ARCHIVUM MATHEMATICUM (BRNO)
Tomus 48 (2012), 139–148

PERIODIC SOLUTIONS FOR A CLASS OF FUNCTIONAL DIFFERENTIAL SYSTEM

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Abstract. In this paper, we study the existence of periodic solutions to a class of functional differential system. By using Schauder's fixed point theorem, we show that the system has aperiodic solution under given conditions. Finally, four examples are given to demonstrate the validity of our main results.

1. Introduction

In this article, we study the existence of ω-periodic solutions to the following functional differential system

$$x_i'(t) = a_i(t)g_i(x_i(t)) - f_i(t, x_1(t-\tau_1(t)), \dots, x_n(t-\tau_n(t))), \quad i = 1, 2, \dots, n, \tag{1.1}$$

where a_i, $\tau_i\colon R \to R$ are ω-periodic continuous functions and $a_i(t) > 0$ for any $t \in [0, \omega]$, $f_i(t, u_1, \dots, u_n)\colon R^{n+1} \to R$ is ω-periodic in t and $g_i\colon R \to R$.

When $n = 1$, the problem (1.1) reduces to the functional differential equation

$$x'(t) = a(t)g\big(x(t)\big) - h\big(t, x(t-\tau(t))\big) . \tag{1.2}$$

The existence of periodic solutions for the special cases of (1.2) have been considered extensively by many authors, because (1.2) includes many important models in mathematical biology, such as, Hematopoiesis models; Nicholson's blowflies models; models for blood cell production, see [2, 3, 4, 8, 9, 7] and the references therein. Recently, Wang [5] investigated existence, multiplicity and nonexistence of positive periodic solutions for the periodic differential equation

$$x'(t) = a(t)p\big(x(t)\big)x(t) - \lambda h(t)h\big(x(t-\tau(t))\big) . \tag{1.3}$$

His approach depended on fixed point theorem in a cone. An essential condition on the function p in [5] is that p is bounded above and below by positive constants on $[0, +\infty)$. Hence, the method in [5] is not necessarily suitable for functional differential equation with general nonlinear term p. For example, to our best knowledge, results about periodic solutions for the following functional differential equation

$$x'(t) = a(t)x^\alpha(t) - \lambda h(t)f\big(x(t-\tau(t))\big) \tag{1.4}$$

2010 *Mathematics Subject Classification*: primary 34K13.

Key words and phrases: functional differential equation, periodic solution, fixed point theorem.
Received September 15, 2011, revised January 2012. Editor O. Došlý.
DOI: 10.5817/AM2012-2-139

are few, here $\alpha \neq 0$ is a constant and $\lambda > 0$ is a positive real parameter.

In the paper, we obtain sufficient conditions for the existence of periodic solutions for the system (1.1) by using Schauder's fixed point theorem. Our results improve and generalize the corresponding results of [1, 6, 10].

2. Main results

The following well-known Schauder's fixed point theorem is crucial in our arguments.

Lemma 2.1. *Let X be a Banach space with $D \subset X$ closed and convex. Assume that $T\colon D \to D$ is a completely continuous map, then T has a fixed point in D.*

Put $C_\omega = \{u \in C(R,R) : u(t+\omega) = u(t), t \in R\}$ with the norm defined by $\|u\|_{C_\omega} = \max_{0 \le t \le \omega} |u(t)|$ and

$$E = \{x = (x_1(t), \dots, x_n(t)) : x_i \in C_\omega\}, \quad \|x\|_E = \sum_{i=1}^{n} \|x_i\|_{C_\omega}.$$

Then C_ω and E are Banach spaces.

Let p, $q \in C_\omega$ and consider the following two differential equations

$$x'(t) = -p(t)x(t) + q(t), \tag{2.1}$$

$$x'(t) = p(t)x(t) - q(t). \tag{2.2}$$

Lemma 2.2. *Assume that $\int_0^\omega p(t)dt \neq 0$, then (2.1) has a unique ω-periodic solution*

$$x(t) = \int_t^{t+\omega} \frac{\exp \int_t^s p(r)dr}{\exp \int_0^\omega p(r)dr - 1} q(s)\, ds$$

and (2.2) has a unique ω-periodic solution

$$x(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} p(r)\, dr}{\exp \int_0^\omega p(r)dr - 1} q(s)\, ds.$$

Let $M \in R$, $m \in R$: $M > m$ and define

$$\prec_{[m,M]} = \{i\colon g_i(m) \le g_i(M), 1 \le i \le n\},$$

$$\succ_{[m,M]} = \{i\colon g_i(m) > g_i(M), 1 \le i \le n\}.$$

By using Schauder's fixed point theorem, we obtain the following existence result on the periodic solution for (1.1).

Theorem 2.1. *Assume that there exist constants $M_i > m_i$, $i = 1, 2, \dots, n$ such that $g_i \in C^1([m_i, M_i], R)$, $f_i \in C(R \times \Lambda, R)$, here $\Lambda = [m_1, M_1] \times \cdots \times [m_n, M_n]$, and for any $u_i \in [m_i, M_i]$ and $t \in [0, \omega]$,*

$$g_i(M_i) \le \frac{f_i(t, u_1, \dots, u_n)}{a_i(t)} \le g_i(m_i) \quad \text{if} \quad i \in \succ_{[m_i, M_i]} \tag{2.3}$$

and

$$g_i(m_i) \le \frac{f_i(t, u_1, \ldots, u_n)}{a_i(t)} \le g_i(M_i) \quad \text{if} \quad i \in \prec_{[m_i,M_i]} \;. \tag{2.4}$$

Then (1.1) *has at least one periodic solution* $(x_1^*(t), \ldots, x_n^*(t)) \in E$ *with* $m_i \le x_i^* \le M_i (1 \le i \le n)$.

Proof. Without loss of the generality, we assume that there exists a $k\colon 0 \le k \le n$ such that

$$i \in \succ_{[m_i,M_i]} \quad \text{for} \quad 1 \le i \le k\,, \qquad i \in \prec_{[m_i,M_i]} \quad \text{for} \quad k+1 \le i \le n\,,$$

here if $i \le 0$, $\succ_{[m_i,M_i]} = \phi$, if $i \ge n+1$, $\prec_{[m_i,M_i]} = \phi$.

Since $g_i \in C^1([m_i, M_i], R)$, there exist $l_i > 0$ such that

$$1 + \frac{1}{l_i} g_i'(u) > 0\,, \quad u \in [m_i, M_i],\ i = 1, 2, \ldots, k\,, \tag{2.5}$$

$$1 - \frac{1}{l_i} g_i'(u) > 0\,, \quad u \in [m_i, M_i],\ i = 1+k, \ldots, n\,. \tag{2.6}$$

Assume that $(x_1(t), \ldots, x_n(t)) \in E$ is a solution of (1.1), then

$$x_i'(t) = -l_i a_i(t) x_i(t) + a_i(t) \Big[g_i(x_i(t)) + l_i x_i(t) - \frac{f_i(t, X(t - \tau(t)))}{a_i(t)} \Big]\,,\ i = 1, 2, \ldots, k\,,$$

$$x_i'(t) = l_i a_i(t) x_i(t) - a_i(t) \Big[l_i x_i(t) - g_i(x_i(t)) + \frac{f_i(t, X(t - \tau(t)))}{a_i(t)} \Big]\,,\ i = k+1, \ldots, n$$

and

$$x_i(t) = \int_t^{t+\omega} \frac{a_i(s) \exp \int_t^s l_i a_i(r)\, dr}{\exp \int_0^\omega l_i a_i(r)\, dr - 1} \Big[g_i\big(x_i(s)\big) + l_i x_i(s) - \frac{f_i(s, X(s - \tau(s)))}{a_i(s)} \Big]\, ds\,,$$
$$i = 1, 2, \ldots, k\,,$$

$$x_i(t) = \int_t^{t+\omega} \frac{a_i(s) \exp \int_s^{t+\omega} l_i a_i(r)\, dr}{\exp \int_0^\omega l_i a_i(r)\, dr - 1} \Big[l_i x_i(s) - g_i\big(x_i(s)\big) + \frac{f_i(s, X(s - \tau(s)))}{a_i(s)} \Big]\, ds\,,$$
$$i = k+1, \ldots, n\,,$$

where $f_i(t, X(t - \tau(t)) = f_i(t, x_1(t - \tau_1(t)), \ldots, x_n(t - \tau_n(t)))$.

Define a set Ω in E and an operator $T\colon E \to E$ by

$$\Omega = \{x \in E : m_i \le x_i \le M_i, i = 1, 2 \ldots, n\}\,,$$

$$(Tx)(t) = \big((Tx_1)(t), (Tx_2)(t), \ldots, (Tx_n)(t)\big)\,, \quad x = \big(x_1(t), \ldots, x_n(t)\big) \in E\,,$$

where

$$(Tx_i)(t) := \int_t^{t+\omega} \frac{a_i(s)\exp\int_t^s l_i a_i(r)\,dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1} \times \Big[g_i(x_i(s)) + l_i x_i(s) - \frac{f_i(s, X(s-\tau(s)))}{a_i(s)}\Big]\,ds\,,\ 1 \le i \le k\,,$$

$$(Tx_i)(t) := \int_t^{t+\omega} \frac{a_i(s)\exp\int_s^{t+\omega} l_i a_i(r)\,dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1} \times \Big[l_i x_i(s) - g_i(x_i(s)) + \frac{f_i(s, X(s-\tau(s)))}{a_i(s)}\Big]\,ds\,,\ k+1 \le i \le n\,.$$

First, we show that $T(\Omega) \subset \Omega$. Using (2.5) and (2.6), we obtain that for $x \in \Omega$,

$$m_i + \frac{1}{l_i} g_i(m_i) \le x_i(t) + \frac{1}{l_i} g_i(x_i(t)) \le M_i + \frac{1}{l_i} g_i(M_i)\,, \quad i = 1, 2, \ldots, k\,,$$

$$m_i - \frac{1}{l_i} g_i(m_i) \le x_i(t) - \frac{1}{l_i} g_i(x_i(t)) \le M_i - \frac{1}{l_i} g_i(M_i)\,, \quad i = k+1, \ldots, n\,.$$

Using (2.3) and (2.4), we have

$$\begin{aligned}(Tx_i)(t) &= \int_t^{t+\omega} \frac{l_i a_i(s)\exp\int_t^s l_i a_i(r)\,dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1} \Big[\frac{1}{l_i} g_i\big(x_i(s)\big) + x_i(s) - \frac{f_i(s, X(s-\tau(s)))}{l_i a_i(s)}\Big]\,ds \\ &\in \Big[m_i \int_t^{t+\omega} \frac{l_i a_i(s)\exp\int_t^s l_i a_i(r)\,dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1}\,ds, M_i \int_t^{t+\omega} \frac{l_i a_i(s)\exp\int_t^s l_i a_i(r)\,dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1}\,ds\Big] \\ &= [m_i, M_i]\,, \quad i = 1, 2, \ldots, k\,,\end{aligned}$$

$$\begin{aligned}(Tx_i)(t) &= \int_t^{t+\omega} \frac{a_i(s)\exp\int_s^{t+\omega} l_i a_i(r)dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1} \Big[l_i x_i(s) - g_i(x_i(s)) + \frac{f_i(s, X(s-\tau(s)))}{a_i(s)}\Big]\,ds \\ &\in \Big[m_i \int_t^{t+\omega} \frac{l_i a_i(s)\exp\int_s^{t+\omega} l_i a_i(r)\,dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1}\,ds, M_i \int_t^{t+\omega} \frac{l_i a_i(s)\exp\int_s^{t+\omega} l_i a_i(r)\,dr}{\exp\int_0^\omega l_i a_i(r)\,dr - 1}\Big] \\ &= [m_i, M_i]\,, \quad i = k+1, k+2, \ldots, n\,.\end{aligned}$$

Next, we show that $T\colon \Omega \to \Omega$ is completely continuous. Obviously, $T(\Omega)$ is a uniformly bounded set and T is continuous on Ω, so it suffices to show $T(\Omega)$ is equi-continuous by Ascoli-Arzela theorem. For any $x \in \Omega$, we have

$$(Tx_i)'(t) = -l_i a_i(t)(Tx_i)(t) + a_i(t)\Big[g_i(x_i(t)) + l_i x_i(t) - \frac{f_i(t, X(t-\tau(t)))}{a_i(t)}\Big]\,, \quad i = 1, 2, \ldots, k\,,$$

$$(Tx_i)'(t) = l_i a_i(t)(Tx_i)(t) - a_i(t)\Big[l_i x_i(t) - g_i(x_i(t)) + \frac{f_i(t, X(t-\tau(t)))}{a_i(t)}\Big]\,, \quad i = k+1, \ldots, n\,.$$

Since $T(\Omega)$ is bounded and f_i, g_i, a_i are continuous, there exists $\rho > 0$ such that

$$|(Tx_i)'(t)| \le \rho\,, \quad x \in \Omega\,, \quad i = 1, 2, \ldots, n\,,$$

which implies that $T(\Omega)$ is equi-continuous. So T is a completely continuous operator on Ω. Clearly, Ω is a close and convex set in E. Therefore, T has a fixed point $x^* \in \Omega$ by Lemma 2.1. Furthermore, $m_i \le x_i^*(t) \le M_i$, which means $(x_1^*(t), \ldots, x_n^*(t)) \in E$ is a ω-periodic solution of (1.1). The proof is complete. □

Remark 2.1. Assume that all conditions of Theorem 2.1 are satisfies. Further suppose that there exist $1 \le i_0 \le n$ and $t_0 \in [0, \omega]$ such that any $u_i \in [m_i, M_i]$,

$$\frac{f_{i_0}(t_0, u_1, \ldots, u_n)}{a_{i_0}(t_0)} < g_{i_0}(m_{i_0}) \quad \text{if} \quad i_0 \in \succ_{[m_{i_0}, M_{i_0}]} \tag{2.7}$$

and

$$\frac{f_{i_0}(t_0, u_1, \ldots, u_n)}{a_{i_0}(t_0)} > g_{i_0}(m_{i_0}) \quad \text{if} \quad i_0 \in \prec_{[m_{i_0}, M_{i_0}]} \ . \tag{2.8}$$

Then $x_{i_0}^* > m_{i_0}$ for any $t \in [0, \omega]$.

Proof. Assume that there is a $t^* \in [0, \omega]$ such that $x_{i_0}^*(t^*) = m_{i_0}$. Then

$$\begin{aligned} m_{i_0} = \int_{t^*}^{t^*+\omega} & \frac{a_{i_0}(s) \exp \int_{t^*}^{s} l_{i_0} a_{i_0}(r)\, dr}{\exp \int_0^\omega l_{i_0} a_{i_0}(r)\, dr - 1} \\ & \times \Big[g_{i_0}(x_{i_0}^*(s)) + l_{i_0} x_{i_0}^*(s) - \frac{f_{i_0}(s, X^*(s - \tau(s)))}{a_{i_0}(s)} \Big]\, ds, \quad i_0 \le k\,, \end{aligned}$$

or

$$\begin{aligned} m_{i_0} = \int_{t^*}^{t^*+\omega} & \frac{a_{i_0}(s) \exp \int_{s}^{t^*+\omega} l_{i_0} a_{i_0}(r)\, dr}{\exp \int_0^\omega l_{i_0} a_{i_0}(r)\, dr - 1} \\ & \times \Big[l_{i_0} x_{i_0}^*(s) - g_{i_0}(x_{i_0}^*(s)) + \frac{f_{i_0}(s, X^*(s - \tau(s)))}{a_{i_0}(s)} \Big]\, ds, \quad i_0 > k\,, \end{aligned}$$

where $f_i(t, X^*(t - \tau(t)) = f_i(t, x_1^*(t - \tau_1(t)), \ldots, x_n^*(t - \tau_n(t)))$.

On the other hand, since for $s \in [0, \omega]$,

$$\frac{g_{i_0}(x_{i_0}^*(s))}{l_{i_0}} + x_{i_0}^*(s) - \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \ge 0 \quad \text{for} \quad i_0 \le k\,,$$

$$x_{i_0}^*(s) - \frac{g_{i_0}(x_{i_0}^*(s))}{l_{i_0}} + \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \ge 0 \quad \text{for} \quad i_0 > k\,,$$

one can obtain that for any $s \in [0, \omega]$,

$$\frac{g_{i_0}(x_{i_0}^*(s))}{l_{i_0}} + x_{i_0}^*(s) - \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \equiv 0 \quad \text{for} \quad i_0 \le k$$

$$x_{i_0}^*(s) - \frac{g_{i_0}(x_{i_0}^*(s))}{l_{i_0}} + \frac{f_{i_0}(s, X^*(s - \tau(s)))}{l_{i_0} a_{i_0}(s)} - m_{i_0} \equiv 0 \quad \text{for} \quad i_0 > k\,,$$

which is a contradiction since

$$0 \geq \frac{g_{i_0}(m_{i_0})}{l_{i_0}} - \frac{f_{i_0}(t_0, X^*(t_0 - \tau(t_0)))}{l_{i_0} a_{i_0}(t_0)} > 0 \quad \text{for} \quad i_0 \leq k\,,$$

$$0 \geq -\frac{g_{i_0}(m_{i_0})}{l_{i_0}} + \frac{f_{i_0}(t_0, X^*(t_0 - \tau(t_0)))}{l_{i_0} a_{i_0}(t_0)} > 0 \quad \text{for} \quad i_0 > k\,.$$

□

Remark 2.2. Assume that all conditions of Theorem 2.1 are satisfies. Further suppose that there exist $1 \leq r_0 \leq n$ and $t_1 \in [0, \omega]$ such that any $u_i \in [m_i, M_i]$,

$$\frac{f_{r_0}(t_1, u_1, \ldots, u_n)}{a_{r_0}(t_1)} > g_{r_0}(M_{r_0}) \quad \text{if} \quad r_0 \in \succ_{[m_{r_0}, M_{r_0}]} \tag{2.9}$$

and

$$\frac{f_{r_0}(t_1, u_1, \ldots, u_n)}{a_{r_0}(t_1)} < g_{r_0}(M_{r_0}) \quad \text{if} \quad r_0 \in \prec_{[m_{r_0}, M_{r_0}]}\,. \tag{2.10}$$

Then $x^*_{r_0} < M_{r_0}$ for any $t \in [0, \omega]$.

Consider the equations

$$x'(t) = -a(t)x(t) + f\big(t, x(t - \tau(t))\big)\,; \tag{2.11}$$

$$x'(t) = a(t)x(t) - f\big(t, x(t - \tau(t))\big) \tag{2.12}$$

where f is ω-periodic in t, a, τ are ω-periodic continuous functions and $a(t) > 0$ for all $t \in R$.

Corollary 2.1. *Assume that there exist constants $M > m$ such that $f \in C(R \times [m, M], R)$ and for any $u \in [m, M]$ and $t \in [0, \omega]$*

$$ma(t) \leq f(t, u) \leq Ma(t)\,.$$

Then (2.11) *(or* (2.12)*) has at least one periodic solution $m \leq x \leq M$.*

Next, we consider the existence of a positive ω-periodic solution for problem (1.4). We give explicit intervals of λ such that (1.4) has at least one positive ω-periodic solution.

In the following, we assume that a, h, $\tau\colon R \to R$ are ω-periodic continuous functions and $a(t) > 0, h(t) > 0$ for any $t \in [0, \omega]$. $f\colon (0, +\infty) \to (0, +\infty)$ is continuous.

Put

$$\overline{f}_0 = \limsup_{t \to 0^+} \frac{f(t)}{t^\alpha}\,, \quad \underline{f}_0 = \liminf_{t \to 0^+} \frac{f(t)}{t^\alpha}\,, \quad \overline{f}_\infty = \limsup_{t \to +\infty} \frac{f(t)}{t^\alpha}\,, \quad \underline{f}_\infty = \liminf_{t \to +\infty} \frac{f(t)}{t^\alpha}\,,$$

$$\delta^* = \max_{t \in [0, \omega]} \frac{h(t)}{a(t)}\,, \quad \delta = \min_{t \in [0, \omega]} \frac{h(t)}{a(t)}\,.$$

Theorem 2.2. *The problem* (1.4) *has at least one positive periodic solution if one of the following conditions holds:*

(H_1) $\alpha < 0$, $\liminf_{t\to 0^+} f(t) > 0$, $\limsup_{t\to+\infty} f(t) < +\infty$ *and* $(\underline{f}_\infty \delta)^{-1} < \lambda < (\overline{f}_0 \delta^*)^{-1}$;

(H_2) $\alpha > 0$, $\limsup_{t\to 0^+} f(t) < +\infty$, $\liminf_{t\to+\infty} f(t) > 0$ *and* $(\underline{f}_0 \delta)^{-1} < \lambda < (\overline{f}_\infty \delta^*)^{-1}$.

Proof. Assume that (H_1) holds. From the definition of $\overline{f}_0$, $\underline{f}_\infty$ and (H_1), there exist $r_1 > 0$ and $\bar{r}_1 > r_1$ such that

$$\frac{\lambda h(t) f(u)}{a(t)} \le u^\alpha\,, \qquad 0 < u \le r_1\,, \qquad \inf_{u\in(0,r_1]} f(u) > 0\,,$$
$$\frac{\lambda h(t) f(u)}{a(t)} \ge u^\alpha\,, \qquad u \ge \bar{r}_1\,, \qquad \sup_{u\in[\bar{r}_1,+\infty)} f(u) < +\infty\,.$$

Let $\lambda \in \big(\frac{1}{\underline{f}_\infty \delta}, \frac{1}{\overline{f}_0 \delta^*}\big)$. It is easy to check that

$$\inf\Big\{\frac{\lambda h(t) f(u)}{a(t)} : t \in [0,\omega], u \in (0, \bar{r}_1]\Big\} := \mu_1 > 0\,,$$

$$\sup\Big\{\frac{\lambda h(t) f(u)}{a(t)} : t \in [0,\omega], u \in [r_1, +\infty)\Big\} := \bar{\mu}_1 < +\infty\,.$$

Put

$$m = \min\Big\{\frac{r_1}{2}, \bar{\mu}_1^{\frac{1}{\alpha}}\Big\}\,, \qquad M = \max\Big\{2\bar{r}_1, \mu_1^{\frac{1}{\alpha}}\Big\}\,,$$

then

$$M^\alpha \le \mu_1 \le \frac{\lambda h(t) f(u)}{a(t)} \le x^\alpha \le m^\alpha\,, \quad m \le u \le r_1\,,$$
$$M^\alpha \le x^\alpha \le \frac{\lambda h(t) f(u)}{a(t)} \le \bar{\mu}_1 \le m^\alpha\,, \quad \bar{r}_1 \le u \le M\,.$$

On the other hand,

$$M^\alpha \le \mu_1 \le \frac{\lambda h(t) f(u)}{a(t)} \le \bar{\mu}_1 \le m^\alpha\,, \quad r_1 \le u \le \bar{r}_1\,.$$

Hence,

$$M^\alpha \le \frac{\lambda h(t) f(u)}{a(t)} \le m^\alpha\,,\ m \le u \le M\,.$$

By Theorem 2.1, (1.4) has at least one periodic solution $x \in C_\omega\colon 0 < m \le x \le M$.

Assume that (H_2) holds. There exist $0 < r_3 < 1$ and $\bar{r}_3 > 1$ such that

$$\frac{\lambda h(t) f(u)}{a(t)} \ge u^\alpha\,, \quad 0 < u \le r_3\,, \qquad \sup_{u\in(0,r_3]} f(u) < +\infty\,,$$

$$\frac{\lambda h(t) f(u)}{a(t)} \le u^\alpha\,, \quad u \ge \bar{r}_3\,, \qquad \inf_{u\in[\bar{r}_3,+\infty)} f(u) > 0\,.$$

Let $\lambda \in \left(\frac{1}{\underline{f}_0\delta}, \frac{1}{\overline{f}_\infty\delta^*}\right)$, then

$$\inf\left\{\frac{\lambda h(t)f(u)}{a(t)} : t \in [0,\omega], u \in [r_3,+\infty)\right\} := \mu_3 > 0\,,$$
$$\sup\left\{\frac{\lambda h(t)f(u)}{a(t)} : t \in [0,\omega], u \in (0,\bar{r}_3]\right\} := \bar{\mu}_3 < +\infty\,.$$

Put

$$m = \min\left\{\frac{r_3}{2}, \mu_3^{\frac{1}{\alpha}}\right\}, \qquad M = \max\left\{2\bar{r}_3, \bar{\mu}_3^{\frac{1}{\alpha}}\right\},$$

then

$$m^\alpha \le \frac{\lambda h(t)f(u)}{a(t)} \le M^\alpha\,, \quad m \le u \le M\,.$$

By Theorem 2.1, (1.4) has at least one periodic solution $x \in C_\omega\colon 0 < m \le x \le M$. The proof is complete. □

Corollary 2.2.

(1) *Assume that* $\alpha < 0$ *and* $0 < \liminf_{t\to 0^+} f(t) \le \limsup_{t\to 0^+} f(t) < +\infty$, *then* (1.4) *has at least one positive periodic solution for sufficiently large* $\lambda > 0$.
(2) *Assume that* $\alpha < 0$ *and* $0 < \liminf_{t\to +\infty} f(t) \le \limsup_{t\to +\infty} f(t) < +\infty$, *then* (1.4) *has at least one positive periodic solution for sufficiently small* $\lambda > 0$.
(3) *Assume that* $\alpha > 0$ *and* $0 < \liminf_{t\to 0^+} f(t) \le \limsup_{t\to 0^+} f(t) < +\infty$, *then* (1.4) *has at least one positive periodic solution for sufficiently small* $\lambda > 0$.
(4) *Assume that* $\alpha > 0$ *and* $0 < \liminf_{t\to +\infty} f(t) \le \limsup_{t\to +\infty} f(t) < +\infty$, *then* (1.4) *has at least one positive periodic solution for sufficiently large* $\lambda > 0$.

Proof. Here we only prove case (1). Since $0 < \liminf_{t\to 0^+} f(t) \le \limsup_{t\to 0^+} f(t) < +\infty$, there exists $0 < r < 1$ such that

$$\mu := \inf_{t\in(0,r]} f(t) \le \sup_{t\in(0,r]} f(t) := \nu < +\infty\,.$$

Let $\lambda > 0$ such that $(\lambda\delta\mu)^{\frac{1}{\alpha}} < r$ and set

$$m = (\lambda\bar{\delta}\nu)^{\frac{1}{\alpha}}\,, \qquad M = (\lambda\delta\mu)^{\frac{1}{\alpha}}\,,$$

then $r > M > m > 0$ and

$$M^\alpha \le \frac{\lambda h(t)f(u)}{a(t)} \le m^\alpha\,, \qquad m \le u \le M\,.$$

By Theorem 2.1, (1.4) has at least one periodic solution $x \in C_\omega\colon 0 < m \le x \le M$. The proof is complete. □

3. Some examples

In this section, we apply the main results obtained in previous section to several examples.

Example 3.1. Consider the differential equation

$$x'(t) = \frac{1}{\sqrt[3]{\sin x(t)}} + b(t)\,, \tag{3.1}$$

where $b(t)$ is a ω-periodic continuous function.

It is easy to verify form Theorem 2.1 that (3.1) has least two periodic solutions $0 < |x_1| < 0.5\pi < |x_2| < \pi$ if $|b(t)| > 1$ for all $t \in R$. Since $x_i + 2k\pi (i = 1, 2, k \in Z)$ is also the periodic solutions of (3.1), (3.1) has infinitely many periodic solutions when $|b(t)| > 1$.

Example 3.2. Consider the differential equation

$$x'(t) = \Big(1 + \frac{\sin t}{100}\Big)x^3(t) - f\big(x(t - \cos t)\big)\,, \tag{3.2}$$

where

$$f(u) = \begin{cases} 0.1\,, & u < \frac{2}{3}\,, \\ u^2 - u + \frac{5}{4}\,, & u > 1\,. \end{cases}$$

In (3.2), $a(t) = 1 + 0.01 \sin t$ and $g(x) = x^3$. Put $m_1 = 0.1$, $M_1 = 0.6$, $m_2 = 1.1$, $M_2 = 2$, then

$$g(m_i) \le \frac{f(u)}{a(t)} \le g(M_i)\,, \qquad \forall u \in [m_i, M_i]\,,\ t \in R\,,\ i = 1, 2\,.$$

By Theorem 2.1, (3.2) has two positive 2π-periodic solutions x_1, x_2 such that $m_1 \le x_1 \le M_1, m_2 \le x_2 \le M_2$.

Example 3.3. Consider the differential equation

$$x'(t) = x^3(t) + \frac{1}{x(t)} - \lambda\Big(1 + \frac{\sin t}{2}\Big)\big(2 - \sin x(t - \cos t)\big) \tag{3.3}$$

where $\lambda > 0$ is a positive real parameter.

In (3.3), $a(t) = 1$, $g(x) = x^3 + x^{-1}$ and $f(t, u) = \lambda\big(1 + \frac{\sin t}{2}\big)(2 - \sin u)$. Put

$$m_1 = \frac{2}{9\lambda}\,, \quad M_1 = 1\,, \quad m_2 = 1\,, \quad M_2 = \sqrt[3]{\frac{9\lambda}{2}}\,,$$

then for sufficiently large $\lambda > 0$,

$$g(M_1) \le f(t, u) \le g(m_1)\,, \qquad u \in [m_1, M_1]\,,\ t \in R\,,$$

$$g(m_2) \le f(t, u) \le g(M_2)\,, \qquad u \in [m_2, M_2]\,,\ t \in R\,.$$

By Theorem 2.1, (3.3) has two positive 2π-periodic solutions $x_1 \in [m_1, M_1]$, $x_2 \in [m_2, M_2]$ for sufficiently large $\lambda > 0$.

Example 3.4. Consider the differential system

$$\begin{cases} x'(t) = (2-\cos t)x(t) - y^2(t)\,, \\ y'(t) = -2\sin y(t) + \exp(0.5x(t) - y(t))\,. \end{cases} \tag{3.4}$$

Put $D = [0.01, 0.29] \times [0.2, 0.53]$, then for $(u_1, u_2) \in D$ and $t \in [0, 2\pi]$,

$$0.01 \le \frac{u_2^2}{2-\cos t} \le 0.29\,, \quad 2\sin 0.2 \le e^{0.5u_1 - u_2} \le 2\sin 0.53\,.$$

By Theorem 2.1, (3.4) has a 2π-periodic solution $(x(t), y(t))$ such that $0.01 \le x(t) \le 0.29$ and $0.2 \le y(t) \le 0.53$.

Acknowledgement. The authors would like to thank the referees for the comments which help to improve the paper. The work is supported by Hunan Provincial Natural Science Foundation of China.

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