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MAXIMAL SOLVABLE EXTENSIONS OF FILIFORM ALGEBRAS

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Abstract. It is already known that any filiform Lie algebra which possesses a codimension 2 solvable extension is naturally graded. Here we present an alternative derivation of this result.

1. Introduction

We present here an alternative derivation of the result of M. Goze and Yu. Khakimdjanov stating that any filiform Lie algebra which possesses a codimension 2 solvable extension is naturally graded.

Filiform Lie algebras are in a sense least nilpotent of nilpotent Lie algebras. At the same time they are generic examples of nilpotent algebras – in any given dimension filiform algebras form an open subset in the variety of all nilpotent algebras.

Let us recall that the *lower central series* of a given Lie algebra $\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots \supseteq \mathfrak{g}^k \supseteq \dots$ is defined recursively

$$\mathfrak{g}^1 = \mathfrak{g}\,, \qquad \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]\,,\ k \geq 2\,. \tag{1}$$

If the lower central series terminates, i.e. there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ is called a *nilpotent Lie algebra*. The largest value of K for which we have $\mathfrak{g}^K \neq 0$ is the *degree of nilpotency* of the nilpotent Lie algebra $\mathfrak{g}$.

A *filiform* Lie algebra $\mathfrak{n}$ is a nilpotent Lie algebra of maximal degree of nilpotency $K = n - 1$ such that $n = \dim \mathfrak{n} \geq 4$. It immediately follows that $\dim \mathfrak{n}/\mathfrak{n}^2 = 2$ and $\dim \mathfrak{n}^k/\mathfrak{n}^{k+1} = 1$ for $k = 2, \dots, n-1$.

Because the 2-dimensional Abelian algebra and the Heisenberg algebra, i.e. 3-dimensional algebra with the Lie bracket $[e_2, e_3] = e_1$, have properties markedly different from filiform algebras, it is convenient to exclude them by definition from the class of filiform algebras, as we did above.

Properties of filiform nilpotent algebras were investigated in great detail in [9, 3, 5, 6].

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2. Basic properties of filiform algebras

Let us recall some basic facts about filiform algebras. Their detailed derivation can be found in [9] or in [6].

The structure of filiform algebras is most transparent in a suitable, so–called *adapted* basis. Definitions of such basis used by various authors differ by some minor variations. We shall use the one given in [9] upon suitable relabeling which brings it to our chosen structural form (7) below. In such a basis $\mathcal{E}$ of an n-dimensional filiform Lie algebra $\mathfrak{n}$ we have

$$
\begin{aligned}
[e_k, e_n] &= e_{k-1}\,, && k = 2, \dots, n-1\,, \\
[e_1, e_j] &= 0\,, && j = 2, \dots, n\,, \\
[e_j, e_{n-j+1}] &= (-1)^j \alpha e_1\,, && j = 2, \dots, n-1\,, \\
[e_j, e_k] &= 0 \bmod \mathfrak{n}^{2n-j-k+1}\,, && 3 \leq j < k \leq n-1\,,\quad n-1 < j+k\,.
\end{aligned}
\tag{2}
$$

(The antisymmetry, $\mathfrak{n}^k = \operatorname{span}\{e_1, \dots, e_{n-k}\}$, $k \geq 2$ and $[\mathfrak{g}^j, \mathfrak{g}^k] \subseteq \mathfrak{g}^{j+k}$ are assumed to hold). By simultaneous rescaling of basis elements $e_1, \dots, e_{n-1}$ we can multiply α by any nonvanishing number; therefore we can assume $\alpha = 0, 1$ without loss of generality. Furthermore, $\alpha = 1$ is possible only when $\dim \mathfrak{n}$ is even.

To any nilpotent Lie algebra one can associate the *graded* Lie algebra $\mathfrak{gr}(\mathfrak{n})$ of $\mathfrak{n}$

$$
\mathfrak{gr}(\mathfrak{n}) = \sum_{k=1}^{K} \mathfrak{n}^k / \mathfrak{n}^{k+1}
$$

with the bracket

$$
[x, y]_{\mathfrak{gr}} = [x, y] \bmod \mathfrak{n}^{k+j+1}, \qquad \forall\, x \in \mathfrak{n}^k, y \in \mathfrak{n}^j
$$

where in $[x, y]_{\mathfrak{gr}}$ an identification of the equivalence class $x \in \mathfrak{n}^k/\mathfrak{n}^{k+1}$ with its representative $x \in \mathfrak{n}^k$ was used.

Due to [9], n-dimensional filiform algebras can be divided into two classes depending on the structure of their graded algebras. One class has

$$
\mathfrak{gr}(\mathfrak{n}) = \mathfrak{n}_{n,1}
$$

where

$$
\mathfrak{n}_{n,1} = \operatorname{span}\{e_1, \dots, e_n\}\,, \qquad [e_k, e_n] = e_{k-1}\,, \qquad k = 2, \dots, n-1\,. \tag{3}
$$

$\mathfrak{n}_{n,1}$ is often called *model filiform algebra* in the literature.

The second class is present only when n is even and has

$$
\mathfrak{gr}(\mathfrak{n}) = \mathcal{Q}_n
$$

where

$$
\begin{gathered}
\mathcal{Q}_n = \operatorname{span}\{e_1, \dots, e_n\}\,, \\
[e_k, e_n] = e_{k-1}\,, \quad [e_k, e_{n-k+1}] = (-1)^k e_1\,, \qquad k = 2\,, \dots, n-1\,.
\end{gathered}
\tag{4}
$$

$\mathcal{Q}_n$ is often called *special filiform algebra* in the literature.

In Eqs. (3), (4) all commutators not listed explicitly vanish. A notation resembling the one in [8, 1] was used here, with a minor modification – the index in $\mathcal{Q}_n$ is equal to the dimension of $\mathcal{Q}_n$, in [1] it was half of it. These two classes correspond to two different values of the parameter α in the adapted basis (2).

When the associated graded algebra $\mathfrak{gr}(\mathfrak{n})$ coincides with the nilpotent algebra $\mathfrak{n}$, the algebra $\mathfrak{n}$ is called *naturally graded*. Obviously, $\mathfrak{n}_{n,1}$ and $\mathcal{Q}_n$ are the only naturally graded filiform algebras.

3. Structure of solvable algebras with a given nilradical

In [7] some general results concerning the structure of any solvable Lie algebra $\mathfrak{s}$ whose nilradical $\mathfrak{n}$ is isomorphic to a given nilpotent algebra were given. Let us briefly review them here.

Let $\mathfrak{s}$ be a solvable Lie algebra with the nilradical $\mathfrak{n}$ (we call any such $\mathfrak{s}$ a *solvable extension* of the nilpotent Lie algebra $\mathfrak{n}$). Let $(e_1, \ldots, e_n, f_1, \ldots, f_p)$ be a basis of $\mathfrak{s}$ such that $(e_1, \ldots, e_n)$ is a basis of $\mathfrak{n}$. Then the adjoint representation of the element f_a restricted to the nilradical $\mathfrak{n}$,

$$D_a = \mathrm{ad}|_{\mathfrak{n}}(f_a)$$

defines a nonnilpotent outer derivation of $\mathfrak{n}$ (were it nilpotent the nilradical would be larger than $\mathfrak{n}$, namely it would contain $\mathfrak{n} \dot{+} \mathrm{span}\{f_a\}$). In fact, the same argument holds for any linear combination of the derivations D_a, i.e. no nonvanishing linear combination of the derivations D_a is nilpotent. We call any such set of derivations *nilindependent*.

At the same time, the well–known property

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{n} \tag{5}$$

shows that $[D_a, D_b]$ must be an inner derivation for any $1 \le a, b \le p$.

In [7] a theorem was proven, stating that

Theorem 1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra and $\mathfrak{s}$ a solvable Lie algebra with the nilradical $\mathfrak{n}$. Let* $\dim \mathfrak{n} = n$, $\dim \mathfrak{s} = n + p$. *Then p satisfies*

$$p \le n - \dim \mathfrak{n}^2 \,. \tag{6}$$

The main ingredient in its proof which is useful also for considerations in this paper is the following simple observation.

Let $\mathfrak{n}$ be a nilpotent Lie algebra. We can write it as a direct sum of subspaces $\mathfrak{m}_j$

$$\mathfrak{n} = \mathfrak{m}_K \dot{+} \mathfrak{m}_{K-1} \dot{+} \ldots \dot{+} \mathfrak{m}_1$$

such that

$$\mathfrak{n}^j = \mathfrak{m}_j \dot{+} \mathfrak{n}^{j+1} \,, \quad \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1] \,.$$

We denote $m_j = \dim \mathfrak{m}_j$.

In the subspaces $\mathfrak{m}_j$ we can find bases $\mathcal{E}_{\mathfrak{m}_k} = (e_{n+1-\sum_{i=1}^{k} m_i}, \ldots, e_{n-\sum_{i=1}^{k-1} m_i})$ such that

$$\forall e_j \in \mathcal{E}_{\mathfrak{m}_k} \quad \exists y_j \in \mathcal{E}_{\mathfrak{m}_{k-1}}, \quad z_j \in \mathcal{E}_{\mathfrak{m}_1} \quad e_j = [y_j, z_j] \,. \tag{7}$$

Together the elements of the bases $\mathcal{E}_{\mathfrak{m}_k}$ form a basis $\mathcal{E} = (e_1, \ldots, e_n)$ of the whole nilpotent algebra $\mathfrak{n}$. The main advantage of the basis $\mathcal{E}$ is that any automorphism

ϕ, or any derivation D, is fully specified once its action on the elements of the basis $\mathcal{E}_{\mathfrak{m}_1}$ of $\mathfrak{m}_1$ is known. This is an immediate consequence of the definition of an automorphism $\phi([x,y]) = [\phi(x), \phi(y)]$ or of a derivation $D([x,y]) = [D(x), y] + [x, D(y)]$, respectively.

In particular this implies that the matrix of any derivation D of $\mathfrak{n}$ is upper block triangular

$$D = \begin{pmatrix} D_{\mathfrak{m}_K\mathfrak{m}_K} & \cdots & D_{\mathfrak{m}_K\mathfrak{m}_2} & D_{\mathfrak{m}_K\mathfrak{m}_1} \\ & \ddots & \vdots & \vdots \\ & & D_{\mathfrak{m}_2\mathfrak{m}_2} & D_{\mathfrak{m}_2\mathfrak{m}_1} \\ & & & D_{\mathfrak{m}_1\mathfrak{m}_1} \end{pmatrix} \tag{8}$$

and the entries in $D_{\mathfrak{m}_j\mathfrak{m}_k}$, $k \leq j = 2, \ldots, K$ are linear functions of entries in the last column blocks $D_{\mathfrak{m}_1\mathfrak{m}_1}, \ldots, D_{\mathfrak{m}_{j-k+1}\mathfrak{m}_1}$.

In addition, a derivation D is nilpotent if and only if its submatrix $D_{\mathfrak{m}_1\mathfrak{m}_1}$ is nilpotent.

Inner derivations have strictly upper triangular block structure because inner derivations by definition map $\mathfrak{n}^k \to \mathfrak{n}^{k+1}$. Consequently, any set of outer derivations $\{D_1, \ldots, D_f\}$ which commutes to inner derivations, i.e. $[D_j, D_k] \in \mathfrak{Inn}(\mathfrak{n})$, must necessarily have commuting $\mathfrak{m}_1\mathfrak{m}_1$-blocks,

$$[(D_j)_{\mathfrak{m}_1\mathfrak{m}_1}, (D_k)_{\mathfrak{m}_1\mathfrak{m}_1}] = 0\,.$$

4. Maximal solvable extensions of filiform algebras

From Theorem 1 we immediately deduce that any solvable algebra with a filiform n-dimensional nilradical has dimension at most $n+2$. We analyze the conditions under which a given filiform algebra $\mathfrak{n}$ possesses an $(n+2)$-dimensional solvable extension. It was shown in [8, 1] by explicit constructions that this bound is saturated and the maximal extension by two nonnilpotent elements is unique in the case of naturally graded filiform nilradicals $\mathfrak{n}_{n,1}$ and $\mathcal{Q}_n$. On the other hand, we know that for other classes of filiform algebras, namely for $\mathbb{N}$-graded [2], often only one nonnilpotent element can be added. In the literature [6] one can find a proposition stating that $\mathfrak{n}_{n,1}$ and $\mathcal{Q}_n$ are the only filiform algebras which possess a codimension 2 solvable extension. Here, we provide an alternative derivation of that result.

We divide our discussion into two cases:

(1) $\mathfrak{gr}(\mathfrak{n}) \simeq \mathfrak{n}_{n,1}$, i.e. we have $\alpha = 0$ in Eq. (2).

Let us assume that we are given $\mathfrak{n}$ such that its $(n+2)$-dimensional solvable extension exists. That means that we have two nilindependent outer derivations D_1, D_2 of $\mathfrak{n}$. We consider first their submatrices

$$(D_1)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} d^{n-1}_{n-1} & d^{n-1}_{n} \\ d^{n}_{n-1} & d^{n}_{n} \end{pmatrix}, \qquad (D_2)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} \tilde{d}^{n-1}_{n-1} & \tilde{d}^{n-1}_{n} \\ \tilde{d}^{n}_{n-1} & \tilde{d}^{n}_{n} \end{pmatrix}$$

where $\mathfrak{m}_1 = \mathrm{span}\{e_{n-1}, e_n\}$. Because $\alpha = 0$ we have $[e_2, e_{n-1}] = 0$ in Eq. (2) and consequently

$$0 = D_1[e_2, e_{n-1}] = [D_1 e_2, e_{n-1}] + [e_2, D_1 e_{n-1}] = 0 + d^n_{n-1}[e_2, e_n] = d^n_{n-1} e_1 \,,$$

i.e. $d^n_{n-1} = 0$ and similarly $\tilde{d}^n_{n-1} = 0$; the matrices $(D_1)_{\mathfrak{m}_1\mathfrak{m}_1}, (D_2)_{\mathfrak{m}_1\mathfrak{m}_1}$ are upper triangular. The nilindependence of D_1, D_2 implies that by a suitable linear combination of D_1, D_2 we can put them into an equivalent form satisfying

$$(D_1)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} 1 & d^{n-1}_n \\ 0 & 0 \end{pmatrix}, \qquad (D_2)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} 0 & \tilde{d}^{n-1}_n \\ 0 & 1 \end{pmatrix}.$$

Their commutativity

$$[(D_1)_{\mathfrak{m}_1\mathfrak{m}_1}, (D_2)_{\mathfrak{m}_1\mathfrak{m}_1}] = 0$$

implies $\tilde{d}^{n-1}_n = -d^{n-1}_n$. Finally, a change of basis $e_n \to e_n + d^{n-1}_n e_{n-1}$ leads to another adapted basis (2) of $\mathfrak{n}$ (recall again that $\alpha = 0$) in which we have

$$(D_1)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (D_2)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

Eq. (9) fully determines the diagonal part of the derivations D_1, D_2. Concerning the off-diagonal part, we firstly add inner derivations to D_1, D_2 in order to arrive to their equivalent but simpler forms. In particular, by adding suitable linear combinations of $\mathrm{ad}_{e_2}, \ldots, \mathrm{ad}_{e_{n-1}}$ to D_1, D_2 we can set to zero off-diagonal entries in the last column

$$d^k_n = 0\,,\ \tilde{d}^k_n = 0\,, \qquad k = 1, \ldots, n-2\,.$$

Similarly, addition of a multiple of ad_{e_n} allows us to set

$$d^{n-2}_{n-1} = 0\,, \quad \tilde{d}^{n-2}_{n-1} = 0\,.$$

The derivations D_1, D_2 have each $n-3$ undetermined parameters left, namely d^k_{n-1}, $\tilde{d}^k_{n-1}$, $k = 1, \ldots, n-3$, respectively. Using

$$D_a e_k = [D_a e_{k+1}, e_n] + [e_{k+1}, D_a e_n]\,, \qquad a \in \{1, 2\}\,, \quad k = 1, \ldots, n-2$$

we find that the matrices of D_1, D_2 have the upper triangular forms

$$D_1 = \begin{pmatrix} 1 & 0 & d_{n-3} & \dots & d_2 & d_1 & 0 \\ & 1 & 0 & \ddots & d_3 & d_2 & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & d_{n-3} & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} n-2 & 0 & \tilde{d}_{n-3} & \dots & \tilde{d}_2 & \tilde{d}_1 & 0 \\ & n-3 & 0 & \ddots & \tilde{d}_3 & \tilde{d}_2 & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & 2 & 0 & \tilde{d}_{n-3} & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 1 \end{pmatrix}. \tag{10}$$

Taking the commutator $[D_1, D_2]$ we immediately see that it contains only zeros in the last column and in its $(n-2, n-1)$-entry – in fact, that was the reason why we chose the particular modification of D_1, D_2 by inner derivations in the previous step. The only inner derivation with these zeros is the vanishing one. Therefore, we must have

$$[D_1, D_2] = 0 \tag{11}$$

by the consequence of Eq. (5).

The condition (11) implies that $d_k = 0$, $k = 1, \dots n-3$, i.e.

$$D_1 = \mathrm{diag}(1, 1, \dots, 1, 0) \tag{12}$$

whereas the parameters $\tilde{d}_k$ in D_2 are unconstrained. The existence of a derivation in the form (12) severely constrains the algebra $\mathfrak{n}$. We have $D_1|_{\mathfrak{n}^2 \dot{+} \mathrm{span}\{e_{n-1}\}} = \mathrm{id}$, i.e.

$$[e_i, e_k] = D_1[e_i, e_k] = [D_1 e_i, e_k] + [e_i, D_1 e_k] = 2[e_i, e_k]\,, \tag{13}$$

leading to $[e_i, e_k] = 0$ for all $i, k \leq n-1$. That means that the algebra $\mathfrak{n}$ must be the model filiform algebra $\mathfrak{n}_{n,1}$ whose solvable extensions were classified in [8]. Using the results contained there, we arrive at the conclusion that its solvable extension by two nonnilpotent elements is unique and its nonvanishing Lie brackets take the form

$$\begin{aligned} &[e_k, e_n] = e_{k-1}\,, && k = 2, \dots, n-1\,, \\ &[f_1, e_j] = e_j\,, && j = 1, \dots, n-1\,, \\ &[f_2, e_j] = (n-j-1)e_j\,, && j = 1, \dots, n-1\,, \\ &[f_2, e_n] = e_n\,. && \end{aligned} \tag{14}$$

(2) n even and $\mathfrak{gr}(\mathfrak{n}) \simeq \mathcal{Q}_n$, i.e. we have $\alpha = 1$ in Eq. (2).

Let us again assume that we are given $\mathfrak{n}$ such that its $(n+2)$-dimensional solvable extension $\mathfrak{s}$ exists. The center $\mathfrak{n}^{n-1} = \operatorname{span}\{e_1\}$ of $\mathfrak{n}$ is also an ideal in $\mathfrak{s}$; therefore, we can consider the factor algebra $\tilde{\mathfrak{s}} = \mathfrak{s}/\mathfrak{n}^{n-1}$. The solvable algebra $\tilde{\mathfrak{s}}$ obviously has an $(n-1)$-dimensional filiform nilradical $\mathfrak{n}/\mathfrak{n}^{n-1}$ and two nonnilpotent elements f_1, f_2. By assumption n is even, $n-1$ is odd, i.e. $\mathfrak{gr}(\mathfrak{n}/\mathfrak{n}^{n-1}) = \mathfrak{n}_{n-1,1}$. Using the results derived above we deduce that the structure of the solvable algebra $\tilde{\mathfrak{s}}$ is as in Eq. (14) when written in a suitable basis.

A minor complication arises from a comparison of allowed transformations in $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$. In both we may add to D_1, D_2 any inner derivation, i.e. conclusions based on suitable additions of inner derivations can be immediately taken over from $\tilde{\mathfrak{s}}$ to $\mathfrak{s}$. On the other hand, the transformation $e_n \to e_n + d_n^{n-1} e_{n-1}$ which brought the blocks $(D_a)_{\mathfrak{m}_1\mathfrak{m}_1}$ to the form (9) causes a problem. It changes one adapted basis in $\tilde{\mathfrak{s}}$ to another (due to $\alpha = 0$) but such a transformation would spoil the adaptation of basis in $\mathfrak{s}$ with its $\alpha = 1$. Therefore, we can only assume

$$(15) \qquad (D_1)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} 1 & d_n^{n-1} \\ 0 & 0 \end{pmatrix}, \qquad (D_2)_{\mathfrak{m}_1\mathfrak{m}_1} = \begin{pmatrix} 0 & -d_n^{n-1} \\ 0 & 1 \end{pmatrix}$$

in $\mathfrak{s}$.

We now attempt to recover as much information as possible about the structure of $\mathfrak{n}$. Any basis $\tilde{\mathcal{E}} = (\tilde{e}_2, \ldots, \tilde{e}_n)$ in $\mathfrak{n}/\mathfrak{n}^{n-1}$ respecting

$$[\tilde{e}_3, \tilde{e}_{n-1}] = 0, \qquad [\tilde{e}_k, \tilde{e}_n] = \tilde{e}_{k-1}\,, \quad k = 3, \ldots, n-2\,,$$

is adapted. The Lie brackets in the model filiform algebra $\mathfrak{n}/\mathfrak{n}^{n-1}$ expressed in terms of $\tilde{e}_k$ take the model form (3)

$$(16) \quad [\tilde{e}_k, \tilde{e}_n] = \tilde{e}_{k-1}\,, \quad k = 3, \ldots, n-1\,, \qquad [\tilde{e}_j, \tilde{e}_k] = 0\,, \quad 2 \le j < k \le n-1$$

because in the model filiform algebra the Lie brackets in an arbitrary adapted basis have the canonical model form (3).

One such basis is obtained setting $\tilde{e}_k = e_k + \mathfrak{n}^{n-1}$. Consequently, the Lie brackets in $\mathfrak{n}$ in the adapted basis must have the form

$$(17) \quad \begin{aligned} [e_k, e_n] &= e_{k-1}\,, && k = 2, \ldots, n-1\,, \\ [e_1, e_j] &= 0\,, && j = 2, \ldots, n\,, \\ [e_j, e_{n-j+1}] &= (-1)^j e_1\,, && j = 2, \ldots, n-1\,, \\ [e_j, e_k] &= 0 \bmod \operatorname{span}\{e_1\}\,, && 3 \le j < k \le n-1\,,\ n-1 < j+k\,. \end{aligned}$$

This structure is the pre-image of the Lie brackets (16) and also takes into account the assumption that the basis $(e_1, \ldots, e_n)$ is adapted, i.e. that the equation (2) holds. Let us check what more we can say about the only Lie brackets not yet completely fixed, i.e. about

$$(18) \qquad [e_j, e_k] = 0 \bmod \operatorname{span}\{e_1\}\,, \quad 3 \le j < k \le n-1\,,\ n-1 < j+k\,.$$

The Jacobi identity (e_{j+1}, e_k, e_n) implies

$$(19) \qquad [e_j, e_k] + [e_{j+1}, e_{k-1}] = 0\,, \qquad 2 \le j, k \le n-2\,.$$

When $k+j=n+1$ the relation (19) holds by virtue of $[e_j,e_{n-j+1}]=(-1)^j e_1$. When $k+j>n+1$ it implies further restrictions on the structure of the Lie brackets (18). We consider separately the cases of $k+j$ even and $k+j$ odd.

– $k+j$ even: Let us assume $k \geq j$ and take $k'=j'=\frac{k+j}{2}$. Then we have

$$0=[e_{k'},e_{k'}]+[e_{k'+1},e_{k'-1}]=[e_{k'+1},e_{k'-1}]$$

and by repeated use of Eq. (19) we find that

$$[e_j,e_k]=0\,,\ 2\leq j<k\leq n-1\,,\ k+j \text{ even.} \tag{20}$$

– $k+j$ odd: we find that all $[e_j,e_k]$ with the same $k+j$ are related through

$$[e_j,e_k]=(-1)^j\alpha_{\frac{1}{2}(j+k-n-1)}e_1\,, \qquad 3\leq j<k\leq n-1\,,\ n+1<j+k \tag{21}$$

for some parameters $\alpha_1,\alpha_2,\ldots,\alpha_{(\frac{n}{2}-2)}$.

To sum up, due to the Jacobi identity the Lie brackets in the adapted basis (17) necessarily have the form

$$\begin{aligned} [e_k,e_n]&=e_{k-1}\,, && k=2,\ldots,n-1\,,\\ [e_1,e_j]&=0\,, && j=2,\ldots,n\,,\\ [e_j,e_{n-j+1}]&=(-1)^j e_1\,, && j=2,\ldots,n-1\,, \end{aligned} \tag{22}$$

$$\begin{aligned} [e_j,e_k]&=0\,, && 3\leq j<k\leq n-1\,,\ j+k \quad \text{even}\\ [e_j,e_k]&=(-1)^j\alpha_{\frac{1}{2}(j+k-n-1)}e_1\,, && 4\leq j<k\leq n-1\,,\ n+1<j+k \quad \text{odd} \end{aligned}$$

for some parameters $\alpha_1,\alpha_2,\ldots,\alpha_{(\frac{n}{2}-2)}$. We may change the adapted basis (22) by a transformation

$$\begin{aligned} \tilde{e}_{n-1}&=e_{n-1}+\sum_{j=1}^{\frac{n}{2}-2}\beta_{\frac{n}{2}-j-1}e_{2j+1}\,,\\ \tilde{e}_n&=e_n\,,\quad \tilde{e}_k=[\tilde{e}_{k+1},\tilde{e}_n]\,,\quad k=n-2,\ldots,1\,. \end{aligned} \tag{23}$$

which preserves its adaptation but changes the values of the parameters α_j. Through a suitable choice of the parameters β_j in the transformation we can set all α_j equal to zero. The easiest way of seeing this is to proceed in steps, using only one nonvanishing β in each of them. Firstly, we use only $\beta_1\neq 0$ in the transformation (23). Setting $\beta_1=-\frac{\alpha_1}{2}$ we have $\tilde{\alpha}_1=0$ in the new basis. Proceeding by induction, assuming that we already have $\alpha_j=0$, $j=1,\ldots,J$ we use the transformation (23) with $\beta_{J+1}=-\frac{\alpha_{J+1}}{2}$ to eliminate $\tilde{\alpha}_{J+1}$.

We arrive at the conclusion that $\mathfrak{n}$ is isomorphic to the special filiform algebra $\mathcal{Q}_n$. Its solvable extensions were classified in [1]. Using the results contained there and converting them to our choice of adapted basis we find that $\mathfrak{s}$ is isomorphic to the algebra with the following nonvanishing Lie brackets

$$\begin{aligned} &[e_k,e_n]=e_{k-1}\,, && [e_k,e_{n-k+1}]=(-1)^k e_1\,,\ k=2,\ldots,n-1\,,\\ &[f_1,e_n]=e_{n-1}\,, && [f_1,e_1]=2e_1\,, \qquad [f_1,e_k]=e_k\,,\ k=2,\ldots,n-1\,,\\ &[f_2,e_n]=e_n\,, && [f_2,e_j]=(n-j)e_j\,,\ j=1,\ldots,n-1\,. \end{aligned} \tag{24}$$

The seemingly anomalous Lie bracket $[f_1, e_n] = e_{n-1}$ is a consequence of Eq. (15), i.e. of our convention for adapted bases. A more convenient linear combination of D_1, D_2 was used instead of D_2 to define the action of f_2.

We recall that the filiform algebras $\mathfrak{n}_{n,1}$ and $\mathcal{Q}_n$ are called *naturally graded* because they coincide with their respective associated graded algebra. We summarize the results of this section in the following theorem

Theorem 2. *Let* $\mathfrak{n}$ *be a filiform Lie algebra, not characteristically nilpotent, and* $\mathfrak{s}$ *be a solvable Lie algebra with the nilradical* $\mathfrak{n}$. *Then* $\dim \mathfrak{s} = \dim \mathfrak{n} + 1$ *or* $\dim \mathfrak{s} = \dim \mathfrak{n} + 2$. *If* $\dim \mathfrak{s} = \dim \mathfrak{n} + 2$ *then* $\mathfrak{n}$ *is naturally graded and* $\mathfrak{s}$ *is determined by* $\mathfrak{n}$ *up to isomorphism. The two possible forms of* $\mathfrak{s}$ *are given in Eq.* (14) *and Eq.* (24)*, respectively.*

5. Conclusions and comparison with the original proof

We have presented an alternative proof of the statement that the only filiform algebras which possess a solvable extension of codimension 2 are $\mathfrak{n}_{n,1}$ and $\mathcal{Q}_n$ (up to isomorphism).

This result was originally obtained in [6] where the the results of [4] were used. In [4] the maximal external torus of derivations[1] and rank[2], of an arbitrary filiform Lie algebra $\mathfrak{n}$ was found (when nonvanishing). It turned out that only $\mathfrak{n}_{n,1}$ and $\mathcal{Q}_n$ have rank 2.

Any such torus spanned by D_a, $a = 1, \ldots, \mathrm{rank}(\mathfrak{n})$ can be used in order to construct a solvable extension of $\mathfrak{n}$ by setting $D_a = \mathrm{ad}|_{\mathfrak{n}}(f_a)$ and $[f_a, f_b] = 0$. Nevertheless, such construction may not necessarily exhaust all solvable extensions (and it indeed doesn't in general) because $\mathrm{ad}|_{\mathfrak{n}}(f_a)$ need neither commute nor be diagonalizable.

Next, in [6] it was shown that the algebra of derivations of any filiform Lie algebra is solvable. Its Chevalley decomposition into a semidirect sum of the maximal torus and the nilpotent ideal then allowed to deduce that the codimension of $\mathfrak{n}$ in any of its solvable extensions $\mathfrak{s}$ must be less or equal to the rank of $\mathfrak{n}$.

Our derivation here is conceptually different. We avoid here the cumbersome construction of the maximal torus of an arbitrary filiform algebra. On the other hand, we rely on the knowledge of the solvable extensions of $\mathfrak{n}_{n,1}$ and $\mathcal{Q}_n$ constructed in [8, 1] and the ideas employed in [7]. Therefore, our derivation provides an alternative, hopefully simpler, re-derivation of Goze's and Khakimdjanov's original result.

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[1] i.e. maximal commuting set of diagonalizable outer derivations

[2] i.e. dimension of the external torus of derivations

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