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INVARIANT VARIATIONAL PROBLEMS ON PRINCIPAL BUNDLES AND CONSERVATION LAWS

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Abstract. In this work, we consider variational problems defined by G-invariant Lagrangians on the r-jet prolongation of a principal bundle P, where G is the structure group of P. These problems can be also considered as defined on the associated bundle of the r-th order connections. The correspondence between the Euler-Lagrange equations for these variational problems and conservation laws is discussed.

1. Introduction

The order reduction of the Euler-Lagrange equations of invariant variational functionals on principal bundles belongs to basic problems of the calculus of variations on these spaces. There are several approaches to this problem. One of them is the *Euler-Poincaré reduction*. Let $\pi\colon P \to X$ be a principal G-bundle, and let J^1P denote the first jet prolongation of P. In [5], [4], [6], G-invariant variational problem on J^1P is represented as a reduced variational problem on the bundle of connections $C(P) = (J^1P)/G$ with constraints on the space of possible variations. Moreover, reduced equations correspond to the Noether's conservation laws plus an additional condition of zero curvature of connection.

For frame bundles, this theory was generalized for higher order invariant variational problems in [2]. It was shown that the variational problem, defined by invariant Lagrangian of any order r, can be equivalently studied on the associated space of connections with some compatibility condition, which gives us order reduction of the corresponding Euler-Lagrange equations.

Another possibility of reduction is presented in [1], where the variational problems defined by the first and the second order invariant Lagrangians on the frame bundles are considered. Using the *Noether's theorem*, the system of the Euler-Lagrange equations is equivalent to the system of the same number of lower order equations which come from the corresponding *Noether's currents*.

In this paper, our aim is to extend this theory for higher order G-invariant variational problems on any principal bundle. For this purpose, in Section 3, we introduce the bundle C^rP of the r-th order connections of a principal bundle P, which has auxiliary character in this work. With its help, a condition for a

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Lagrangian on J^rP to be G-invariant is stated. Our main result is presented in Section 4. The equivalence between the system of the Euler-Lagrange equations on one side, and the system of conservation laws on the other side is proved. This gives us an order reduction for solving of these equations. Section 5 is devoted to an illustration of presented theory for the first prolongation of frame bundle.

2. Preliminaries and notation

In this section we briefly recall basic notions and objects of presented theory. Our main reference concerning the calculus of variations on fibred manifolds is [3]; we also use some results contained in [9], [10], [11], and [13]. All manifolds and mappings in this work are considered to be smooth.

For the formulation of variational principles on fibred manifolds we use the following notation. Let Y be a *fibred manifold* with oriented *base manifold* X and *projection* π. We denote $n = \dim X$, $m = \dim Y - n$. J^rY denotes *r-jet prolongation* of Y. The *r-jet* of a section γ of Y at a point $x \in X$, is denoted by $J^r_x\gamma$; and $x \mapsto J^r\gamma(x) = J^r_x\gamma$ is the *r-jet prolongation* of γ. If $J^r_x\gamma \in J^rY$, the *canonical jet projections* $\pi^{r,s}\colon J^rY \to J^sY$, $1 \le s \le r$, $\pi^{r,0}\colon J^rY \to Y$ (the target projection), $\pi^r\colon J^rY \to X$ (the source projection), are defined by $\pi^{r,s}(J^r_x\gamma) = J^s_x\gamma$, $\pi^{r,0}(J^r_x\gamma) = \gamma(x)$, and $\pi^r(J^r_x\gamma) = x$, respectively.

Any *fibred chart* (V, ψ), $\psi = (x^i, y^\sigma)$, on Y, where $1 \le i \le n$, $1 \le \sigma \le m$, induces the *associated charts* on X and on J^rY, (U, φ), $\varphi = (x^i)$, and (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})$, respectively; here $U = \pi(V)$, and $V^r = (\pi^{r,0})^{-1}(V)$.

For any open set $W \subset Y$, Ω^r_0W denotes the ring of smooth functions on W^r. The Ω^r_0W-module of differential q-forms on W^r is denoted by Ω^r_qW. The concept of *horizontalization*, as the exterior algebra morphism $h : \Omega^rW \to \Omega^{r+1}W$, allows us to define a form η to be contact by condition $h\eta = 0$. For any fibred chart (V, ψ), $\psi = (x^i, y^\sigma)$, we can introduce examples of contact 1-forms $\omega^\sigma_{j_1j_2\dots j_l} = dy^\sigma_{j_1j_2\dots j_l} - y^\sigma_{j_1j_2\dots j_lk}dx^k$, where $1 \le l \le r-1$.

It is known that a form $\eta \in \Omega^r_kW$ has a unique decomposition

$$(\pi^{r+1,r})^*\eta = h\eta + p_1\eta + p_2\eta + \dots + p_k\eta\,, \tag{1}$$

such that $p_i\eta$ contains, in any fibred chart, exactly i exterior factors $\omega^\sigma_{j_1j_2\dots j_l}$, $1 \le l \le r$.

$h\eta$ ($p_i\eta$) is the *horizontal* (*i-th contact*) *component* of η. The decomposition (1) is *invariant*, and is called the *canonical decomposition* of η. η is π^r-*horizontal* if and only if $(\pi^{r+1,r})^*\eta = h\eta$. We say that η is *k-contact*, if $(\pi^{r+1,r})^*\eta = p_k\eta$; in this case k is the *order of contactness* of η. We say that the order of contactness of η is $\le l$ ($\ge l$), if $p_{l+1}\eta = p_{l+2}\eta = \dots = p_k\eta = 0$ ($h\eta = p_1\eta = \dots = p_{l-1}\eta = 0$).

A *Lagrangian* (of *order* r) for Y is any π^r-horizontal n-form on some $W^r \subset J^rY$. Its chart expression is

$$\lambda = \mathcal{L}\omega_0\,,$$

where $\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$, and $\mathcal{L}\colon V^r \to \mathbb{R}$ is a *Lagrange function*. The corresponding *variational functional* is a real-valued function $\Gamma_\Omega(\pi) \ni \gamma \to \lambda_\Omega(\gamma) \in$

$\mathbb{R}$, defined by

$$\lambda_\Omega(\gamma) = \int\limits_\Omega J^r\gamma^*\lambda\,,$$

where $\Omega \subset X$ is a compact, n-dimensional submanifold with boundary, and $\Gamma_\Omega(\pi)$ is the set of sections of Y defined on Ω.

The *Euler-Lagrange form* E_λ, associated with a Lagrangian λ of order r, is $\pi^{2r,0}$-horizontal $(n+1)$-form, in fibred chart defined by

$$E_\lambda = E_\sigma(\mathcal{L})\,\omega^\sigma \wedge \omega_0\,,$$

where

$$E_\sigma(\mathcal{L}) = \sum_{k=0}^{r}(-1)^k d_{i_1}d_{i_2}\dots d_{i_k}\frac{\partial\mathcal{L}}{\partial y^\sigma_{i_1i_2\dots i_k}}$$

are the *Euler-Lagrange expressions* associated with λ.

A section γ is an extremal of a Lagrangian λ of order r, if and only if, for every fibred chart on Y, γ satisfies the system of partial differential equations

$$E_\sigma(\mathcal{L}) \circ J^{2r}\gamma = 0\,.$$

A differential form $\rho \in \Omega^s_n W$ is called a *Lepage form*, if $p_1 d\rho$ is $\pi^{s+1,0}$-horizontal $(n+1)$-form. A Lepage form ρ is a *Lepage equivalent* of a Lagrangian $\lambda \in \Omega^r_{n,X}W$, if $h\rho = \lambda$ (possibly up to a jet projection). Now we recall a theorem on the structure of Lepage forms on higher order jet prolongations of fibred manifold ([11]). Let $W \subset Y$ be an open set, (V,ψ), $\psi = (x^i, y^\sigma)$, a fibred chart on Y such that $V \subset W$, and let $\lambda \in \Omega^r_{n,X}W$ be a Lagrangian expressed by $\lambda = \mathcal{L}\omega_0$. A Lepage equivalent $\rho \in \Omega^s_n W$ of a Lagrangian λ has the chart expression

$$(\pi^{s+1,s})^*\rho = \Theta_\lambda + d\tau + \nu\,,$$

where its *principal component* Θ_λ is given by

$$(2)\qquad \Theta_\lambda = \mathcal{L}\omega_0 + \sum_{k=0}^{r-1}\Big(\sum_{l=0}^{r-1-k}(-1)^l d_{i_1}d_{i_2}\dots d_{i_l}\frac{\partial\mathcal{L}}{\partial y^\sigma_{i_1i_2\dots i_lj_1j_2\dots j_ki}}\Big)\omega^\sigma_{j_1j_2\dots j_k}\wedge\omega_i\,,$$

τ is a contact $(n-1)$-form, order of contactness of an n-form ν is at least 2, and we denote $\omega_{i_1i_2\dots i_k} = i_{\partial/\partial x^{i_k}}\dots i_{\partial/\partial x^{i_2}}i_{\partial/\partial x^{i_1}}\omega_0$.

An example of a Lepage equivalent is the well-known *Poincaré-Cartan form* for the first order variational problem. It is a unique Lepage equivalent of a first order Lagrangian λ whose order of contactness is ≤ 1. If λ is expressed in a fibred chart by $\lambda = \mathcal{L}\omega_0$, then

$$(3)\qquad \Theta_\lambda = \mathcal{L}\omega_0 + \frac{\partial\mathcal{L}}{\partial y^\sigma_i}\omega^\sigma\wedge\omega_i\,.$$

An analogous example can be given for second order Lagrangian. We have

$$(4)\qquad \Theta_\lambda = \mathcal{L}\omega_0 + \Big(\frac{\partial\mathcal{L}}{\partial y^\sigma_i} - d_p\frac{\partial\mathcal{L}}{\partial y^\sigma_{pi}}\Big)\omega^\sigma\wedge\omega_i + \frac{\partial\mathcal{L}}{\partial y^\sigma_{ji}}\omega^\sigma_j\wedge\omega_i$$

(see [9]). Expression (4) defines a differential form on J^3Y. We note that for higher order Lagrangians ($r \geq 3$), there is no global Lepage equivalents analogous to (3) and (4).

By an *automorphism* of Y we mean a diffeomorphism $\alpha\colon W \to Y$, where $W \subset Y$ is an open set, for which there exists a diffeomorphism $\alpha_0\colon \pi(W) \to X$ such that $\pi\alpha = \alpha_0\pi$. The *r-jet prolongation* of α is an automorphism $J^r\alpha\colon W^r \to J^rY$ of J^rY, defined by

$$J^r\alpha(J^r_x\gamma) = J^r_{\alpha_0(x)}(\alpha\gamma\alpha_0^{-1})\,.$$

Using this concept, we can introduce the *r-jet prolongation* of any vector field ξ, whose 1-parameter group is formed by automorphisms of Y; such vector fields are called *projectable.* The r-jet prolongation of ξ is denoted by $J^r\xi$.

An automorphism $\alpha : W \to Y$ is said to be an *invariant transformation* of a form $\eta \in \Omega^s_pW$, if

$$J^s\alpha^*\eta = \eta\,.$$

We also say that η is *invariant* with respect to α. Let ξ be a π-projectable vector field on Y. We say that ξ is the *generator* of invariant transformations of η, if

$$\partial_{J^s\xi}\eta = 0\,,$$

where $\partial_{J^s\xi}$ denotes Lie derivative by s-jet prolongation of a vector field ξ. In this case η is said to be *invariant* with respect to ξ. These definitions also include the notion of invariance of *Lagrangians.*

Let $\lambda \in \Omega^r_{n,X}W$ be a Lagrangian, and let $\rho \in \Omega^s_nW$ be a Lepage equivalent of λ. By the *Noether's current* associated with a Lepage equivalent ρ and a vector field ξ we mean the $(n-1)$-form $\Psi_{\rho,\xi} = i_{J^s\xi}\rho$. The *Noether theorem* (see, e.g., [13]) says that if ξ leaves invariant the Lagrangian λ, then for every extremal γ,

$$dJ^s\gamma^*\Psi_{\rho,\xi} = 0\,.$$

3. Prolongations of principal bundles

Let P be a (right) principal fibre bundle with n-dimensional base X and structure group G and let $\pi : P \to X$ be its projection. The *right action* $R : P \times G \to P$ of G on P, will be written as

$$R(u,g) \equiv R_g(u) \equiv R^u(g) \equiv ug\,, \tag{5}$$

for all $u \in P$, $g \in G$. This action can be canonically prolonged to the action $J^rR\colon J^rP \times G \to J^rP$ on r-jet prolongation J^rP of P by

$$J^rR(J^r_x\gamma, g) \equiv (J^rR)_g(J^r_x\gamma) = J^r_x(R_g \circ \gamma)\,, \tag{6}$$

for all $J^r_x\gamma \in J^rP$, $g \in G$. We define

$$C^rP = (J^rP)/G \tag{7}$$

to be the set of orbits of (6). J^rP is the fibre bundle over X with standard fibre $J^r_0(\mathbb{R}^n, G)$ of all jets with source in $0 \in \mathbb{R}^n$ and target in G. Since the projection $\pi^r\colon J^rP \to X$ is G-invariant, we have the projection $\kappa^r\colon C^rP \to X$. The smooth structure on J^rP induces the smooth structure on C^rP such that C^rP becomes the fibre bundle over X with standard fiber $J^r_{(0,e)}(\mathbb{R}^n, G)$, of all jets with source in $0 \in \mathbb{R}^n$ and target in identity element $e \in G$, and the canonical projection

$$\pi^{(r)} : J^r P \to C^r P \tag{8}$$

becomes a surjective submersion (see [8]). Moreover, $J^r P$ is a fibre bundle over $C^r P$ with standard fibre G.

For $r = 1$, the bundle $C^1 P$ can be identified with the bundle of (principal) connections of P, and for $r > 1$, $C^r P$ is called the bundle of r-th order connections. We have the following

Theorem 1. *$J^r P$ is a right principal G-bundle over $C^r P$ such that the diagram*

$$\begin{tikzcd} J^r P \arrow[r, "\pi^{(r)}"] \arrow[d, "\pi^{r,0}"'] & C^r P \arrow[d, "\kappa^r"] \\ P \arrow[r, "\pi"] & X \end{tikzcd} \tag{9}$$

commutes.

Proof. The proof is based on a verification of the definition of a principal G-bundle.

Relation $(J^r R)_g(J^r_x\gamma) = J^r_x\gamma$ for some $J^r_x\gamma \in J^r P$ implies $(R_g \circ \gamma)(x) = \gamma(x)$, which gives us $g = e$, and the action (6) of G on $J^r P$ is free.

Since $\pi : P \to X$ is a principal bundle with structure group G, every $x \in X$ has a neighborhood $U \in X$, for which there is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ such that $\phi(u) = (\pi(u), \chi(u))$, where a mapping $\chi : \pi^{-1}(U) \to G$ satisfies $\chi(R_g(u)) = \chi(u) \cdot g$ for all $u \in \pi^{-1}(U)$ and $g \in G$ ("$\cdot$" denotes the operation in G).

Let us denote $W^r = (\kappa^r)^{-1}(U) \subset C^r P$, and consider a bijective smooth mapping $\theta : (\pi^{(r)})^{-1}(W^r) \to W^r \times G$, defined by $J^r_x\gamma \mapsto ((J^r R)_{g^{-1}}(J^r_x\gamma), \nu(J^r_x\gamma))$, where $\nu(J^r_x\gamma) = \chi(\gamma(x)) \equiv g$. The inverse $\theta^{-1} : W^r \times G \to (\pi^{(r)})^{-1}(W^r)$ is defined by $\theta^{-1} : (J^r_x\gamma, h) \mapsto (J^r R)_h(J^r_x\gamma)$. We have

$$\nu((J^r R)_g(J^r_x\gamma)) = \nu(J^r_x(R_g \circ \gamma)) = \chi(R_g(\gamma(x)) = \chi(\gamma(x)) \cdot g = \nu(J^r_x\gamma) \cdot g$$

for all $J^r_x\gamma \in (\pi^{(r)})^{-1}(W^r)$, $g \in G$, which gives us the local trivialization of the principal bundle $\pi^{(r)} : J^r P \to C^r P$, associated with the local trivialization ϕ of $\pi : P \to X$.

Commuting of the diagram (9) is obvious. □

Consider the canonical jet projections $\pi^{s,s-1} \colon J^s P \to J^{s-1} P$ ($\kappa^{s,s-1} \colon C^s P \to C^{s-1} P$), and the canonical quotient projections $\pi^{(s)} \colon J^s P \to C^s P$, given by (8), for every s, $1 \leq s \leq r$. Then we have the commutative diagram

$$\begin{array}{ccccccccccc} J^rP & \longrightarrow & J^{r-1}P & \longrightarrow & \cdots & \longrightarrow & J^2P & \longrightarrow & J^1P & \longrightarrow & P \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ C^rP & \longrightarrow & C^{r-1}P & \longrightarrow & \cdots & \longrightarrow & C^2P & \longrightarrow & C^1P & \longrightarrow & X \end{array}$$

We say that a form η on J^rP is G-invariant if $(J^rR)^*_g\eta = \eta$ holds for every $g \in G$. According to the identification (7) and Theorem 1 we have

Lemma 1. *A form η on J^rP is G-invariant if and only if there exists a form $\tilde{\eta}$ on C^rP such that $\eta = (\pi^{(r)})^*\tilde{\eta}$.*

This Lemma also covers G-invariant Lagrangians on J^rP.

It is easy to determine the fundamental vector fields of the right action (5). Let $\mathfrak{g}$ be the Lie algebra of G. The *fundamental vector field* ξ on P, associated with an element $\xi_0 \in \mathfrak{g}$, is defined, for every $u \in P$, by the formula

$$\xi(u) = T_eR^u \cdot \xi_0 \tag{10}$$

(see, e.g., [7], [12]).

By the same way, the r-jet prolongation of the right action of G on P (6) generates the fundamental vector fields on J^rP. The r-jet prolongation $J^r\xi$ of the fundamental vector field ξ, defined by (10), on P, coincides with the fundamental vector field on J^rP, associated with ξ_0.

The construction of a fundamental vector field on J^rP gives us that ξ is a generator of invariant transformations of differential forms which are invariant with respect to the right action of G on J^rP.

4. Invariant variational principles on principal bundles

Now we discuss consequences of G-invariance of the r-th order Lagrangians on principal bundles for the Euler-Lagrange equations. Our basic tool is the first variation formula (see [3], [11]).

We denote by $\Psi_{\lambda,\xi}$ the Noether's current associated with the principal component Θ_λ (2), of a Lepage equivalent of a Lagrangian λ, and a vector field ξ, i.e.,

$$\Psi_{\lambda,\xi} = i_{J^{2r-1}\xi}\Theta_\lambda\,.$$

Let P be a principal bundle over an n-dimensional manifold X, and let π be the bundle projection. Suppose that we have a Lagrangian $\lambda \in \Omega^r_{n,X}P$ and a π-*vertical* vector field ξ on P. Then in our standard notation

$$\partial_{J^r\xi}\lambda = i_{J^{2r}\xi}E_\lambda + hdi_{J^{2r-1}\xi}\Theta_\lambda\,. \tag{11}$$

Theorem 2. *Let $\lambda \in \Omega^r_{n,X}P$ be a G-invariant Lagrangian, let $n \geq 2$, and let γ be a section of P. The following conditions are equivalent:*

(a) *γ satisfies the Euler-Lagrange equations,*

$$E_\sigma(\mathcal{L}) \circ J^{2r}\gamma = 0\,.$$

(b) *For any fundamental vector field ξ on P, γ satisfies the conservation law*

$$d(J^{2r-1}\gamma)^* i_{J^{2r-1}\xi}\Theta_\lambda = 0\,.$$

(c) *For every $x \in X$ there exist a neighborhood U of x and $(n-2)$-form η defined on U such that*

$$J^{2r-1}\gamma^*(\Psi_{\lambda,\xi} - d\eta) = 0\,.$$

Proof. By hypothesis, for any fundamental vector field ξ on P, $\partial_{J^{2r-1}\xi}\lambda = 0$. Consequently, since ξ is always π-vertical, the first variation formula (11) reduces to

$$i_{J^{2r}\xi}E_\lambda + hd\Psi_{\lambda,\xi} = 0\,. \tag{12}$$

Suppose that a section γ satisfies the Euler-Lagrange equations. Then the form $i_{J^{2r}\xi}E_\lambda$ vanishes along $J^{2r}\gamma$, so we have $J^{2r}\gamma^* d\Psi_{\lambda,\xi} = dJ^{2r}\gamma^*\Psi_{\lambda,\xi} = 0$. Integrating we can find an $(n-2)$-form η on U such that

$$J^{2r-1}\gamma^*\Psi_{\lambda,\xi} = d\eta\,. \tag{13}$$

Conversely, if a section γ satisfies condition (13), then by (12), γ is necessarily an extremal. □

5. Example: First order invariant variational problem on frame bundles

In this Section the consequences of the above stated theory for frame bundles are shown. We recall some basic objects of the theory of frame bundles. For other related notions we refer to [1], [7], and [12].

Let X be an n-dimensional smooth manifold, and let $\mu : FX \to X$ be the *frame bundle* over an n-dimensional manifold X, which has the structure of a principal fiber bundle with the structure group $Gl_n(\mathbb{R})$. A *frame* at a point $x \in X$ is a pair $\Xi = (x, \zeta)$, where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ is an ordered basis of the tangent space T_xX. For every chart (U, φ), $\varphi = (x^i)$, on X, the *associated chart* on FX, (V, ψ), $\psi = (x^i, x^i_j)$, is defined by $V = \mu^{-1}(U)$, and

$$x^i(\Xi) = x^i(\mu(\Xi)), \quad \zeta_j = x^i_j(\Xi)\Big(\frac{\partial}{\partial x^i}\Big)_x,$$

where $\Xi \in V$. We denote by y^j_k the *inverse matrix* of x^i_j. The right action $FX \times Gl_n(\mathbb{R}) \ni (\Xi, A) \mapsto R_A(\Xi) \in FX$ is given by the equations

$$\bar{x}^i = x^i \circ R_A = x^i, \quad \bar{x}^i_j = x^i_j \circ R_A = x^i_k a^k_j\,,$$

where $A = (a^i_j)$ is an element of the group $Gl_n(\mathbb{R})$.

For the formulation of variational principles defined by a first order Lagrangian on the frame bundles we use the manifolds J^1FX and J^2FX. To the charts (U, φ), and (V, ψ), introduced above, we associate a chart (V^2, ψ^2), $\psi^2 = (x^i, x^i_j, x^i_{j,k}, x^i_{j,kl})$, on J^2FX, defined in a standard way. The general linear group acts on J^2FX on the right by the formula $J^2FX \times Gl_n(\mathbb{R}) \ni (J^2_x\gamma, A) \mapsto J^2_x(R_A \circ \gamma) \in J^2FX$; the action is expressed by the equations

$$\bar{x}^i = x^i\,, \quad \bar{x}^i_j = x^i_k a^k_j\,, \quad \bar{x}^i_{j,k} = x^i_{m,k} a^m_j, \quad \bar{x}^i_{j,kl} = x^i_{m,kl} a^m_j\,. \tag{14}$$

It is easy to determine the orbits of this action. Denoting

$$\Gamma^i_{kp} = -y^m_p x^i_{m,k}, \quad \Gamma^i_{klp} = -y^m_p x^i_{m,kl}\,, \tag{15}$$

we obtain $Gl_n(\mathbb{R})$-*invariant functions* on J^2FX, and *equations of* $Gl_n(\mathbb{R})$-*orbits* are

$$\Gamma^i_{kp} = c^i_{kp}, \quad \Gamma^i_{klp} = c^i_{klp}\,,$$

where $c^i_{kp}, c^i_{klp} \in \mathbb{R}$ are arbitrary numbers. The functions Γ^i_{klp} are symmetric in k, l. We have the following results.

Lemma 2. *Every $Gl_n(\mathbb{R})$-invariant function on J^2FX depends on x^i, Γ^i_{kp}, Γ^i_{klp}.*

In other words Lemma 2 says that $Gl_n(\mathbb{R})$-invariant functions coincide with the functions on the *bundle of second order connections* $C^2FX = (J^2FX)/Gl_n(\mathbb{R})$ over X.

From equations (14) we can obtain an extension of Lemma 2 to differential forms.

Lemma 3. *A k-form η on J^2FX is $Gl_n(\mathbb{R})$-invariant if and only if it has an expression*

$$\eta = \Delta_0 + y^{q_1}_{r_1} dx^{p_1}_{q_1} \wedge \Delta^{r_1}_{p_1} + y^{q_1}_{r_1} y^{q_2}_{r_2} dx^{p_1}_{q_1} \wedge dx^{p_2}_{q_2} \wedge \Delta^{r_1 r_2}_{p_1 p_2}$$
$$+ \cdots + y^{q_1}_{r_1} y^{q_2}_{r_2} \ldots y^{q_k}_{r_k} dx^{p_1}_{q_1} \wedge dx^{p_2}_{q_2} \cdots \wedge dx^{p_k}_{q_k} \wedge \Delta^{r_1 r_2 \ldots r_k}_{p_1 p_2 \ldots p_k}\,,$$

where Δ_0, $\Delta^{r_1}_{p_1}$, $\Delta^{r_1 r_2}_{p_1 p_2}$, $\ldots$, $\Delta^{r_1 r_2 \ldots r_k}_{p_1 p_2 \ldots p_k}$ are arbitrary forms defined on C^2FX.

Further, we need expressions for the fundamental vector fields. Let

$$\xi_0 = \xi^i_j \Big(\frac{\partial}{\partial a^i_j} \Big)_e$$

be a vector belonging to the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $Gl_n(\mathbb{R})$. Then the corresponding fundamental vector field of the action (14), on J^2FX, is given by

$$\xi = \xi^i_s \Big(x^t_i \frac{\partial}{\partial x^t_s} + x^t_{i,k} \frac{\partial}{\partial x^t_{s,k}} + x^t_{i,kl} \frac{\partial}{\partial x^t_{s,kl}} \Big)\,.$$

For the following results we use $Gl_n(\mathbb{R})$-adapted chart (V^2, Ψ^2), $\Psi^2 = (x^i, x^i_j, \Gamma^i_{kp}, \Gamma^i_{klp})$, on J^2FX, associated with a chart (V^2, ψ^2), where coordinate functions are defined by (15). We also denote $\omega^i_j = dx^i_j - x^i_{j,k} dx^k$.

Lemma 4. (a) *A Lagrangian $\lambda = \mathcal{L}\omega_0$ is $Gl_n(\mathbb{R})$-invariant if and only if $\mathcal{L}$ depends on x^i, Γ^i_{kj} only.*

(b) *The Euler-Lagrange form of a $Gl_n(\mathbb{R})$-invariant Lagrangian has an expression $E_\lambda = y^j_l E^l_i(\mathcal{L}) \omega^i_j \wedge \omega_0$, where*

$$E^l_i(\mathcal{L}) = -\Gamma^p_{qi} \frac{\partial \mathcal{L}}{\partial \Gamma^p_{ql}} + \Gamma^l_{pq} \frac{\partial \mathcal{L}}{\partial \Gamma^i_{pq}} + \frac{\partial^2 \mathcal{L}}{\partial x^p \partial \Gamma^i_{pl}} + (\Gamma^k_{mpr} + \Gamma^k_{mq}\Gamma^q_{pr}) \frac{\partial^2 \mathcal{L}}{\partial \Gamma^k_{mr} \partial \Gamma^i_{pl}}\,. \tag{16}$$

(c) *If λ is $Gl_n(\mathbb{R})$-invariant, then the Noether's current associated with the Poincaré-Cartan form Θ_λ and any fundamental vector field ξ is given by*

$$\Psi_{\lambda,\xi} = -\xi_j^m y_l^j x_m^i \frac{\partial \mathcal{L}}{\partial \Gamma_{kl}^i} \omega_k \,.$$

Using the Noether's theorem we get the following correspondence between two sets of equations for the sections of FX.

Theorem 3. *Let $\lambda \in \Omega^1_{n,X} FX$ be a $Gl_n(\mathbb{R})$-invariant Lagrangian, $n \geq 2$, and let γ be a section of FX. The following conditions are equivalent:*

(a) *γ satisfies the Euler-Lagrange equations,*

$$E_i^l(\mathcal{L}) \circ J^2\gamma = 0 \,.$$

(b) *For any fundamental vector field ξ on FX, γ satisfies the conservation law*

$$d(J^1\gamma)^* i_{J^1\xi} \Theta_\lambda = 0 \,.$$

(c) *For any chart (U, φ), $\varphi = (x^i)$, on X, and all j, k, there exist $(n-2)$-forms η_k^j such that*

$$J^1\gamma^* \left(y_l^j x_k^i \frac{\partial \mathcal{L}}{\partial \Gamma_{ml}^i} \omega_m - d\eta_k^j \right) = 0 \,.$$

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