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δ-IDEALS IN PSEUDO-COMPLEMENTED DISTRIBUTIVE LATTICES

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Abstract. The concept of δ-ideals is introduced in a pseudo-complemented distributive lattice and some properties of these ideals are studied. Stone lattices are characterized in terms of δ-ideals. A set of equivalent conditions is obtained to characterize a Boolean algebra in terms of δ-ideals. Finally, some properties of δ-ideals are studied with respect to homomorphisms and filter congruences.

Introduction

The theory of pseudo-complements was introduced and extensively studied in semi-lattices and particularly in distributive lattices by Orrin Frink [4] and G. Birkhoff [2]. Later the problem of characterizing Stone lattices has been studied by several authors like R. Balbes [1], O. Frink [4], G. Grätzer [5] etc.

In this paper, the concept of δ-ideals is introduced in a distributive lattice in terms of pseudo-complementation and filters. Some properties of these δ-ideals are studied and then proved that the set of all δ-ideals can be made into a complete distributive lattice. We derive a set of equivalent conditions for the class of all δ-ideals to become a sublattice to the lattice of all ideals, which leads to a characterization of Stone lattices. A set of equivalent conditions are established for every prime ideal to become a δ-ideal which leads to a characterization of a Boolean algebra. Finally, the set of δ-ideals of a pseudo-complemented distributive lattice is characterized in terms of filter congruences.

1. Preliminaries

In this section, we recall certain definitions and important results taken from [6] for the ready reference to the reader.

Definition 1.1 ([6]). For any element a of a distributive lattice L, the pseudo-complement a^* of a is an element satisfying the following property for all $x \in L$:

$$a \wedge x = 0 \;\Leftrightarrow\; a^* \wedge x = x \;\Leftrightarrow\; x \leq a^* .$$

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A distributive lattice L in which every element has a pseudo-complement is called a pseudo-complemented distributive lattice. Throughout this paper L stands for a pseudo-complemented distributive lattice $(L, \vee, \wedge, ^*, 0, 1)$.

Theorem 1.2 ([6])**.** *For any two elements a, b of a pseudo-complemented distributive lattice, we have the following:*

(1) $0^{**} = 0$,

(2) $a \wedge a^* = 0$,

(3) $a \leq b$ *implies* $b^* \leq a^*$,

(4) $a \leq a^{**}$,

(5) $a^{***} = a^*$,

(6) $(a \vee b)^* = a^* \wedge b^*$,

(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element x of a pseudo-complemented lattice L is called dense [6] if $x^* = 0$ and the set $D(L)$ of all dense element of L forms a filter of L.

Definition 1.3 ([1])**.** A pseudo-complemented distributive lattice L is called a Stone lattice if, for all $x \in L$, it satisfies the property: $x^* \vee x^{**} = 1$.

Theorem 1.4 ([6])**.** *Let I be an ideal and F a filter of a distributive lattice L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

A prime ideal P of a distributive lattice L is called a minimal prime ideal [8] if there exists no prime ideal Q such that $Q \subset P$. A prime ideal P is minimal if and only if to each $x \in P$ there exists $y \notin P$ such that $x \wedge y = 0$.

2. δ-ideals

In this section, the concept of δ-ideals is introduced in a pseudo-complemented distributive lattice. Stone lattice and Boolean algebras are characterized in terms of δ-ideals. Finally, δ-ideals are characterized in terms of congruences.

Definition 2.1. Let L be a pseudo-complemented distributive lattice. Then for any filter F of L, define the set $\delta(F)$ as follows:

$$\delta(F) = \{x \in L \mid x^* \in F\}.$$

In the following, some basic properties of $\delta(F)$ can be observed.

Lemma 2.2. *Let L be a pseudo-complemented distributive lattice. Then for any filter F of L, $\delta(F)$ is an ideal of L.*

Proof. Since $0^* \in F$, we get that $0 \in \delta(F)$. Let $x, y \in \delta(F)$. Then x^*, $y^* \in F$. Hence $(x \vee y)^* = x^* \wedge y^* \in F$. Again, let $x \in \delta(F)$ and $r \in L$. Then $x^* \in F$. Hence $(x \wedge r)^* = (x \wedge r)^{***} = (x^{**} \wedge r^{**})^* = (x^* \vee r^*)^{**} \in F$(because $x^* \vee r^* \in F$). Hence we get that $x \wedge r \in \delta(F)$. Therefore $\delta(F)$ is an ideal in L. □

Lemma 2.3. *Let L be a pseudo-complemented distributive lattice. For any two filters F, G of L, we have the following:*

(1) $F \cap \delta(F) = \emptyset$,

(2) $x \in \delta(F) \Rightarrow x^{**} \in \delta(F)$,

(3) $F = L$ *if and only if* $\delta(F) = L$,

(4) $F \subseteq G \Rightarrow \delta(F) \subseteq \delta(G)$,

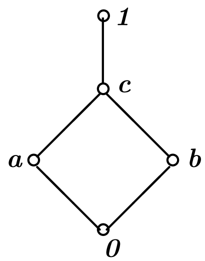
(5) $\delta(F \cap G) = \delta(F) \cap \delta(G)$.

Proof. (1) Suppose $x \in F \cap \delta(F)$. Then $x \in F$ and $x^* \in F$. Since F is a filter, we get $0 = x^* \wedge x \in F$, which is a contradiction. Therefore $F \cap \delta(F) = \emptyset$.
(2) Since $x^{***} = x^*$, it is clear.
(3) Assume that $F = L$. Then we have $0^{**} = 0 \in F$. Hence $0^* \in \delta(F)$. Therefore $\delta(F) = L$. Converse is an easy reverse of the above.
(4) Suppose $F \subseteq G$. Let $x \in \delta(F)$. Then $x^* \in F \subseteq G$. Therefore $x \in \delta(G)$.
(5) Clearly $\delta(F \cap G) \subseteq \delta(F) \cap \delta(G)$. Conversely, let $x \in \delta(F) \cap \delta(G)$. Then $x^* \in F \cap G$. Hence $x \in \delta(F \cap G)$. Therefore $\delta(F) \cap \delta(G) \subseteq \delta(F \cap G)$. $\square$

The concept of δ-ideals is now introduced in the following.

Definition 2.4. Let L be a pseudo-complemented distributive lattice. An ideal I of L is called a δ-ideal if $I = \delta(F)$ for some filter F of L.

Example 2.5. *Consider the distributive lattice* $L = \{0, a, b, c, 1\}$ *whose Hasse diagram is given in the following figure:*



Consider $I = \{0, a\}$ *and* $F = \{b, c, 1\}$.
Clearly I *is an ideal and* F *a filter of* L.
Now $\delta(F) = \{x \mid x^* \in F\} = \{0, a\}$.
Therefore I *is a* δ*-ideal of* L.
But $J = \{0, a, b, c\}$ *is not a* δ*-ideal of* L.
Suppose $J = \delta(F)$ *for some filter* F
Then $0 = c^* \in F$. *Hence* $F = L$, *which yields that* $J = \delta(F) = L$.

The following lemmas produce some more examples for δ-ideals.

Lemma 2.6. *For each* $x \in L$, $(x^*]$ *is a* δ*-ideal of* L.

Proof. It is enough to show that $(x^*] = \delta([x))$. Let $a \in (x^*]$. Then $a \wedge x = 0$ and hence $a^* \wedge x = x \in [x)$. Thus $a^* \in [x)$. Therefore $a \in \delta([x))$. Conversely, suppose that $a \in \delta([x))$. Then $a^* \wedge x = x$ and hence $a \wedge x = a \wedge a^* \wedge x = 0$. Thus $a \wedge x^* = a$, which yields that $a \in (x^*]$. Therefore $(x^*]$ is a δ-ideal of L. $\square$

Lemma 2.7. *Every prime ideal without dense element is a* δ*-ideal.*

Proof. Let P be a prime ideal with out dense element. Let $x \in P$. Then clearly $x \wedge x^* = 0$ and $x \vee x^*$ is dense. Hence $x \vee x^* \notin P$. Thus we get $x^* \notin P$, which yields that $x^* \in L - P$. Thus $x \in \delta(L - P)$. Therefore $P \subseteq \delta(L - P)$. Conversely, let $x \in \delta(L - P)$. Then $x^* \in L - P$, which implies that $x^* \notin P$. Hence $x \in P$. Thus $P = \delta(L - P)$. Therefore P is a δ-ideal of L. $\square$

Corollary 2.8. *Every minimal prime ideal is a δ-ideal.*

Proof. Let P be a minimal prime ideal of L. Suppose $x \in P \cap D(L)$. Then there exists $y \notin P$ such that $x \wedge y = 0$. Hence $y \leq x^* = 0 \in P$, which is a contradiction. Thus $P \cap D(L) = \emptyset$. Therefore, by above lemma, P is a δ-ideal. □

In the following, a simple property of δ-ideals is observed.

Lemma 2.9. *A proper δ-ideal contains no dense element.*

Proof. Let $\delta(F)$ be a proper δ-ideal of L. Suppose $x \in \delta(F) \cap D(L)$. Then we get $0 = x^* \in F$, which is a contradiction. Therefore $\delta(F) \cap D(L) = \emptyset$. □

Let us denote the set of all δ-ideals of L by $\mathcal{I}^\delta(L)$. Then by Example 2.5, it can be observed that $\mathcal{I}^\delta(L)$ is not a sublattice of $\mathcal{I}(L)$ of all ideals of L. Consider $F = \{b, c, 1\}$ and $G = \{a, c, 1\}$. Clearly F and G are filters of L. Now $\delta(F) = \{0, a\}$ and $\delta(G) = \{0, b\}$. But $\delta(F) \vee \delta(G) = \{0, a, b, c\}$ is not a δ-ideal of L, because $c \in \delta(F) \vee \delta(G)$ is a dense element. However, in the following theorem, we prove that $\mathcal{I}^\delta(L)$ forms a complete distributive lattice.

Theorem 2.10. *Let L be a pseudo-complemented distributive lattice. Then the set $\mathcal{I}^\delta(L)$ forms a complete distributive lattice on its own.*

Proof. For any two filters F, G of L, define two binary operations $\cap$ and $\sqcup$ as follows:

$$\delta(F) \cap \delta(G) \;=\; \delta(F \cap G) \quad \text{and} \quad \delta(F) \sqcup \delta(G) \;=\; \delta(F \vee G)\,.$$

It is clear that $\delta(F \cap G)$ is the infimum of $\delta(F)$ and $\delta(G)$ in $\mathcal{I}^\delta(L)$. Also $\delta(F) \sqcup \delta(G)$ is a δ-ideal of L. Clearly $\delta(F), \delta(G) \subseteq \delta(F \vee G) = \delta(F) \sqcup \delta(G)$. Let $\delta(H)$ be a δ-ideal of L such that $\delta(F) \subseteq \delta(H)$ and $\delta(G) \subseteq \delta(H)$, where H is a filter of L. Now we claim that $\delta(F \vee G) \subseteq \delta(H)$. Let $x \in \delta(F \vee G)$. Then $x^* \in F \vee G$. Hence $x^* = f \wedge g$ for some $f \in F$ and $g \in G$. Since $f \in F$ and $g \in G$, we can get $f^* \in \delta(F) \subseteq \delta(H)$ and $g^* \in \delta(G) \subseteq \delta(H)$. Now

$$\begin{aligned} f^* \in \delta(H),\ g^* \in \delta(H) \quad &\Rightarrow f^* \vee g^* \in \delta(H) \\ &\Rightarrow (f^* \vee g^*)^{**} \in \delta(H) \\ &\Rightarrow (f^{**} \wedge g^{**})^* \in \delta(H) \\ &\Rightarrow x^{**} \in \delta(H) \\ &\Rightarrow x \in \delta(H) \end{aligned}$$

Hence $\delta(F) \sqcup \delta(G) = \delta(F \vee G)$ is the supremum of both $\delta(F)$ and $\delta(G)$ in $\mathcal{I}^\delta(L)$. Therefore $\langle \mathcal{I}^\delta(L), \cap, \sqcup \rangle$ is a lattice. Distributivity of δ-ideals can be easily followed by using the above operations of $\mathcal{I}^\delta(L)$.

It is clear that $\mathcal{I}^\delta(L)$ is a partially ordered set with respect to set-inclusion. Then by the extension of the property of Lemma 2.3(5), we can obtain that $\mathcal{I}^\delta(L)$ is a complete lattice. Therefore $\mathcal{I}^\delta(L)$ is a complete distributive lattice. □

From Lemma 2.6, we have already observed that each $(x^*]$(for $x \in L$) is a δ-ideal of L. Now let us denote that $\mathcal{A}^*(L) = \{(x^*] \mid x \in L\}$. Then, in the following theorem, it is proved that $\mathcal{A}^*(L)$ is a Boolean algebra.

Theorem 2.11. *For any pseudo-complemented distributive lattice L, $\mathcal{A}^*(L)$ is a sublattice of the lattice $\mathcal{I}^\delta(L)$ of all δ-ideals of L and hence is a Boolean algebra. Moreover, the mapping $x \longmapsto (x^*]$ is a dual homomorphism from L onto $\mathcal{A}^*(L)$.*

Proof. Let $(x^*]$, $(y^*] \in \mathcal{A}^*(L)$ for some $x, y \in L$. Then clearly $(x^*] \cap (y^*] \in \mathcal{A}^*(L)$. Again, $(x^*] \sqcup (y^*] = \delta([x)) \sqcup \delta([y)) = \delta([x) \vee [y)) = \delta([x \wedge y)) = ((x \wedge y)^*] \in \mathcal{A}^*(L)$. Hence $\mathcal{A}^*(L)$ is a sublattice of $\mathcal{I}^\delta(L)$ and hence a distributive lattice. Clearly $(0^{**}]$ and $(0^*]$ are the least and greatest elements of $\mathcal{A}^*(L)$. Now for any $x \in L$, $(x^*] \cap (x^{**}] = (0]$ and $(x^*] \sqcup (x^{**}] = \delta([x)) \sqcup \delta([x^*)) = \delta([x) \vee [x^*)) = \delta([x \wedge x^*)) = \delta([0)) = \delta(L) = L$. Hence $(x^{**}]$ is the complement of $(x^*]$ in $\mathcal{A}^*(L)$. Therefore $\langle \mathcal{A}^*(L), \sqcup, \cap \rangle$ is a bounded distributive lattice in which every element is complemented. The remaining part can be easily observed. □

It was already observed that $\mathcal{I}^\delta(L)$ is not a sublattice of the ideal lattice $\mathcal{I}(L)$. However, we establish some equivalent conditions for $\mathcal{I}^\delta(L)$ to become a sublattice of $\mathcal{I}(L)$, which leads to a characterization of Stone lattices. For this, we need the following lemma.

Lemma 2.12. *Every proper δ-ideal is contained in a minimal prime ideal.*

Proof. Let I be a proper δ-ideal of L. Then $I = \delta(F)$ for some filter F of L. Clearly $\delta(F) \cap D(L) = \emptyset$. Then there exists a prime ideal P of L such that $\delta(F) \subseteq P$ and $P \cap D(L) = \emptyset$. Let $x \in P$. We have always $x \wedge x^* = 0$. Suppose $x^* \in P$. Then $x \vee x^* \in P \cap D(L)$, which is a contradiction. Thus P is a minimal prime ideal of L. □

Since the minimal prime ideals are precisely the complements of maximal filter of L, the following corollary is a direct consequence.

Corollary 2.13. *The minimal prime ideals of a pseudo-complemented distributive lattice L are maximal elements of the complete lattice $\mathcal{I}^\delta(L)$.*

We now characterize Stone lattices in terms of δ-ideals.

Theorem 2.14. *Let L be a pseudo-complemented distributive lattice. Then the following are equivalent:*

(1) *L is a Stone lattice,*
(2) *For any $x, y \in L$, $(x \wedge y)^* = x^* \vee y^*$,*
(3) *For any two filters F, G of L, $\delta(F) \vee \delta(G) = \delta(F \vee G)$,*
(4) *$\mathcal{I}^\delta(L)$ is a sublattice of $\mathcal{I}(L)$.*

Proof. (1) $\Rightarrow$ (2): It is obtained in Lemma 3 of [6] pp.113.
(2) $\Rightarrow$ (3): Assume the condition (2). Let F, G be two filters of L. We have always $\delta(F) \vee \delta(G) \subseteq \delta(F \vee G)$. Conversely, let $x \in \delta(F \vee G)$. Then

$$\begin{aligned}
x^* \in F \vee G &\Rightarrow x^* = f \wedge g \quad \text{for some} \quad f \in F, g \in G \\
&\Rightarrow x^{**} = (f \wedge g)^* \\
&\Rightarrow x^{**} = f^* \vee g^* \\
&\Rightarrow x^{**} = f^* \vee g^* \in \delta(F) \vee \delta(G) \quad \text{since} \quad f^{**} \in F, g^{**} \in G \\
&\Rightarrow x \in \delta(F) \vee \delta(G)\,.
\end{aligned}$$

Hence $\delta(F \vee G) \subseteq \delta(F) \vee \delta(G)$. Therefore $\delta(F) \vee \delta(G) = \delta(F \vee G)$.
(3) $\Rightarrow$ (4): It is obvious.
(4) $\Rightarrow$ (1): Assume that $\mathcal{I}^\delta(L)$ is a sublattice of $\mathcal{I}(L)$. Let $x \in L$. By Lemma 2.6, $(x^*]$ and $(x^{**}]$ are both δ-ideals of L. Suppose $x^* \vee x^{**} \neq 1$. Then by condition (4), $(x^*] \vee (x^{**}]$ is a proper δ-ideal of L. Hence there exists a minimal prime ideal P such that $(x^*] \vee (x^{**}] \subseteq P$. Since P is minimal, we get that $x^{**} \notin P$, which is a contradiction. Therefore L is a Stone lattice. □

In the following theorem, a set of equivalent conditions are obtained for every prime ideal of L to become a δ-ideal which in turn leads to establish some equivalent conditions for a pseudo-complemented distributive lattice to become a Boolean algebra. Let us recall that an element x is called closed [6] if $x^{**} = x$.

Theorem 2.15. *Let L be a pseudo-complemented distributive lattice. Then the following conditions are equivalent:*

(1) *L is a Boolean algebra,*
(2) *Every element of L is closed,*
(3) *Every principal ideal is a δ-ideal,*
(4) *For any ideal I, $x \in I$ implies $x^{**} \in I$,*
(5) *For any proper ideal I, $I \cap D(L) = \emptyset$,*
(6) *For any prime ideal P, $P \cap D(L) = \emptyset$,*
(7) *Every prime ideal is a minimal prime ideal,*
(8) *Every prime ideal is a δ-ideal,*
(9) *For any $x, y \in L$, $x^* = y^*$ implies $x = y$,*
(10) *$D(L)$ is a singleton set.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a Boolean algebra. Then clearly L has a unique dense element, precisely the greatest element. Let $x \in L$. Then $x^* \wedge x = 0 = x^* \wedge x^{**}$. Also $x^* \vee x, x^* \vee x^{**} \in D(L)$. Hence $x^* \vee x = x^* \vee x^{**}$. By the cancelation property of L, we get $x = x^{**}$. Therefore every element of L is closed.
(2) $\Rightarrow$ (3): Let I be a principal ideal of L. Then $I = (x]$ for some $x \in L$. Then by condition (2), $x = x^{**}$. Now, $(x] = (x^{**}] = \delta([x^*))$. Therefore $(x]$ is a δ-ideal.
(3) $\Rightarrow$ (4): Let I be a proper ideal of L. Let $x \in I$. Then $(x] = \delta(F)$ for some filter F of L. Hence we get $x^{***} = x^* \in F$. Therefore $x^{**} \in \delta(F) = (x] \subseteq I$.
(4) $\Rightarrow$ (5): Let I be a proper ideal of L. Suppose $x \in I \cap D(L)$. Then $x^{**} \in I$ and $x^* = 0$. Therefore $1 = 0^* = x^{**} \in I$, which is a contradiction.
(5) $\Rightarrow$ (6): It is clear.
(6) $\Rightarrow$ (7): Let P be a prime ideal of L such that $P \cap D(L) = \emptyset$. Let $x \in P$. Then clearly $x \wedge x^* = 0$ and $x \vee x^* \in D(L)$. Hence $x \vee x^* \notin P$. Thus $x^* \notin P$. Therefore P is a minimal prime ideal of L.
(7) $\Rightarrow$ (8): Let P be a minimal prime ideal of L. Then clearly $L - P$ is a filter of L. Let $x \in P$. Since P is minimal, there exists $y \notin P$ such that $x \wedge y = 0$. Hence $x^* \wedge y = y$, which implies that $x^* \notin P$. Thus $x^* \in L - P$, which yields $x \in \delta(L - P)$. Conversely, let $x \in \delta(L - P)$. Then we get $x^* \notin P$. Hence we have $x \in P$. Thus

$P = \delta(L - P)$ and therefore P is a δ-ideal of L.
(8) $\Rightarrow$ (9): Assume that every prime ideal of L is a δ-ideal. Let $x, y \in L$ be such that $x^* = y^*$. Suppose $x \neq y$. Then there exists a prime ideal P of L such that $x \in P$ and $y \notin P$. By hypothesis, P is a δ-ideal of L. Hence $P = \delta(F)$ for some filter F of L. Since $x \in P = \delta(F)$, we get $y^* = x^* \in F$. Hence $y \in \delta(F) = P$, which is a contradiction. Therefore $x = y$.
(9) $\Rightarrow$ (10): Suppose a, b be two elements of $D(L)$. Then $a^* = 0 = b^*$. Hence $a = b$. Therefore $D(L)$ is a singleton set.
(10) $\Rightarrow$ (1): Assume that $D(L) = \{d\}$ is a singleton set. Let $x \in L$. We have always $x \vee x^* \in D(L)$. Therefore $x \wedge x^* = 0$ and $x \vee x^* = d$. This true for all $x \in L$. Also $0 \leq x \leq x \vee x^* = d$. Therefore L is a bounded distributive lattice in which every element is complemented. $\square$

We now prove that the homomorphic image of a δ-ideal is again a δ-ideal. By a homomorphism [6] on a bounded lattice, we mean a homomorphism which preserves 0 and 1. We now start our observation with the following fact.

Unlike in rings, if an onto homomorphism of a distributive lattice L into another lattice L' such that $\ker f = \{x \in L \mid f(x) = 0\} = \{0\}$, then f need not be an isomorphism. For this, we consider two chains $L = \{0, a, 1\}$ and $L' = \{0', 1'\}$. Now, define a mapping $f \colon L \longrightarrow L'$ by $f(0) = 0'$ and $f(a) = f(1) = 1'$. Then clearly f is a homomorphism from L into L' and also f is onto. Also $\operatorname{Ker} f = \{0\}$. But f is not one-one. Hence f is not an isomorphism.

Lemma 2.16. *Let L and L' be two pseudo-complemented distributive lattices with pseudo-complementation $*$ and $f \colon L \longrightarrow L'$ an onto homomorphism. If $\operatorname{Ker} f = \{0\}$, then $f(x^*) = \{f(x)\}^*$ for all $x \in L$.*

Proof. We have always $f(x) \wedge f(x^*) = f(x \wedge x^*) = f(0) = 0$. Suppose $f(x) \wedge f(t) = 0$ for some $t \in L$. Then $f(x \wedge t) = 0$ and hence $x \wedge t \in \ker f = \{0\}$. Thus $x \wedge t = 0$. Hence $x^* \wedge t = t$, which yields $f(x^*) \wedge f(t) = f(x^* \wedge t) = f(t)$. Therefore $f(x^*)$ is the pseudo-complement of $f(x)$ in L'. $\square$

In the following, we prove that the image of a δ-ideal of L under the above homomorphism is again a δ-ideal.

Theorem 2.17. *Let L, L' be two pseudo-complemented distributive lattices with pseudo-complementation $*$ and $f \colon L \longrightarrow L'$ an onto homomorphism such that $\operatorname{Ker} f = \{0\}$. If I is a δ-ideal of L, then $f(I)$ is a δ-ideal of L'.*

Proof. Let I be a δ-ideal of L. Then $I = \delta(G)$ for some filter G of L. Since the homomorphism f preserves 1, we can get that $f(G)$ is a filter in L'. Now, it is enough to show that $f\{\delta(G)\} = \delta\{f(G)\}$. Let $a \in f\{\delta(G)\}$. Then $a = f(x)$ for some $x \in \delta(G)$. Hence $x^* \in G$. Now $f(x) \wedge f(x^*) = f(x \wedge x^*) = f(0) = 0$. Hence $\{f(x)\}^* \wedge f(x^*) = f(x^*) \in f(G)$. Thus $\{f(x)\}^* \in f(G)$. Therefore $a = f(x) \in \delta\{f(G)\}$. Therefore $f\{\delta(G)\} \subseteq \delta\{f(G)\}$. Conversely, let $y \in \delta\{f(G)\}$. Since f is on-to, there exists $x \in L$ such that $y = f(x)$. Then $\{f(x)\}^* \in f(G)$. Hence

$\{f(x)\}^* = f(a)$ for some $a \in G$. Now

$$\begin{aligned} f(x) \wedge \{f(x)\}^* = 0 &\Rightarrow f(x) \wedge f(a) = 0 \\ &\Rightarrow f(x \wedge a) = 0 \\ &\Rightarrow x \wedge a \in \operatorname{Ker} f = \{0\} \\ &\Rightarrow x^* \wedge a = a \in G \\ &\Rightarrow x^* \in G \\ &\Rightarrow x \in \delta(G) \\ &\Rightarrow y = f(x) \in f\{\delta(G)\}\,. \end{aligned}$$

Thus $\delta\{f(G)\} \subseteq f\{\delta(G)\}$. Therefore $\delta\{f(G)\} = f\{\delta(G)\}$. □

We now characterize δ-ideals in terms of congruences. For this, we consider a well known filter congruence introduced by T. P. Speed [7].

Theorem 2.18 ([7]). *For any filter F of L, define a relation $\theta(F)$ as follows:*

$$(a, b) \in \theta(F) \;\Leftrightarrow\; a \wedge f = b \wedge f \quad \text{for some} \quad f \in F\,.$$

Then $\theta(F)$ is a congruence relation on L.

Lemma 2.19. *Let L be a pseudo-complemented distributive lattice. Then for any ideal I of L, $F_I = \{x \in L \mid x^* \wedge a^* = 0$ for some $a \in I\}$ is a filter of L.*

Proof. Clearly $0^* \in F_I$. Let $x, y \in F_I$. Then $x^* \wedge a^* = 0$ and $y^* \wedge b^* = 0$ for some $a, b \in I$. Hence $x^{**} \wedge a^* = a^*$ and $y^{**} \wedge b^* = b^*$. Now $(x \wedge y)^{**} \wedge (a \vee b)^* = x^{**} \wedge y^{**} \wedge a^* \wedge b^* = a^* \wedge b^*$. Thus $(x \wedge y)^* \wedge (a \vee b)^* = 0$. Therefore $x \wedge y \in F_I$. Again, let $x \in F_I$ and $s \in L$. Then $x^* \wedge a^* = 0$ for some $a \in I$. Now $(x \vee s)^* \wedge a^* \leq x^* \wedge a^* = 0$. Thus $x \vee s \in F_I$. Therefore F_I is a filter of L. □

Theorem 2.20. *For any ideal I of a pseudo-complemented distributive lattice L, the following conditions are equivalent:*

(1) *I is a δ-ideal,*

(2) $I = \operatorname{Ker} \theta(F_I)$,

(3) *$I = \operatorname{Ker} \theta(F)$ for some filter F of L.*

Proof. (1) $\Rightarrow$ (2): Assume that I is a δ-ideal of L. Then $I = \delta(F)$ for some filter F of L. Let $x \in I$. Since $x^{**} \wedge x^* = 0$, we can get $x^* \in F_I$. Since $x \wedge x^* = 0$ and $x^* \in F_I$, we thus get $x \in \operatorname{Ker} \theta(F_I)$. Therefore $I \subseteq \operatorname{Ker} \theta(F_I)$. Conversely, let $x \in \operatorname{Ker} \theta(F_I)$. Then $(x, 0) \in \theta(F_I)$. Thus $x \wedge f = 0$ for some $f \in F_I$. Since $f \in F_I$, we get that $f^* \wedge a^* = 0$ for some $a \in I$. Hence $x \leq f^* \leq a^{**} \in \delta(F) = I$. Therefore $I = \operatorname{Ker} \theta(F_I)$.
(2) $\Rightarrow$ (3): It is clear.
(3) $\Rightarrow$)1): Assume that $I = \operatorname{Ker} \theta(F)$ for some filter F of L. Let $x \in I = \operatorname{Ker} \theta(F)$. Then $x \wedge f = 0$ for some $f \in F$. Hence $x^* \wedge f = f \in F$. Thus $x^* \in F$, which yields that $x \in \delta(F)$. Therefore $I \subseteq \delta(F)$. Conversely, let $x \in \delta(F)$. Then $x^* \in F$. Since $x \wedge x^* = 0$ and $x^* \in F$, we get $(x, 0) \in \theta(F)$. Thus $x \in \operatorname{Ker} \theta(F) = I$. Therefore I is a δ-ideal of L. □

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