedge \alpha \otimes s)$$

for each $\alpha \otimes s \in \Gamma(M, \mathcal{E}^i)$.

In order to prove the ellipticity of the appropriate parts of the symplectic twistor complexes, we need to compare the kernels and the images of the symbols maps σ_i^ξ for any $\xi \in \Gamma(M, T^*M) \setminus \{0\}$. Therefore, we prove the following statement in which the projections p^i are more specified.

Lemma 6. *For $i = 0, \ldots, l-1$, $\xi \in \mathbb{V}^*$ and $\alpha \otimes s \in \mathbf{E}^i$, we have*

$$p^{i+1}(\xi \wedge \alpha \otimes s) = \xi \wedge \alpha \otimes s + \beta F^+(\alpha \otimes \xi^\sharp \cdot s) + \gamma(E^+ \iota_{\xi^\sharp} \alpha \otimes s) \tag{6}$$

where $\beta = \frac{2}{i-l}$ and $\gamma = \frac{\imath}{i-l}$.

For $i = l+1, \ldots, 2l$ and $\psi \in \mathbf{E}^{i-1, m_{i-1}} \oplus \mathbf{E}^{i-1, m_{i-1}-1} \oplus \mathbf{E}^{i-1, m_{i-1}-2}$, we have

$$p^{i-1}\psi = \psi + \frac{4}{l-i} F^- F^+ \psi + \frac{1}{l-i} E^- E^+ \psi\,. \tag{7}$$

Proof. We prove the first relation only. The second formula can be derived following the same lines of reasoning used for proving the first one. We split the proof of (6) into four parts.

1. In this item, we prove that for a fixed $i \in \{0, \ldots, l\}$ and any $k = 0, \ldots, i$, there exists $\alpha_k^i \in \mathbb{C}$ such that

   $$p^i = \sum_{k=0}^{i} \alpha_k^i (F^+)^k (F^-)^k$$

   with $\alpha_0^i = 1$ for each $i = 0, \ldots, l$. Because for each $i = 0, \ldots, l$, the projections p^i are $\tilde{G}$-equivariant, they can be expressed as (finite) linear combinations of the elements of the finite dimensional vector space $\mathrm{End}_{\tilde{G}}(\mathbf{E})$. Due to the Lemma 3 item 3 (cf. also Krýsl [12]), we know that the complex associative algebra $\mathrm{End}_{\tilde{G}}(\mathbf{E})$ is generated by F^+ and F^- and by the projections $p_\pm$. It is easy to see that the projections $p_\pm$ can be omitted from any expression for p^i and thus, each projection p^i can be expressed just using F^+ and F^-. Due to the defining relation $H = 2\{F^+, F^-\}$ and the relation (4) on the values of H on homogeneous elements, one can order the operators F^+ and F^- in an expression for p^i in the way that the operators F^+ appear on the left-hand and the operators F^- on the right-hand side. In this way, we express p^i as a linear combination of the expressions of type $(F^+)^a(F^-)^b$ for some $a, b \in \mathbb{N}_0$. Since the projection p^i does not change the form degree of a symplectic spinor valued exterior form and F^- and F^+ decreases and increases the form degree by one, respectively, the relation $a = b$ follows. Because the operator F^- decreases the form degree by one, the summands $(F^+)^k(F^-)^k$ for $k > i$ actually do not occur in the expression for the projection p^i written above. Thus,

   $$p^i = \sum_{k=0}^{i} \alpha_k^i (F^+)^k (F^-)^k \tag{8}$$

   for some $\alpha_k^i \in \mathbb{C}$, $k = 0, \ldots, i$.

Now, we shall prove the equation $\alpha_0^i = 1$, $i = 0, \dots, l$. By evaluating the left-hand side of (8) on an element $\phi \in \mathbf{E}^i$ we get ϕ, whereas at the right-hand side the only summand which remains is the one indexed by zero. (The other summands vanish because F^- is $\tilde{G}$-equivariant, decreases the form degree by one and there is no summand in $\bigwedge^{i-1} \mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}^i_+$ or to $\mathbf{E}^i_-$. See the Remark item 3 below the Theorem 1.)

2. Now, suppose $\xi \in \mathbb{V}^*$ and $\alpha \otimes s \in \mathbf{E}^i$, $i = 0, \dots, l-1$. Due to the Theorem 2, we know that $\phi := \xi \wedge \alpha \otimes s \in \mathbf{E}^{i+1,i-1} \oplus \mathbf{E}^{i+1,i} \oplus \mathbf{E}^{i+1,i+1}$. Applying p^{i+1} to the element ϕ, only the zeroth, first, and second summand in the expression $p^{i+1}\phi = \sum_{k=0}^{i+1} \alpha_k^{i+1} (F^+)^k (F^-)^k \phi$ remains. (For $k > 2$, the k^{th} summand vanishes in the expression for $p^{i+1}\phi$ because F^- is $\tilde{G}$-equivariant, decreases the form degree by one and there is no summand in $\bigwedge^{i-2} \mathbb{V}^* \otimes \mathbf{S}$ isomorphic to $\mathbf{E}_\pm^{i+1,i-1}$ or $\mathbf{E}_\pm^{i+1,i}$ or $\mathbf{E}_\pm^{i+1,i+1}$. See the item 3 of the Remark below the Theorem 1.)

3. Due to the previous item, we already know that for the element $\phi = \xi \wedge \alpha \otimes s$ chosen above, we get

$$p^{i+1}\phi = \sum_{k=0}^{2} \alpha_k^{i+1} (F^+)^k (F^-)^k \phi \,.$$

Using the relations (4) and (2), we may write

$$\begin{aligned} p^{i+1}(\xi \wedge \alpha \otimes s) &= \xi \wedge \alpha \otimes s + \alpha_1^{i+1} F^+ F^- (\xi \wedge \alpha \otimes s) \\ &\quad + \alpha_2^{i+1} (F^+)^2 (F^-)^2 (\xi \wedge \alpha \otimes s) \\ &= \xi \wedge \alpha \otimes s + \alpha_1^{i+1} \frac{1}{2} F^+ \omega^{ij} [(\iota_{e_i}\xi)\alpha \otimes e_j \cdot s - \xi \wedge \iota_{e_i}\alpha \otimes e_j \cdot s] \\ &\quad - \alpha_2^{i+1} E^+ \frac{\imath}{32} \omega^{ij} \iota_{e_i} \iota_{e_j} (\xi \wedge \alpha \otimes s) \\ &= \xi \wedge \alpha \otimes s - \alpha_1^{i+1} \frac{1}{2} F^+ [\alpha \otimes \xi^\sharp \cdot s + 2\xi \wedge F^-(\alpha \otimes s)] \\ &\quad - \alpha_2^{i+1} E^+ \frac{\imath}{32} \omega^{ij} \iota_{e_i} (\xi_j \alpha \otimes s - \xi \wedge \iota_{e_j}\alpha \otimes s) \,. \end{aligned}$$

Because $\alpha \otimes s \in \mathbf{E}^i$, we get $F^-(\alpha \otimes s) = 0$ by Lemma 3 item 2. Using the last written equation, we may write

$$\begin{aligned} p^{i+1}(\xi \wedge \alpha \otimes s) &= \xi \wedge \alpha \otimes s - \frac{\alpha_1^{i+1}}{2} F^+(\alpha \otimes \xi^\sharp \cdot s) \\ &\quad - \frac{\imath \alpha_2^{i+1}}{32} E^+ (2\xi^i \iota_{e_i} \alpha \otimes s + \frac{2\alpha_2^{i+1}}{\imath} \xi \wedge E^- \alpha \otimes s) \,. \end{aligned}$$

The last summand in this expression vanishes due to the Lemma 3 item 2 because first $E^- = -4F^-F^-$ (Eqn. (2)) and second $\alpha \otimes s \in \mathbf{E}^i$. Summing-up, we have

$$p^{i+1}\phi = \xi \wedge \alpha \otimes s - \alpha_1^{i+1} \frac{1}{2} F^+(\alpha \otimes \xi^\sharp \cdot s) - \alpha_2^{i+1} \frac{\imath}{16} E^+ \iota_{\xi^\sharp} \alpha \otimes s \,,$$

which is a formula of the form written in the statement of the lemma.

4. In this item, we shall determine the numbers $\beta, \gamma \in \mathbb{C}$. Using the fact that p^{i+1} is an idempotent ($(p^{i+1})^2 = p^{i+1}$), we get $\alpha_1^{i+1} = 4/(l-i)$ and $\alpha_2^{i+1} = 16/(l-i)$ after a tedious but straightforward calculation.

   Thus, comparing the last written formula of the preceding item and the Eqn. (6), we get $\beta = 2/(i-l)$ and $\gamma = \imath/(i-l)$.

$\square$

Remark. For $i = l, \ldots, 2l$, $\xi \in \mathbb{V}^*$ and $\alpha \otimes s \in \mathbf{E}^i$, the formula for p^{i+1} reads simply

$$p^{i+1}(\xi \wedge \alpha \otimes s) = \xi \wedge \alpha \otimes s$$

because of the Theorem 2 and the items 1 and 2 of the Remark below the Theorem 1. (Notice that one may also use the relation (7).)

Now, we are prepared to prove the ellipticity of the truncated symplectic twistor complexes.

Theorem 7. *Let (M^{2l}, ω, ∇) be a Fedosov manifold of Ricci type admitting a metaplectic structure, $l \geq 2$. Then the truncated symplectic twistor complexes*

$$0 \longrightarrow \Gamma(M, \mathcal{E}^0) \xrightarrow{T_0} \Gamma(M, \mathcal{E}^1) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-2}} \Gamma(M, \mathcal{E}^{l-1})$$

and

$$\Gamma(M, \mathcal{E}^l) \xrightarrow{T_l} \Gamma(M, \mathcal{E}^{l+1}) \xrightarrow{T_{l+1}} \cdots \xrightarrow{T_{2l-1}} \Gamma(M, \mathcal{E}^{2l}) \longrightarrow 0$$

are elliptic.

Proof. We should prove the equations $\mathrm{Ker}(\sigma_i^\xi)_m = \mathrm{Im}(\pi_{i-1}^\xi)_m$ for the appropriate indices i and for each point $m \in M$. Here the constituents of the previous equation are fibers of the corresponding shaeves.

1. First, we prove that the sequences mentioned in the formulation of the theorem are complexes. For $i = 0, \ldots, l-2, l, \ldots, 2l-1$, $\psi \in \Gamma(M, \mathcal{E}^i)$ and a differential 1-form $\xi \in \Gamma(M, T^*M)$, we may write $0 = p^{i+2}(0) = p^{i+2}((\xi \wedge \xi) \wedge \psi) = p^{i+2}(\xi \wedge \mathrm{Id}(\xi \wedge \psi)) = p^{i+2}(\xi \wedge \sum_{j=0}^{m_{i+1}} p^{i+1,j}(\xi \wedge \psi))$. Due to the Theorem 2, we know that the last written expression equals $p^{i+2}(\xi \wedge p^{i+1}(\xi \wedge \psi)) = \sigma_{i+1}^\xi \sigma_i^\xi \psi$ and thus $\sigma_{i+1}^\xi \sigma_i^\xi = 0$.
2. Second, we prove the relation $\mathrm{Ker}(\sigma_i^\xi)_m \subseteq \mathrm{Im}(\sigma_{i-1}^\xi)_m$ for each $0 \neq \xi \in T_m^*M$ and $i = 0, \ldots, l-2$. Here $\sigma_{-1}^\xi = 0$ is to be understood. Suppose a homogeneous element $\alpha \otimes s \in \mathcal{E}_m^i$ is given such that $\sigma_i^\xi(\alpha \otimes s) = 0$. (In the next item, we will treat the general non-homogeneous case.) Due to the paragraph below the Lemma 5, we know that $0 = \sigma_i^\xi(\alpha \otimes s) = p^{i+1}(\xi \wedge \alpha \otimes s)$. We shall find an element $\psi \in \mathcal{E}_m^{i-1}$ such that $p^i(\xi \wedge \psi) = \alpha \otimes s$.

   Using formula (6) for the projection (Lemma 6), we may rewrite the equation $p^{i+1}(\xi \wedge \alpha \otimes s) = 0$ into

$$\xi \wedge \alpha \otimes s + \beta F^+(\alpha \otimes \xi^\sharp \cdot s) + \gamma E^+ \iota_{\xi^\sharp} \alpha \otimes s = 0\,. \tag{9}$$

Applying the operator E^- (formula (2)) on the both sides of the previous equation and using the first commutation relation in the row (3) from Lemma 3, we get

$$\frac{\imath}{2}\omega^{ij}\iota_{e_i}\iota_{e_j}(\xi\wedge\alpha)\otimes s+\beta E^-F^+(\alpha\otimes\xi^\sharp\cdot s)+\gamma(E^+E^--2H)\iota_{\xi^\sharp}\alpha\otimes s=0\,.$$

Using the graded Leibniz property of $\iota_{\xi^\sharp}$, the relation (4) for the values of H on form-homogeneous elements and the second relation in the row (3) from Lemma 3, we obtain

$$\frac{\imath}{2}(-2\iota_{\xi^\sharp}-2\imath\xi\wedge E^-)(\alpha\otimes s)+\beta F^+E^-(\alpha\otimes\xi^\sharp\cdot s)-\beta F^-(\alpha\otimes\xi^\sharp\cdot s)$$
$$+\gamma E^+E^-\iota_{\xi^\sharp}\alpha\otimes s+\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.$$

The operator E^- commutes with the operator of the symplectic Clifford multiplication (by the vector field $\xi^\sharp$) and also with the contraction $\iota_{\xi^\sharp}$ because $E^-=\frac{\imath}{2}\omega^{ij}\iota_{e_i}\iota_{e_j}$ (formula (2)). Using these two facts, we get

$$\frac{\imath}{2}(-2\iota_{\xi^\sharp}-2\imath\xi\wedge E^-)(\alpha\otimes s)+\beta F^+\xi^\sharp\cdot E^-(\alpha\otimes s)-\beta F^-(\alpha\otimes\xi^\sharp.s)$$
$$+\gamma E^+\iota_{\xi^\sharp}E^-\alpha\otimes s+\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.$$

Because $F^-(\alpha\otimes s)=0$ (Lemma 3 item 2), we have $E^-\alpha\otimes s=4F^-F^-(\alpha\otimes s)=0$. Thus, we obtain the identity

$$-\imath\iota_{\xi^\sharp}\alpha\otimes s-\beta F^-(\alpha\otimes\xi^\sharp\cdot s)+\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.$$

Substituting the second relation in the row (5) into the previous equation and using the fact $F^-(\alpha\otimes s)=0$ again, we get

$$-\imath\iota_{\xi^\sharp}\alpha\otimes s-\beta\xi^\sharp\cdot F^-(\alpha\otimes s)-\beta\frac{\imath}{2}\iota_{\xi^\sharp}\alpha\otimes s$$
$$+\gamma(l-i+1)\iota_{\xi^\sharp}\alpha\otimes s=0\,.$$

Using the prescription for the numbers β and γ (Lemma 6) and the already twice used relation $F^-(\alpha\otimes s)=0$, we get $(-\imath+\gamma(l-i+1)-\beta\frac{\imath}{2})\iota_{\xi^\sharp}\alpha\otimes s=-2\imath\iota_{\xi^\sharp}\alpha\otimes s=0$ from which the equation

$$\iota_{\xi^\sharp}\alpha\otimes s=0 \tag{10}$$

follows.

Substituting this relation into the prescription for the projection p^i (Eqn. (9)), we get for $i=0,\dots,l-2$ the equation

$$0=p^{i+1}(\xi\wedge\alpha\otimes s)=\xi\wedge\alpha\otimes s+\beta F^+(\alpha\otimes\xi^\sharp\cdot s)\,. \tag{11}$$

Applying the contraction operator $\iota_{\xi^\sharp}$ to the previous equation and using the first formula in the row (5) from Lemma 3, we obtain

$$0=-\xi\wedge\iota_{\xi^\sharp}\alpha\otimes s-\beta F^+\iota_{\xi^\sharp}(\alpha\otimes\xi^\sharp\cdot s)+\beta\frac{\imath}{2}\alpha\otimes\xi^\sharp.(\xi^\sharp\cdot s)\,.$$

Using the fact that the contraction and symplectic Clifford multiplication commute, we have

$$0=-\xi\wedge\iota_{\xi^\sharp}\alpha\otimes s-\beta F^+\xi^\sharp\cdot(\iota_{\xi^\sharp}\alpha\otimes s)+\beta\frac{\imath}{2}\alpha\otimes\xi^\sharp\cdot(\xi^\sharp\cdot s)\,.$$

Substituting the Eqn. (10) into the previous equation, we obtain

$$\alpha \otimes \xi^\sharp \cdot (\xi^\sharp \cdot s) = 0\,.$$

Substituting the definition of F^+ into the equation (11) multiplying it by $\xi^\sharp$ and using the equation $\iota_{\xi^\sharp}\alpha \otimes s = 0$ (Eqn. (10)) again, we get

$$0 = \xi \wedge \alpha \otimes \xi^\sharp \cdot s + \beta \frac{\imath}{2}\epsilon^i \wedge \alpha \otimes \xi^\sharp \cdot e_i \cdot \xi^\sharp \cdot s\,,$$
$$0 = \xi \wedge \alpha \otimes \xi^\sharp \cdot s + \beta \frac{\imath}{2}\epsilon^i \wedge \alpha \otimes (e_i \cdot \xi^\sharp \cdot \xi^\sharp \cdot - \imath\omega_0(\xi^\sharp, e_i)\xi^\sharp \cdot)s\,.$$

Substituting the identity $\alpha \otimes \xi^\sharp \cdot \xi^\sharp \cdot s = 0$ into the previous equation, we obtain

$$0 = (1 + \frac{1}{2}\beta)\xi \wedge \alpha \otimes \xi^\sharp \cdot s\,.$$

If $i = 0, \ldots, l-2$, the coefficient $1 + \beta/2 \neq 0$, and thus by dividing, we get $\xi \wedge \alpha \otimes \xi^\sharp \cdot s = 0$. Because the symplectic Clifford multiplication by a non-zero vector is injective (see the subsection 2.2), we have

$$0 = \xi \wedge \alpha \otimes s\,. \tag{12}$$

3. In this item, we will still suppose $i = 0, \ldots, l-2$. Let us consider a general element $\phi \in \mathrm{Ker}(\sigma_i^\xi)_m \subseteq \mathcal{E}^i_m$ and denote the basis of $\bigwedge^i T^*_m M$ by $(\alpha_{ik})_{k=1}^{n_i}$, $n_i \in \mathbb{N}$. Due to the finite dimensionality of $\bigwedge^i T^*_m M$, there exist complex numbers a_{jk}, $j \in \mathbb{N}$, $k = 1, \ldots, n_i$, such that $\phi = \sum_{k=1}^{n_i}\sum_{j=1}^{\infty} a_{jk}\alpha^{ik} \otimes h_j$ where $(h_j)_{j\in\mathbb{N}}$ is the Schauder basis of $\mathcal{S}_m$ corresponding to the Schauder basis of $\mathcal{S}(\mathbb{L}) \simeq \mathcal{S}_m$. Because the operators $F^\pm$, H, $E^\pm$, ι_ξ and $\xi\wedge$ are continuous on $\mathcal{E}_m$, we get $0 = \sum_{k=1}^{n_i}\sum_{j=1}^{\infty} a_{jk}\xi \wedge \alpha^{ik} \otimes h_j$ precisely in the same way as we obtained the formula (12) in the homogeneous situation (item 2 of this proof). Using the definition of the Schauder basis again, we have for each $j \in \mathbb{N}$ the equation $\sum_{k=1}^{n_i} a_{jk}\xi \wedge \alpha^{ik} = 0$. Using the Cartan lemma on exterior differential systems, we get the existence of a family $(\beta_j)_{j\in\mathbb{N}}$ of $(i-1)$ forms such that $\xi \wedge \beta_j = \sum_{k=1}^{n_i} a_{jk}\alpha^{ik}$. It is possible to see (e.g. by taking the standard Hodge-type metric on the space of forms) that one can choose the family $(\beta_j)_{j\in\mathbb{N}}$ in such a way that $\psi := \sum_{j=1}^{\infty}\beta_j \otimes h_j$ converges. Thus, we may write $\sigma^\xi_{i-1}(\sum_{j=1}^{\infty}\beta_j \otimes h_j) = p^i(\sum_{j=1}^{\infty}\xi \wedge \beta_j \otimes h_j) = p^i(\sum_{j=1}^{\infty}\sum_{k=1}^{n_i} a_{jk}\alpha^{ik} \otimes h_j) = p^i(\phi) = \phi$. Summing-up, we have that $\psi = \sum_{j=1}^{\infty}\beta_j \otimes h_j$ is the desired preimage. Thus, $\phi \in \mathrm{Im}(\sigma^\xi_{i-1})_m$.
4. Now, we prove that $\mathrm{Ker}(\sigma^\xi_i)_m \subseteq \mathrm{Im}(\sigma^\xi_{i-1})_m$ for $i = l+1, \ldots, 2l$, $0 \neq \xi \in \Gamma(M, T^*M)$. If $\phi = \alpha \otimes s \in \mathrm{Ker}(\sigma^\xi_i)_m$, then $0 = p^{i+1}(\xi \wedge \phi) = \xi \wedge \alpha \otimes s$. Due to the Cartan lemma, we know that there is a form $\beta \in \bigwedge^{i-1} T^*_m M$ such that $\xi \wedge \beta \otimes s = \alpha \otimes s$. Define $\psi := p^{i-1}(\beta \otimes s)$. Using the formula (7), the equation $\xi \wedge \beta = \alpha$ and the assumption $F^+(\alpha \otimes s) = 0$ (implied by $\alpha \otimes s \in \mathbf{E}^{im_i}$), one can prove that $\xi \wedge \psi = \alpha \otimes s$ in an analogous way as we proceeded the item 2 of this proof. The dehomogenization goes in the steps similar to that ones written in the preceding item.

□

In the future, we would like to interpret the appropriate (reduced) cohomology groups of the truncated symplectic twistor complexes. Eventually, one can search for an application of the symplectic twistor complexes in representation theory. One can also try to prove that the full (i.e., not truncated) symplectic twistor complexes are not elliptic by finding an example of a suitable Ricci type Fedosov manifold admitting a metaplectic structure.

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