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# $\phi$-LAPLACIAN BVPS WITH LINEAR BOUNDED OPERATOR CONDITIONS

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Abstract. The aim of this paper is to present new existence results for $\phi$-Laplacian boundary value problems with linear bounded operator conditions. Existence theorems are obtained using the Schauder and the Krasnosel'skii fixed point theorems. Some examples illustrate the results obtained and applications to multi-point boundary value problems are provided.

## 1. Introduction

This paper is concerned with the existence of positive solutions to the following boundary value problem with linear bounded operator conditions:

$$\begin{cases} -\big(\phi(u')\big)'(x) = \lambda f\big(x, u(x), u'(x)\big)\,, & 0 < x < 1 \\ u(0) = L_0(u)\,, \quad u(1) = L_1(u)\,, \end{cases} \tag{1}$$

where $\lambda > 0$, $f\colon [0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is $L^1$-Carathéodory function, i.e.

(a) the map $x \longmapsto f(x,u,v)$ is measurable for all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}$,

(b) the map $(u,v) \longmapsto f(x,u,v)$ is continuous for a.e. $x \in [0,1]$.

(c) For every $r > 0$, there exists $h_r \in L^1([0,1],\mathbb{R}^+)$ such that $0 \le f(x,u,v) \le h_r(x)$, for a.e. $x \in [0,1]$ and for all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}$ with $0 \le u \le r$ and $|v| \le r$.

The nonlinear derivation operator $\phi\colon \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism such that $\phi$ is sub-multiplicative, i.e. $\forall\, \alpha, \beta \in \mathbb{R}^+,\ \phi(\alpha \cdot \beta) \le \phi(\alpha) \cdot \phi(\beta)$, extending the $p$-Laplacian derivation operator $\phi(s) = |s|^{p-2}s$, $p > 1$. More generally, one may consider as well the class of sub-multiplicative-like functions introduced in [10] (see, also [11]), that is increasing homeomorphisms $\phi$ of the real line, vanished at 0, such that there exists an increasing homeomorphism $\Phi$ of $[0,+\infty)$ with $\phi(\alpha\cdot\beta) \le \Phi(\alpha)\cdot\phi(\beta)$, for all $\alpha, \beta \in \mathbb{R}^+$. Notice that (see [2]) if $\phi$ is sub-multiplicative, then $\phi^{-1}$ is super-multiplicative, i.e.

$$\forall\, \alpha, \beta \in \mathbb{R}^+\,, \quad \phi^{-1}(\alpha \cdot \beta) \ge \phi^{-1}(\alpha) \cdot \phi^{-1}(\beta)\,. \tag{2}$$

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Moreover, there exists $\Phi^* \in (0,1)$ such that

$$\forall\, \alpha, \beta \in \mathbb{R}^+ , \quad \phi^{-1}(\alpha) + \phi^{-1}(\beta) \geq \Phi^* \phi^{-1}(\alpha + \beta) . \tag{3}$$

Finally $L_0, L_1$ are linear bounded increasing operators from $E := C([0,1], \mathbb{R}^+)$ to $\mathbb{R}^+$ such that $L_i(1) < 1$ ($i = 1,2$). Here $E$ denotes the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}^+$ with the norm

$$\|u\|_0 = \sup \big\{ |u(x)|,\ 0 \leq x \leq 1 \big\} .$$

$E^1 := C^1([0,1], \mathbb{R}^+)$ will refer to the space of continuously differentiable functions from $[0,1]$ to $\mathbb{R}^+$; equipped with the norm $\|u\| = \max\big(\|u\|_0, \|u'\|_0\big)$, this is a Banach space. The boundary value problem (bvp in short) (1) was studied in [12] where the author proved existence of positive solutions under appropriate conditions on the level of growth of the response operator $F$ defined by $Fu(x) = f(x, u(x))$. In this paper, new conditions including sub-linear and super-linear growth nonlinearities are assumed to prove existence of solutions lying either in balls or in positive cones of Banach spaces. In [3, 2, 4], the authors studied two-point Dirichlet bvps associated to the $\phi$-Laplacian equation $-(\phi(u')'(x) = f(x, u(x))$; the Schauder fixed point theorem is used in [4] while existence of positive solutions is obtained via the Krasnosel'skii fixed point theorem in [3]; [2] is mainly concerned with multiplicity results via the Leggett-Williams fixed point theorem. Notice that multi-point bvps with the classical $p$-Laplacian as a nonlinear derivation operator are intensively studied in the literature; see [6, 9, 15, 19] and the references therein. In [19], existence of solution is obtained for the equation $(pu'(x))' + f(x, u) = 0$, $0 < x < 1$. In [6], existence of positive solutions in a cone of a Banach space is obtained via the Krasnosel'skii fixed point theorem for the equation $(\phi(u'(x)))' + q(x) f(x, u) = 0$, $0 < x < 1$. The same equation is investigated in [15] where the proofs of the existence results involve computation of the fixed point index on a special cone of a Banach space. The case when $f = f(x, u(x), u'(x))$ is also studied by the same authors in [16]. To our knowledge, only Karakostas [12] extends the multi-point boundary conditions to more general bounded linear conditions. Thus the main motivation of this work is to provide new existence results for (1) which extend similar results in [3, 2, 4, 12]. The plan of the paper is organized as follows. Section 2 is devoted to the functional setting useful to study bvp (1); this includes fixed point formulation and a compactness criterion. Some existence results are then presented in Section 3 when $f = f(x, u)$. The first one uses the Schauder fixed point theorem while in the second one existence of positive solutions is obtained via the Krasnosel'skii fixed point theorem; then we deal with some consequences regarding the sub-linear and super-linear growth of the nonlinearity $f$. The case when the nonlinearity also depends on the first derivative is dealt with in Section 4; a recent variant of the Krasnosel'skii fixed point theorem is employed. Each existence result is illustrated by means of an example of application.

## 2. Preliminaries and auxiliary lemmas

In order to transform bvp (1) into a fixed point problem, we need some preliminary results which we collect in this section. For any fixed $u \in E^1$, and $\theta \in [0,1]$, define the quantity

$$\begin{aligned}\zeta(\theta,u) = aL_0\Big(\int_0^{\cdot} \phi^{-1}\Big(\int_s^{\theta} f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\Big)\,ds\Big)\\ +\int_0^1 \phi^{-1}\Big(\int_s^{\theta} f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\Big)\,ds\\ +bL_1\Big(\int_{\cdot}^1 \phi^{-1}\Big(\int_s^{\theta} f\big(\tau,u(\tau),u'(\tau)\big)\,d\tau\Big)\,ds\Big)\,,\end{aligned}$$

where

$$a=(1-L_0(1))^{-1}>0 \quad\text{and}\quad b=(1-L_1(1))^{-1}>0\,. \tag{4}$$

**Lemma 2.1** ([12, Lemma 3.2]).

(a) $\zeta(\cdot,\cdot)$ *is continuous.*

(b) *For each* $u\in E^1$*, the correspondence* $\theta\mapsto\zeta(\theta,u)$ *is strictly increasing.*

(c) *For any* $u\in E^1$*, there is a unique* $\theta(u)\in[0,1]$ *such that* $\zeta(\theta(u),u)=0$.

(d) *The function* $u\mapsto\theta(u)$ *depends continuously on* $u$.

**Lemma 2.2** ([12, Lemma 3.3]). *Let* $u\in L^1([0,1],\mathbb{R}^+)$*. Then the boundary value problem*

$$\begin{cases}-(\phi(v'))'=u(x)\,, & 0<x<1\\ v(0)=L_0(v)\,, & v(1)=L_1(v)\end{cases}$$

*has a unique solution given by*

$$v(x)=\begin{cases} aL_0\big(\int_0^{\cdot}\phi^{-1}\big(\int_s^{\theta(u)}u(\tau)\,d\tau\big)\,ds\big)\\ \quad+\int_0^x\phi^{-1}\big(\int_s^{\theta(u)}u(\tau)\,d\tau\big)\,ds\,, & \text{if}\quad 0\le x\le\theta(u)\\ bL_1\big(\int_{\cdot}^1\phi^{-1}\big(\int_{\theta(u)}^s u(\tau)\,d\tau\big)\,ds\big)\\ \quad+\int_x^1\phi^{-1}\big(\int_{\theta(u)}^s u(\tau)\,d\tau\big)\,ds\,, & \text{if}\quad \theta(u)\le x\le 1\,,\end{cases}$$

*where* $\theta(u)$ *satisfies the implicit algebraic equation* $\zeta(\theta(u),u)=0$*. Moreover, the solution* $v$ *has the following properties:*
(a) *it is a concave function,*
(b) *it is a nonnegative function,*
(c) *its maximum is attained at some point of* $(0,1)$.

**Remark 2.1.** We can see that the function $u\in E^1$ is a solution of the boundary value problem (1) if and only if it is a solution of the operator equation $u=Tu$

with $T$ defined by:

$$Tu(x) = \begin{cases} aL_0\big(\int_0^{\cdot} \phi^{-1}\big(\lambda \int_s^{\theta(u)} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\big) \\ \quad + \int_0^x \phi^{-1}\big(\lambda \int_s^{\theta(u)} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\,, \\ \quad\ \text{if} \quad 0 \leq x \leq \theta(u) \\ bL_1\big(\int_{\cdot}^1 \phi^{-1}\big(\lambda \int_{\theta(u)}^s f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\big) \\ \quad + \int_x^1 \phi^{-1}\big(\lambda \int_{\theta(u)}^s f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\,, \\ \quad\ \text{if} \quad \theta(u) \leq x \leq 1\,, \end{cases} \tag{5}$$

where $\theta(u)$ is as defined in Lemma 2.2. Hence

$$(Tu)'(x) = \begin{cases} \phi^{-1}\big(\lambda \int_x^{\theta(u)} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\,, & \text{if} \quad 0 \leq x \leq \theta(u) \\ -\phi^{-1}\big(\lambda \int_{\theta(u)}^x f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\,, & \text{if} \quad \theta(u) \leq x \leq 1\,. \end{cases} \tag{6}$$

Then $(Tu)'(\theta(u)) = 0$. This and the concavity of $Tu$ imply that $Tu(x)$ achieves its maximum for $x = \theta(u)$. As a consequence

$$\begin{aligned} \|Tu\| &= aL_0\Big(\int_0^{\cdot} \phi^{-1}\Big(\int_s^{\theta(u)} \lambda f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) \\ &\quad + \int_0^{\theta(u)} \phi^{-1}\Big(\int_s^{\theta(u)} \lambda f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds \\ &= bL_1\Big(\int_{\cdot}^1 \phi^{-1}\Big(\int_{\theta(u)}^s \lambda f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) \\ &\quad + \int_{\theta(u)}^1 \phi^{-1}\Big(\int_{\theta(u)}^s \lambda f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\,. \end{aligned} \tag{7}$$

**Lemma 2.3.** *The operator* $T\colon E^1 \longrightarrow E^1$ *defined by* (5) *is completely continuous.*

Since this lemma is only sketched in [12], we present the proof in detail, in particular the continuity of $T$.

**Proof.**
(a) $T$ is continuous. Let $\lim_{n\to+\infty} \|u_n - u_0\|_{E^1} = 0$. Then there exists some $M > 0$ such that $\|u_n\| \leq M$, for all $n \in \mathbb{N}$. Let $v_n(\cdot) = \lambda f(\cdot, u_n(\cdot), u_n'(\cdot))$. Since $f$ is Carathéodory, $v_n(\cdot) \to v(\cdot) = \lambda f(\cdot, u(\cdot), u'(\cdot))$ a.e. on $[0,1]$ as $n \to +\infty$. By the Lebesgue dominated convergence theorem, for a.e. $s \in (0, \theta_n)$, we have

$$0 \leq \lim_{n\to\infty} \int_s^{\theta_n} |v_n(\tau) - v(\tau)|\, d\tau \leq \lim_{n\to\infty} \int_0^1 |v_n(\tau) - v(\tau)|\, d\tau = 0\,,$$

where $\theta_n = \theta(u_n)$ is as defined in Lemma 3.2. Since $0 < \theta_n < 1$, then $\theta_n$ converges, up to a subsequence, to some limit $\theta_* \in [0,1]$. Assume $0 < \theta_* < 1$. Again by the Lebesgue dominated convergence theorem, the integral $\int_0^x \phi^{-1}\big(\int_s^{\theta_n} v_n(\tau)\, d\tau\big)\, ds$ converges to $\int_0^x \phi^{-1}\big(\int_s^{\theta_*} v(\tau)\, d\tau\big)\, ds$ because $\phi$ is a homeomorphism. Also, the integral $L_0\big(\int_0^{\cdot} \phi^{-1}\big(\int_s^{\theta_n} v_n(\tau)\, d\tau\big)\, ds\big)$ converges to $L_0\big(\int_0^{\cdot} \phi^{-1}\big(\int_s^{\theta_*} v(\tau)\, d\tau\big)\, ds\big)$ because $\phi$ is an homeomorphism and $L_0$ is continuous. The same holds for the second term in (5) with $\theta = \theta_n$. $Tu_n(x)$ converges to $Tu(x)$ uniformly on $[0,1]$ with

$$Tu(x) = \begin{cases} aL_0\big(\int_0^{\cdot} \phi^{-1}\big(\lambda \int_s^{\overline{\theta}} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\big) & \\ \quad + \int_0^x \phi^{-1}\big(\lambda \int_s^{\overline{\theta}} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\,, & \text{if} \quad 0 \le x \le \overline{\theta} < 1 \\ bL_1\big(\int_{\cdot}^1 \phi^{-1}\big(\lambda \int_{\overline{\theta}}^s f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\big) & \\ \quad + \int_x^1 \phi^{-1}\big(\lambda \int_{\overline{\theta}}^s f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\big)\, ds\,, & \text{if} \quad 0 < \overline{\theta} \le x \le 1\,, \end{cases}$$

where $\overline{\theta} = \overline{\theta}(u)$ is uniquely defined in Lemma 2.1. Since

$$\begin{aligned} &aL_0\Big(\int_0^{\cdot} \phi^{-1}\Big(\lambda \int_s^{\theta_n} f(\tau, u(\tau), u'(\tau))\, d\tau\Big)\, ds\Big) \\ &\quad + \int_0^1 \phi^{-1}\Big(\lambda \int_s^{\theta_n} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds \\ &\quad + bL_1\Big(\int_{\cdot}^1 \phi^{-1}(\lambda \int_s^{\theta_n} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) = 0\,, \end{aligned}$$

invoking once again the Lebesgue dominated convergence theorem, and passing to the limit as $n \to +\infty$, we find that

$$\begin{aligned} &aL_0\Big(\int_0^{\cdot} \phi^{-1}\Big(\lambda \int_s^{\theta_*} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) \\ &\quad + \int_0^1 \phi^{-1}\Big(\lambda \int_s^{\theta_*} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds \\ &\quad + bL_1\Big(\int_{\cdot}^1 \phi^{-1}\Big(\lambda \int_s^{\theta_*} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) = 0\,. \end{aligned}$$

By uniqueness of $\overline{\theta}$, we get $\theta_* = \overline{\theta}$. Now, assume that $\theta_* = 0$. Then

$$\begin{aligned} &aL_0\Big(\int_0^{\cdot} \phi^{-1}\Big(\lambda \int_0^s f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) \\ &\quad + \int_0^1 \phi^{-1}\Big(\lambda \int_0^s f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds \\ &\quad + bL_1\Big(\int_{\cdot}^1 \phi^{-1}\Big(\lambda \int_0^s f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) = 0\,. \end{aligned}$$

Since all the terms are nonnegative, we obtain

$$\phi^{-1}\Big(\lambda \int_0^t f\big(\cdot, u(\cdot), u'(\cdot)\big)\, ds\Big) = 0\,, \quad t \in [0,1]$$

and $f\big(\cdot, u(\cdot), u'(\cdot)\big) = 0$ a.e. on $[0,1]$, leading to a contradiction. Analogously, we can check that $\theta_* \neq 1$. In the same way, we prove the uniform convergence of $(Tu_n)'(x)$ to $(Tu)'(x)$, proving the continuity of $T$ and ending the proof of our claim.

(b) $T$ is totally bounded. Let $B$ be a bounded subset in $E^1$ and $M > 0$ a constant such that $\|u\| \leq M$ for all $u \in B$. We have

$$\begin{aligned}\int_0^{\cdot} \phi^{-1}\Big(\lambda \int_s^{\theta(u)} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds &\leq \int_0^1 \phi^{-1}\Big(\lambda \int_0^1 f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds \\ &\leq \phi^{-1}\Big(\lambda \int_0^1 f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big) \\ &\leq \phi^{-1}(\lambda |h_M|_1)\,,\end{aligned}$$

where $|h_M|_1 = \int_0^1 h_M(\tau) d\tau$. Since $L_0$ is increasing, we deduce that

$$\begin{aligned}L_0\Big(\int_0^{\cdot} \phi^{-1}\Big(\lambda \int_s^{\theta(u)} f\big(\tau, u(\tau), u'(\tau)\big)\, d\tau\Big)\, ds\Big) \\ \leq L_0\big(\phi^{-1}(\lambda |h_M|_1)\big) = \phi^{-1}(\lambda |h_M|_1) L_0(1)\,.\end{aligned}$$

From (6) and (7), we deduce that

$$\|Tu\|_0 \leq (aL_0(1) + 1)\phi^{-1}(\lambda |h_M|_1) \quad \text{and} \quad \|(Tu)'\|_0 \leq \phi^{-1}(\lambda |h_M|_1)\,.$$

This implies the boundedness of $T(B)$. To show the equicontinuity of $T(B)$, notice that for $x \in [0, 1]$ and $u \in B$, we have

$$|(Tu)'(x)| \leq \phi^{-1}\Big(\int_0^1 \lambda f\big(x, u(x), u'(x)\big)\, dx\Big) \leq \phi^{-1}\big(\lambda |h_M|_1\big)\,.$$

Therefore, if $x_1, x_2 \in [0, 1]$, then $|(Tu)(x_1) - (Tu)(x_2)| \leq \phi^{-1}\big(\lambda |h_M|_1\big)|x_1 - x_2|$ and the right hand-side term tends to 0 as $|x_1 - x_2| \to 0$. Finally (6) gives the estimate:

$$|(\phi(Tu))'(x_1) - (\phi(Tu))'(x_2)| \leq \Big|\int_{x_1}^{x_2} h_M(\tau) d\tau\Big|$$

which also tends to 0 when $|x_1 - x_2| \to 0$ for $h_M \in L^1([0, 1], \mathbb{R}^+)$. Since $\phi$ is a homeomorphism, this shows the equicontinuity of $T(B)$. Finally, the Arzéla-Ascoli theorem then concludes the proof. $\square$

## 3. THE CASE $f = f(x, u)$

The following classical theorems will be the main tools used in this section.

**Theorem A** (Schauder's fixed point theorem. (See [5, Thm. 8.8, p. 60], [14, Thm. 2.3.7, p. 15], [18, Thm. 2.A, p. 57])). *Let $X$ be a Banach space and $C \subset X$ a bounded, closed, convex subset of $E$. If $T \colon C \to C$ is a completely continuous operator, then $T$ has a fixed point in $C$.*

**Theorem B** (Krasnosel'skii's fixed point theorem. (See [13, 8])). *Let $X$ be a Banach space, $K \subset X$ a cone and $\Omega_1, \Omega_2$ two bounded open subsets satisfying $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let $T \colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that:*

(a) *either $\|Tv\| \leq \|v\|$ for $v \in K \cap \partial\Omega_1$ and $\|Tv\| \geq \|v\|$ for $v \in K \cap \partial\Omega_2$,*

(b) *or* $\|Tv\| \geq \|v\|$ *for* $v \in K \cap \partial\Omega_1$ *and* $\|Tv\| \leq \|v\|$ *for* $v \in K \cap \partial\Omega_2$.
*Then* $T$ *has at least a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

### 3.1. An existence theorem by the Schauder fixed point theorem.

Our first existence result in this section is:

**Theorem 3.1.** *Assume that there exists* $R \geq 1$ *such that*

$$\int_0^1 f(x,R)\,dx \geq R\,. \tag{8}$$

*Then, for sufficiently small* $\lambda$, *bvp* (1) *has at least one nonnegative solution* $u$ *such that* $\|u\|_0 \leq R$.

**Proof.** Let $g_R(x) = \max\limits_{0\leq y\leq R} f(x,y)$, then

$$\int_0^1 g_R(\tau)\,d\tau \geq \int_0^1 f(\tau,R)\,d\tau \geq R \geq 1\,.$$

Let $a$ be given by (4) and

$$\lambda^\star = \frac{\phi(1/aL_0(1)+1)}{|g_R|_1}\,.$$

Let $u \in B := \{u \in E,\ \|u\|_0 \leq R\}$. Arguing as in the proof of Lemma 2.3, we find that, for $0 < \lambda \leq \lambda^\star$, we have

$$\|Tu\|_0 \leq (aL_0(1)+1)\phi^{-1}(\lambda|g_R|_1) \leq 1 \leq R\,.$$

Therefore, the operator $T$ maps the ball $B$ into itself. By Theorem A and Lemma 2.3, $T$ has a fixed point $u$ such that $\|u\|_0 \leq R$. □

**Example 3.1.** Consider the boundary value problem:

$$\begin{cases} -\left(|u'|u'\right)'(x) = \lambda|(x-\frac{1}{4})(e^u-2)|\ln(|u|+1)\,, & 0<x<1 \\ u(0) = \int_0^1 u(s)d\mu_1(s)\,, \quad u(1) = \int_0^1 u(s)d\mu_2(s)\,. \end{cases} \tag{9}$$

Here $\phi = \phi_3$, $f(x,u) = |(x-\frac{1}{4})(e^u-2)|\ln(|u|+1)$, and $\mu_1$, $\mu_2$ are two nondecreasing functions on $[0,1]$ of bounded variation $V_0^1(\mu_i) < 1$, $(i=0,1)$. This condition ensures that the Stieltjes integrals do exist. Then, for sufficiently small $\lambda > 0$, bvp (9) has a solution $u$ such that $\|u\|_0 \leq 3$. Indeed, for $R = 3$ we have

$$\int_0^1 f(x,R)\,dx = \frac{5}{16}(e^R-2)\ln(R+1) \geq R\,.$$

### 3.2. Existence results by the Krasnosel'skii fixed point theorem.

Let the operator $T$ be as defined in (5) and consider the positive cone

$$K = \{u \in E \text{ and } u \text{ is concave on } (0,1)\}\,. \tag{10}$$

It is clear that $T$ maps $K$ into itself and $(Tu)(0) \geq 0$, $(Tu)(1) \geq 0$. To prove existence of positive solutions, we need some preliminary results:

**Lemma 3.2** ([3, Lemma 2.3] or [12, Lemma 3.1]). *Let* $p(x) = \min(x, 1-x)$, $x \in [0,1]$. *If* $u \in K$, *then for all* $x \in [0,1]$

$$u(x) \geq p(x)\|u\|_0 , \quad \forall\, x \in [0,1] .$$

**Lemma 3.3** ([3], [2, Lemma 2.6]). *Let* $0 < \sigma < \frac{1}{2}$ *an arbitrary real number. Then for every* $u \in E$, *the operator* $T$ *verifies*

$$\|Tu\|_0 \geq \begin{cases} \sigma\phi^{-1}\big(\int_{\theta(u)}^{1-\sigma} \lambda f\big(\tau, u(\tau)\big)\, d\tau\big) , & \text{if} \quad \theta(u) \leq \sigma \\ \sigma\phi^{-1}\big(\int_{\sigma}^{\theta(u)} \lambda f\big(\tau, u(\tau)\big)\, d\tau\big) , & \text{if} \quad \theta(u) \geq 1-\sigma \\ \frac{\sigma}{2}\phi^{-1}\big(\int_{\sigma}^{\theta(u)} \lambda f\big(\tau, u(\tau)\big)\, d\tau\Big) + \frac{\sigma}{2}\phi^{-1}\big(\int_{\theta(u)}^{1-\sigma} \lambda f\big(\tau, u(\tau)\big)\, d\tau\big) , & \\ \qquad \text{if} \quad \sigma \leq \theta(u) \leq 1-\sigma , & \end{cases}$$

*where* $\theta(u)$ *is as defined in Lemma 2.2.*

### 3.3. The super-linear-like case.

**Theorem 3.4.** *Suppose that the following condition holds:*

$$\limsup_{u\to 0^+} \frac{f(x,u)}{\phi(u)} = 0 \quad \text{and} \quad \liminf_{u\to +\infty} \frac{f(x,u)}{\phi(u)} = +\infty , \quad \text{uniformly in} \quad x \in [0,1] .$$

*Then bvp* (1) *has at least one positive solution* $u \in E$ *for all positive* $\lambda$.

**Proof.**
*Claim 1.* Let $\varepsilon > 0$ satisfy

$$0 < \varepsilon \leq \frac{1}{\lambda\phi(aL_0(1)+1)} . \tag{11}$$

Since $\lim\limits_{u\to 0^+} \frac{f(x,u)}{\phi(u)} = 0$, uniformly in $x \in [0,1]$, then there exists $r > 0$ such that $0 \leq f(x,u) \leq \varepsilon\phi(u)$, for $x \in [0,1]$ and $0 \leq u \leq r$. Let $\Omega_1 := \{u \in E, \|u\|_0 < r\}$ and $u \in K \cap \partial\Omega_1$, then $\phi(u(s)) \leq \phi(\|u\|_0) = \phi(r)$, for all $s \in [0,1]$. So, for $\varepsilon$ satisfying (11) and using (2), we have the estimates

$$\begin{aligned} \|Tu\|_0 \leq aL_0\Big(\int_0^1 \phi^{-1}\Big(\int_0^1 \lambda f\big(\tau, u(\tau)\big)\, d\tau\Big)\, ds\Big) \\ + \int_0^1 \phi^{-1}\Big(\int_0^1 \lambda f\big(\tau, u(\tau)\big)\, d\tau\Big)\, ds \\ &\leq (aL_0(1)+1) \int_0^1 \phi^{-1}\Big(\int_0^1 \lambda\varepsilon\phi(r)\, d\tau\Big)\, ds \\ &= \big(aL_0(1)+1\big)\phi^{-1}\big(\varepsilon\lambda\phi(r)\big) \\ &= \phi^{-1}\big(\phi(aL_0(1)+1)\big) \cdot \phi^{-1}\big(\varepsilon\lambda\phi(r)\big) \\ &\leq \phi^{-1}\big(\phi(aL_0(1)+1) \cdot \varepsilon\lambda\phi(r)\big) = r = \|u\|_0 . \end{aligned}$$

*Claim 2.* Let $\lambda > 0$, $0 < \sigma < 1/2$ be arbitrary and let $k$ satisfy

$$k \geq \max(\phi(1/\sigma^2) , \phi(2/\sigma^2\Phi^*))/\lambda(1-2\sigma) . \tag{12}$$

Since $\liminf\limits_{u\to\infty}\frac{f(x,u)}{k\phi(u)}=+\infty$ uniformly in $x\in[0,1]$, then there exists $R>0$ such that $f(x,u)\geq k\phi(u)$, for $x\in[0,1]$ and $u\geq R$. Let $\widetilde{R}\geq R/\sigma$ and define the open set $\Omega_2:=\Big\{u\in E:\ \|u\|_0<\widetilde{R}\Big\}$. Then $u\in K$ and $\|u\|_0=\widetilde{R}$ imply that $u(x)\geq p(x)\|u\|_0\geq\sigma\widetilde{R}\geq R$, for all $x\in[\sigma,1-\sigma]$. Two distinct cases are then discussed separately.

*Case (a):* If $\theta(u)<\sigma$ or $\theta(u)>1-\sigma$, then by Lemma 3.3 and using (2), (12), and the fact that $\phi$ is increasing, we get

$$\begin{aligned}\|Tu\|_0&\geq\sigma\phi^{-1}\Big(\int_\sigma^{1-\sigma}\lambda f\big(\tau,u(\tau)\big)\,d\tau\Big)\\&\geq\sigma\phi^{-1}\Big(\int_\sigma^{1-\sigma}k\lambda\phi\big(u(\tau)\big)\,d\tau\Big)\\&\geq\sigma\phi^{-1}\big(k\lambda(1-2\sigma)\phi(\sigma\widetilde{R})\big)\\&\geq\sigma^2\widetilde{R}\phi^{-1}\big(k\lambda(1-2\sigma)\big)\geq\widetilde{R}=\|u\|_0\,.\end{aligned}$$

*Case (b):* If $\theta(u)\in[\sigma,1-\sigma]$, then again by Lemma 3.3 together with (3) and (12), we have the estimates:

$$\begin{aligned}\|Tu\|_0&\geq\frac{\sigma}{2}\phi^{-1}\Big(\int_\sigma^{\theta(u)}\lambda f\big(\tau,u(\tau)\big)\,d\tau\Big)+\frac{\sigma}{2}\phi^{-1}\Big(\int_{\theta(u)}^{1-\sigma}\lambda f\big(\tau,u(\tau)\big)\,d\tau\Big)\\&\geq\frac{\sigma}{2}\Phi^*\phi^{-1}\Big(\int_\sigma^{1-\sigma}\lambda f\big(\tau,u(\tau)\big)\,d\tau\Big)\\&\geq\frac{\sigma}{2}\Phi^*\phi^{-1}\Big(\int_\sigma^{1-\sigma}k\lambda\phi\big(u(\tau)\big)\,d\tau\Big)\\&\geq\frac{\sigma}{2}\Phi^*\phi^{-1}\big(k\lambda(1-2\sigma)\phi(\sigma\widetilde{R})\big)\\&\geq\frac{\sigma^2}{2}\widetilde{R}\Phi^*\phi^{-1}\big(k\lambda(1-2\sigma)\big)\geq\widetilde{R}=\|u\|_0\,.\end{aligned}$$

Therefore, in both cases, we have $\forall\,u\in K\cap\partial\Omega_2$, $\|Tu\|_0\geq\|u\|_0$. By Theorem B, bvp (1) admits a positive solution $u$ such that $\min(r,\widetilde{R})\leq\|u\|_0\leq\max(r,\widetilde{R})$. ☐

**Corollary 3.5.** *Assume there exist continuous nonnegative functions $\varphi$, $\psi$ on $\mathbb{R}^+$ and $\omega,\rho\in L^1([0,1],\mathbb{R}^+)$ such that*

$$\rho(x)\varphi(u)\leq f(x,u)\leq\omega(x)\psi(u)\,,\quad\text{on}\quad[0,1]\times\mathbb{R}^+$$

*and*

$$\lim_{u\to0^+}\frac{\psi(u)}{\phi(u)}=0\,,\qquad\lim_{u\to+\infty}\frac{\varphi(u)}{\phi(u)}=+\infty\,.$$

*Then bvp (1) has at least one positive solution for every $\lambda>0$.*

Also, we have

**Corollary 3.6.** *Let* $q \in C([0,1],\mathbb{R}^+)$ *with* $\min\limits_{x\in[0,1]} q(x) > 0$ *and* $F\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *satisfies*

$$\limsup_{s\to 0^+} \frac{F(s)}{\phi(s)} = 0 \quad \textit{and} \quad \liminf_{s\to+\infty} \frac{F(s)}{\phi(s)} = +\infty\,.$$

*Then, the boundary value problem*

$$\begin{cases} -(\phi(u'))' = \lambda q(x)F(u)\,, \quad 0<x<1\,, \\ u(0) = L_0(u)\,, \quad u(1) = L_1(u) \end{cases} \tag{13}$$

*has at least one positive solution for every* $\lambda > 0$.

**Example 3.2.** Consider the boundary value problem:

$$\begin{cases} -\big(\phi_3(u')\big)'(x) = q(x)\big(\phi_4(u) + \phi_5(u)\big)\,, \quad 0<x<1 \\ u(0) = \int_0^1 u(s)\,d\mu_1(s)\,, \quad u(1) = \int_0^1 u(s)\,d\mu_2(s)\,, \end{cases} \tag{14}$$

where the function $q \in C([0,1],(0,+\infty))$. $\mu_1$, $\mu_2$ are two nondecreasing functions on $[0,1]$ of bounded variation $V_0^1(\mu_i) < 1$, $(i = 0,1)$. Let $\psi(u) = \phi_4(u) + \phi_5(u)$ and $\varphi(u) = \phi_4(u)$. Then,

$$\lim_{u\to0^+} \frac{\psi(u)}{\phi(u)} = \lim_{u\to0^+} (u+u^2) = 0 \quad \text{and} \quad \lim_{u\to+\infty} \frac{\varphi(u)}{\phi(u)} = \lim_{u\to+\infty} u = +\infty\,.$$

By Corollary 3.5, bvp (14) has at least one positive solution.

### 3.4. **The sub-linear-like case.**

**Theorem 3.7.** *Suppose that the following condition holds:*

$$\liminf_{u\to0^+} \frac{f(x,u)}{\phi(u)} = +\infty \quad \textit{and} \quad \limsup_{u\to+\infty} \frac{f(x,u)}{\phi(u)} = 0\,, \qquad \textit{uniformly in } x \in [0,1]\,.$$

*Then bvp* (1) *has at least one positive solution for sufficiently small* $\lambda > 0$.

**Proof.**

*Claim 1.* Let $\lambda > 0$, $0 < \sigma < 1/2$ be fixed constants, and pick $k$ such that (12) is satisfied. Since $\liminf\limits_{u\to0^+} \frac{f(x,u)}{k\phi(u)} = +\infty$, uniformly in $x \in [0,1]$, then there exists $r > 0$ such that $\lambda f(x,u) \geq k\phi(u)$, for $u \in [0,r]$. Consider the open ball $\Omega_1 := B(0,r)$ and let $u \in K \cap \partial\Omega_1$, that is $u \in K$ and $\|u\|_0 = r$. Then, in one hand, we have that $u(x) \geq p(x)\|u\|_0 \geq \sigma\|u\|_0$ for any $x \in [\sigma, 1-\sigma]$ and in the other hand, the following discussion holds true:

*Case (a):* If $\theta(u) < \sigma$ or $\theta(u) > 1-\sigma$, then by Lemma 3.3 we get, since $\phi$ is increasing

$$\begin{aligned} \|Tu\|_0 &\geq \sigma\phi^{-1}\Big(\int_\sigma^{1-\sigma} \lambda f(\tau,u(\tau))d\tau\Big) \\ &\geq \sigma\phi^{-1}\Big(\int_\sigma^{1-\sigma} k\lambda\phi(u(\tau))d\tau\Big) \\ &\geq \sigma\phi^{-1}\Big(k\lambda(1-2\sigma)\phi(\sigma\|u\|_0)\Big)\,. \end{aligned}$$

Owing to (2) and (12), we deduce that

$$\|Tu\|_0 \geq \sigma^2 \|u\|_0 \phi^{-1}\left(k\lambda(1-2\sigma)\right) \geq \|u\|_0 = r\,.$$

*Case (b):* If $\theta(u) \in [\sigma, 1-\sigma]$, then again by Lemma 3.3 both with (3), we obtain the estimates:

$$\begin{aligned}
\|Tu\|_0 &\geq \frac{\sigma}{2}\phi^{-1}\Big(\int_\sigma^\theta \lambda f(\tau, u(\tau))d\tau\Big) + \frac{\sigma}{2}\phi^{-1}\Big(\int_\theta^{1-\sigma} \lambda f(\tau, u(\tau))d\tau\Big) \\
&\geq \frac{\sigma}{2}\phi^{-1}\Big(\int_\sigma^\theta k\lambda\phi(u(\tau))d\tau\Big) + \frac{\sigma}{2}\phi^{-1}\Big(\int_\theta^{1-\sigma} k\lambda\phi(u(\tau))d\tau\Big) \\
&\geq \frac{\sigma}{2}\phi^{-1}\big(k\lambda(\theta-\sigma)\phi(\sigma\|u\|_0)\big) + \frac{\sigma}{2}\phi^{-1}\big(k\lambda(1-\sigma-\theta)\phi(\sigma\|u\|_0)\big) \\
&\geq \frac{\sigma}{2}\Phi^*\phi^{-1}\big(k\lambda(1-2\sigma)\phi(\sigma\|u\|_0)\big)\,.
\end{aligned}$$

Hence

$$\|Tu\|_0 \geq \frac{\sigma^2}{2}\Phi^*\|u\|_0\phi^{-1}\big(k\lambda(1-2\sigma)\big) \geq \|u\|_0 = r\,.$$

Therefore, in both cases, we arrive at the estimate

$$\forall\, u \in K \cap \partial\Omega_1,\ \|Tu\|_0 \geq \|u\|_0\,.$$

*Claim 2.* Since $\limsup\limits_{u\to\infty} \frac{f(x,u)}{\phi(u)} = 0$ uniformly in $x \in [0,1]$. then there exists $R > 0$ such that $0 \leq f(x,u) \leq \phi(u)$ for $x \in [0,1]$ and $u \geq R$. So, there exists $C > 0$ such that $0 \leq f(x,u) \leq \phi(u) + C$ for $(x,u) \in [0,1] \times \mathbb{R}^+$. Now let the open ball $\Omega_2 := B(0,R)$ and $u \in K \cap \partial\Omega_2$. If $v = Tu$, then $v$ verifies

$$\begin{cases} -(\phi(v'))'(x) = \lambda f(x,u)\,, \quad 0 < x < 1 \\ v(0) = L_0(u)\,, \qquad v(1) = L_1(u)\,. \end{cases}$$

By Lemma 2.2(c), there exists $x_m \in (0,1)$ such that $v'(x_m) = 0$. Then for any $s \in [0,1]$. We have

$$\phi\big(v'(s)\big) = \lambda\int_s^{x_m} f\big(\tau, u(\tau)\big)\,d\tau\,.$$

Hence

$$\begin{aligned}
|\phi(v'(s))| = \phi(|v'(s)|) &\leq \lambda\int_0^1 f(\tau, u(\tau))d\tau \\
&\leq \lambda\int_0^1 \left(\phi(u(\tau)) + C\right)d\tau \leq \lambda\big(\phi(R) + C\big)\,.
\end{aligned}$$

Thus $|v'(s)| \leq \phi^{-1}\big(\lambda(\phi(R)+C)\big)$, $\forall\, s \in [0,1]$. Since $L_0$ is increasing, we deduce that

$$\begin{aligned}
v(t) = v(0) + \int_0^t v'(s)ds &= L_0(u) + \int_0^t v'(s)\,ds \\
&\leq L_0(R) + \sup_{t\in[0,1]} |v'(t)| \leq L_0(R) + \phi^{-1}\big(\lambda\big(\phi(R) + C\big)\big)\,.
\end{aligned}$$

Since $0 < L_0(R) < R$, choose $0 < \lambda \leq \lambda^\star := \phi(R - L_0(R))/(\phi(R) + C)$ to obtain that $v(t) \leq R = \|u\|_0$, that is $\forall\, u \in K \cap \partial\Omega_2$, $\|Tu\|_0 \leq \|u\|_0$. By Theorem B, for $0 < \lambda \leq \lambda^\star$, bvp (1) admits a positive solution $u$ such that $\min(r, R) \leq \|u\|_0 \leq \max(r, R)$. □

As a consequence, we deduce

**Corollary 3.8.** *Let* $q \in C([0,1], \mathbb{R}^+)$ *with* $\min\limits_{x\in[0,1]} q(x) > 0$ *and* $F\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *satisfies*

$$\liminf_{s\to 0^+} \frac{F(s)}{\phi(s)} = +\infty \quad \textit{and} \quad \limsup_{s\to+\infty} \frac{F(s)}{\phi(s)} = 0\,.$$

*Then, the boundary value problem*

$$\begin{cases} -(\phi(u'))' = \lambda q(x) F(u)\,, \quad 0 < x < 1\,, \\ u(0) = L_0(u)\,, \quad u(1) = L_1(u) \end{cases} \tag{15}$$

*has at least one positive solution for sufficiently small* $\lambda > 0$.

**Example 3.3.** Consider the multi-point boundary value problem:

$$\begin{cases} -\left(\phi(u')\right)'(x) = \lambda q(x) g(u(x))\,, & 0 < x < 1 \\ u(0) = \sum\limits_{i=1}^{n} a_i u(\xi_i)\,, \quad u(1) = \sum\limits_{i=1}^{n} b_i u(\xi_i)\,, \end{cases} \tag{16}$$

where $q \in C([0,1], (0,+\infty))$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, and $a_i, b_i \geq 0$ are such that $\sum\limits_{i=1}^{n} a_i < 1$ and $\sum\limits_{i=1}^{n} b_i < 1$. With $L_0(u) = u(0)$ and $L_1(u) = u(1)$, we have that $L_0(1) = \sum\limits_{i=1}^{n} a_i < 1$ and $L_1(1) = \sum\limits_{i=1}^{n} b_i < 1$. Let $\phi(u) = k_3\phi_p(u) + k_4\phi_q(u)$, $g(u) = k_1\phi_s(u) + k_2\phi_t(u)$, for some positive constants $k_i$, $(i = 1, \ldots, 4)$, and $1 < s < p < t < q$. The latter condition yields that

$$\begin{aligned} \lim \frac{g(u)}{\phi(u)} &= \lim \frac{k_1 u^s + k_2 u^t}{k_3 u^p + k_4 u^q} = \lim \frac{k_1 u^{s-p} + k_2 u^{t-p}}{k_3 + k_4 u^{q-p}} \\ &= \begin{cases} 0\,, & \text{if} \quad u \to +\infty \\ +\infty & \text{if} \quad u \to 0\,. \end{cases} \end{aligned}$$

By Corollary 3.8, we obtain the existence of at least one positive solution for small parameter $\lambda > 0$.

## 4. THE CASE $f = f(x, u, v)$

When the nonlinearity $f$ also depends on the first derivative, application of the classical Krasnosel'skii fixed point theorem turns out to be difficult. Indeed, it not always so easy to perform the two inequalities $\|Tu\| \leq \|u\|$ and $\|Tu\| \geq \|u\|$ for instance when $\|Tu\|$ is the sup-norm in the Banach space $C^1([0,1], \mathbb{R})$. An alternative way consists in employing the following recent fixed point theorem. First, we present the general framework. Let $X$ be a linear space such that there is a norm $\|u\|_1$ under which $X$ is a normed linear space (not necessarily complete) and there is a semi-norm $\|\cdot\|_2$ such that under $\|u\| = \max(\|u\|_1, \|u\|_2)$, $X$ is a

Banach space. For example, equipped with $\|u\|_1 = \|u\|_0$, $X = E^1 := C^1([0,1],\mathbb{R})$ is an incomplete normed linear space. If $\|u\|_2 = \|u'\|_0$, then $\|u\|_2$ is a semi-norm of $E^1$. Finally, with $\|u\| = \max(\|u\|_1, \|u\|_2)$, $E^1$ is a Banach space.

**Theorem C** ([17, Theorem 2.8]). *Let $\Omega_1 = \{u \in X, \|u\|_1 < r\}$ and $\Omega_2 = \{u \in X, \|u\|_1 < R\}$ be two open sets in $X$ with $r < R$. Let $T\colon K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a continuous map with relatively compact image $T(K \cap (\bar{\Omega}_2 \setminus \Omega_1))$. Suppose that one of the following two conditions is satisfied:*
(a) *$\|Tv\| \le \|v\|$ for $v \in K \cap \partial\Omega_1$ and $\|Tv\|_1 \ge \|v\|_1$ for $v \in K \cap \partial\Omega_2$,*
(b) *$\|Tv\|_1 \ge \|v\|_1$ for $v \in K \cap \partial\Omega_1$ and $\|Tv\| \le \|v\|$ for $v \in K \cap \partial\Omega_2$.*
*Then $T$ has at least a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$.*

Notice that $\Omega_1$ and $\Omega_2$ need not be bounded. Then we can prove existence of positive solutions which are only bounded with respect to the norm $\|u\|_0$. Arguing as in Theorems 3.4 and 3.7, we have the following two existence theorems:

**Theorem 4.1** (The super-linear-like case). *Suppose*

(a) *there exists a nondecreasing function $\psi_1 \in C(\mathbb{R},(0,+\infty))$ with $0 < \int_1^{+\infty} \frac{\psi_1(t)}{t^2}dt < +\infty$ such that*

$$\limsup_{u\to 0^+} \frac{f(x,u,v)}{\phi(u)\psi_1(v)} = 0, \text{ uniformly in } x \in [0,1] \quad \text{and} \quad v \in \mathbb{R}\,.$$

(b) $$\liminf_{u\to+\infty} \frac{f(x,u,v)}{\phi(u)} = +\infty, \text{ uniformly in } x \in [0,1] \text{ and } v \in \mathbb{R}\,.$$

*Then bvp (1) has at least one positive solution in $E^1$ for all positive $\lambda$.*

**Example 4.1.** Consider the multi-point boundary value problem:

$$\begin{cases} -\left(\phi(u')\right)'(x) = \lambda q(x) g(u(x)) h(u'(x))\,, & 0 < x < 1 \\ u(0) = \sum\limits_{i=1}^{n} a_i u(\xi_i)\,, \quad u(1) = \sum\limits_{i=1}^{n} b_i u(\xi_i)\,, \end{cases} \tag{17}$$

where $q \in C([0,1],(0,+\infty))$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, and $a_i, b_i \ge 0$ are such that $\sum\limits_{i=1}^{n} a_i < 1$ and $\sum\limits_{i=1}^{n} b_i < 1$. With $L_0(u) = u(0)$ and $L_1(u) = u(1)$, we have that $L_0(1) = \sum\limits_{i=1}^{n} a_i < 1$ and $L_1(1) = \sum\limits_{i=1}^{n} b_i < 1$. Let $g(u) = c_1\phi_{5/2}(u) + c_2\phi_3(u)$ and $\phi(u) = c_3\phi_{3/2}(u) + c_4\phi_2(u)$ for some positive constants $c_i$, $(i = 1,4)$. Let $h(v) = 1 + e^v$ and $\psi_1(v) = \sqrt{1+\vartheta(v)}$ where $\vartheta(s) = \begin{cases} s\,, & \text{if } s \ge 0 \\ 0\,, & \text{otherwise}\,. \end{cases}$ Then $\int_1^{+\infty} \frac{\sqrt{1+t}}{t^2}dt < +\infty$ and the ratio

$$\frac{g(u)h(v)}{\phi(u)\psi_1(v)} = \Big(\frac{c_1u^{5/2} + c_2u^3}{c_3u^{3/2} + c_4u^2}\Big)\Big(\frac{1+e^v}{\sqrt{1+\vartheta(v)}}\Big)$$

tends to 0 as $u \to 0$ uniformly in $v \in \mathbb{R}$. Also, the ratio

$$\frac{g(u)h(v)}{\phi(u)} = (1+e^v)\Big(\frac{c_1 u^{5/2} + c_2 u^3}{c_3 u^{3/2} + c_4 u^2}\Big)$$

tends to positive infinity if $u \to +\infty$ uniformly in $v \in \mathbb{R}$. By Theorem 4.1, we obtain the existence of at least one positive solution for all positive $\lambda$.

**Theorem 4.2** (The sub-linear-like case)**.** *Suppose*

(a) *there exists a nondecreasing function* $\psi_1 \in C(\mathbb{R}, (0,+\infty))$ *with* $0 < \displaystyle\int_1^{+\infty} \frac{\psi_1(t)}{t^2}dt < +\infty$ *such that*

$$\limsup_{u\to+\infty} \frac{f(x,u,v)}{\phi(u)\psi_1(v)} = 0\,, \quad \textit{uniformly in } x \in [0,1] \textit{ and } v \in \mathbb{R}\,.$$

(b) $$\liminf_{u\to 0} \frac{f(x,u,v)}{\phi(u)} = +\infty\,, \quad \textit{uniformly in } x \in [0,1] \textit{ and } v \in \mathbb{R}\,.$$

*Then bvp* (1) *has at least one positive solution in* $E^1$ *for every* $\lambda > 0$ *small enough.*

**Sketch of the proof.**
*Claim 1.* As in the proof of Theorem 3.7, Assumption (b) yields some $R_1 > 0$ such that

$$\forall\, u \in K \cap \partial\Omega_1\,, \ \|Tu\|_0 \geq \|u\|_0\,,$$

where $\Omega_1 := \big\{u \in E^1 : \|u\|_0 < R_1\big\}$ and $K := \big\{u \in E^1 : u \text{ concave}\big\}$.

*Claim 2.* Here we first notice that since $u \in E^1$ is concave, then for any $x \in (0,1)$, there exists $\eta \in \mathbb{R}$, $(0 < \eta < x)$ such that $u(x) \geq xu'(\eta)$. Hence $\frac{u(x)}{x} \geq u'(\eta) \geq u'(x)$. This with Assumption (a) and since $\psi_1$ is nondecreasing, there exists some $C > 0$ such that

$$0 \leq f(t, u(t), u'(t)) \leq \phi(\|u\|_0)\psi_1\Big(\frac{\|u\|_0}{t}\Big) + C\,,$$

for $u \in K$. Moreover, there exists $R_2 > 0$ such that for every $u \in K \cap \partial\Omega_2$, we deduce the estimates: $\|(Tu)'\|_0 \leq \|u\|_0 \leq \|u\|$ and for $\lambda$ small enough $\|Tu\|_0 \leq \|u\|_0 \leq \|u\|$. Here $\Omega_2 := \big\{u \in E^1 : \|u\|_0 < R_2\big\}$. Hence

$$\|Tu\| \leq \|u\|_0 \leq \|u\|\,.$$

By Theorem C, we obtain the existence of at least one positive solution $u$ for small parameter $\lambda$. In addition, $\min(R_1, R_2) \leq \|u\|_0 \leq \min(R_1, R_2)$. □

Finally, we mention a third existence result proved in [1] for homogeneous Dirichlet boundary conditions.

**Theorem 4.3.** *Suppose that*
(a) *there exist* $r_0 > 0$, $q_1 \in C([0,1], \mathbb{R}^+)$, $\varphi_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$, *and* $\psi_1 \in C(\mathbb{R}, \mathbb{R}^+)$ *where* $\varphi_1, \psi_1$ *are nondecreasing with*

$$\int_0^1 q_1(s)\psi_1\Big(\frac{r_0}{s}\Big)\, ds \leq \frac{1}{\varphi_1(r_0)}\phi\Big(\frac{r_0}{1+a}\Big)$$

*such that*

$$0 \le \lambda f(x,u,v) \le q_1(x)\varphi_1(u)\psi_1(v), \quad \textit{for all} \;\; x \in [0,1],\, 0 \le u \le r_0, \quad \textit{and} \quad v \in \mathbb{R}\,.$$

(b) *There exist* $0 < \sigma < 1/2$*,* $R_0 \neq r_0$*,* $q_2 \in C([0,1],\mathbb{R}^+)$*,* $\varphi_2 \in C(\mathbb{R}^+,\mathbb{R}^+)$*, and* $\psi_2 \in C(\mathbb{R},\mathbb{R}^+)$ *with* $\varphi_2$ *nondecreasing,* $\psi_2$ *nonincreasing and*

$$0 < R_0 \le \frac{\sigma D(\sigma)}{2} \quad \textit{and} \quad \int_0^1 q_2(s)\psi_2\Big(\frac{R_0}{s}\Big)\,ds < \infty$$

*such that*

$$\lambda f(x,u,v) \ge q_2(x)\varphi_2(u)\psi_2(v)\,, \quad \textit{for all} \;\; x \in [0,1]\,, \sigma R_0 \le u \le R_0\,, \quad \textit{and} \quad v \in \mathbb{R}\,.$$

*Then, for every* $\lambda > 0$*, bvp* (1) *has at least one positive solution* $u \in E^1$ *satisfying*

$$\min(r_0,R_0) \le \|u\|_0 \le \max(r_0,R_0)\,.$$

*Here*

$$D(\sigma) := \Phi^*\phi^{-1}\Big(\int_\sigma^{1-\sigma} q_2(s)\varphi_2(\sigma R_0)\psi_2\Big(\frac{R_0}{s}\Big)\,ds\Big)\,. \tag{18}$$

**Example 4.2.** Consider the boundary value problem:

$$\begin{cases} -\big(\phi_{\frac{7}{2}}(u')\big)'(x) = q(x)\varphi(u(x))h(u'(x))\,, & 0 < x < 1 \\ u(0) = \int_0^1 u(s)\,d\mu_1(s)\,, & u(1) = \int_0^1 u(s)\,d\mu_2(s)\,, \end{cases} \tag{19}$$

where $q \in C([0,1],(0,+\infty))$ with $\|q\|_0 \le 1$, $h(s) = \psi(s) + \psi_2(s)$ with $\psi_2(s) = e^{-s}$, $\psi(s) = \sqrt{1+\vartheta(s)}$, and $\vartheta(s) = \begin{cases} s\,, & \text{if } \; s \ge 0 \\ 0\,, & \text{otherwise.} \end{cases}$ The functions $\mu_1$, $\mu_2$ are nondecreasing on $[0,1]$ of bounded variation $V_0^1(\mu_i) < 1$, $(i = 0,1)$ which ensures that the Stieltjes integrals do exist. Letting $L_0(u) = u(0)$ and $L_1(u) = u(1)$, we obtain that $L_0(1) = \int_0^1 d\mu_1(s) = V_0^1(\mu_1) < 1$ and $L_1(1) = \int_0^1 d\mu_2(s) = V_0^1(\mu_2) < 1$. Since $h(s) \le \psi(s) + 1$, put $\psi_1(s) = 1 + \psi(s)$, then

$$\begin{aligned} \int_0^1 q(s)\psi_1\Big(\frac{r_0}{s}\Big)\,ds &\le \|q\|_0 \int_0^1 \psi_1\Big(\frac{r_0}{s}\Big)\,ds \\ &\le \int_0^1 \psi_1\Big(\frac{r_0}{s}\Big)\,ds = \int_0^1 1\,ds + \int_0^1 \sqrt{1+\frac{r_0}{s}}\,ds \\ &= 1 + \frac{r_0}{2}\Big(\frac{1}{\sqrt{1+r_0}-1} + \frac{1}{\sqrt{1+r_0}+1} - \ln\Big(\frac{\sqrt{1+r_0}-1}{\sqrt{1+r_0}+1}\Big)\Big)\,. \end{aligned}$$

With $\varphi_1(s) = \varphi(s) = (1+s)^{2/3}$, assumption (a) in Theorem 3.1 is fulfilled whenever there exists $r > 0$ such that

$$\frac{1}{\sqrt{1+r_0}-1} + \frac{1}{\sqrt{1+r_0}+1} - \ln\Big(\frac{\sqrt{1+r_0}-1}{\sqrt{1+r_0}+1}\Big) \le \frac{2}{(1+a)^{\frac{5}{2}}}\frac{r_0^{3/2}}{(1+r_0)^{2/3}} - \frac{2}{r_0}$$

which is satisfied for $r_0$ large enough. Moreover, with $\psi_2(s) = e^{-s}$, $\varphi_2(s) = \varphi(s) = (1+s)^{2/3}$, and since $\phi^{-1}(s) = s^{2/7}$ for $s \geq 0$, we find that

$$D(\sigma) = \Phi^* \left( (1+\sigma R_0)^{2/3} \int_{\sigma}^{1-\sigma} q(s) e^{-R_0/s}\, ds \right)^{2/7} .$$

Hence a sufficient condition for (b) in Theorem 4.3 be satisfied is

$$\frac{2R_0}{\Phi^*} \leq \sigma (1+\sigma R_0)^{4/21} \Big( \int_{\sigma}^{1-\sigma} q(s) e^{-R_0/s}\, ds \Big)^{2/7}$$

that is if

$$\frac{2R_0}{\Phi^*} \leq \sigma (1+\sigma R_0)^{4/21} q^{2/7} (1-2\sigma)^{2/7} e^{-2R_0/7\sigma} ,$$

where $q := \min\left(q(x), \sigma \leq x \leq 1-\sigma\right)$ is positive. Notice that the latter condition is satisfied for small $R_0$. Therefore, all assumptions in Theorem 4.3 are met, hence Problem (19) has at least one positive solution $u$ with $R_0 \leq \|u\|_0 \leq r_0$.

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