ARCHIVUM MATHEMATICUM (BRNO)
Tomus 48 (2012), 107–112

ON THE SUBGROUPS OF COMPLETELY DECOMPOSABLE TORSION-FREE GROUPS THAT ARE IDEALS IN EVERY RING

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Abstract. In this paper we consider completely decomposable torsion-free groups and we determine the subgroups which are ideals in every ring over such groups.

1. Introduction

All groups considered here are abelian, with addition as the group operation. Given an abelian group A, we call R a ring over A if the group A is isomorphic to the additive group of R. In this situation we write $R = (A, *)$, where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative. Every group may be turned into a ring in a trivial way, by setting all products equal to zero; such a ring is called a zero-ring. If this is the only multiplication over A, then A is said to be a nil group. Fried [2] gives a criterion for the subgroups of an abelian group that are ideals in every ring. In [4], the authors use the type set of a rank one or an indecomposable rank two abelian group A, to give necessary and sufficient conditions for the subgroups of A which are ideals in every ring over A. In this paper, we discuss about the subgroups of completely decomposable torsion-free groups which are ideals in every ring.

2. Notations and preliminaries

Let A be a torsion-free abelian group. Given a prime p, the p-height of $x \in A$, denoted by $h_p^A(x)$, is the largest integer k such that p^k divides x in A; if no such maximal integer exists, we set $h_p^A(x) = \infty$. Now let $p_1, p_2, \dots$ be an increasing sequence of all primes. Then the sequence

$$\chi_A(x) = (h_{p_1}^A(x), h_{p_2}^A(x), \dots, h_{p_n}^A(x), \dots)\,,$$

is said to be the height-sequence of x. We omit the subscript A if no ambiguity arises. For any two height-sequences $\chi = (k_1, k_2, \dots, k_n, \dots)$ and $\mu = (l_1, l_2, \dots, l_n, \dots)$ we set $\chi \geq \mu$ if $k_n \geq l_n$ for all n. Moreover, χ and μ will be considered equivalent if

2010 *Mathematics Subject Classification*: primary 20K15.
Key words and phrases: completely decomposable group, type, ring.
Received October 10, 2011. Editor J. Trlifaj.
DOI: 10.5817/AM2012-2-107

$\sum_n |k_n - l_n|$ is finite [we set $\infty - \infty = 0$]. An equivalence class of height-sequences is called a type. If $\chi(x)$ belongs to the type $\mathbf{t}$, then we say that x is of type $\mathbf{t}$. For two types t_1, t_2 we have $t_1 \le t_2$ if there exists $\chi \in t_1$ and $\mu \in t_2$ such that $\chi \le \mu$. The type set of A is the partially ordered set of types, i.e.,

$$\mathrm{T(A) = \{t(x) \mid x \in A \setminus 0\}}\,.$$

A torsion-free group A in which all non-zero elements are of the same type $\mathbf{t}$ is called homogeneous of type $\mathbf{t}$, or $\mathbf{t}$-homogeneous. For example every rank one group A is homogeneous. We use the symbol $t(A)$ for the type set of a rank one group A, which is indeed the type of any non-zero element of A. For two types $t_1 = (l_1, l_2, \dots)$ and $t_2 = (k_1, k_2, \dots)$ we set

$$t_1 \cap t_2 = (\mathrm{min\{l_1, k_1\}, min\{l_2, k_2\}}, \dots)$$

and

$$t_1 t_2 = (l_1 + k_1, l_2 + k_2, \dots)\,.$$

If C is a subgroup of A and S a subset of C, then

$$\langle S\rangle_*^C = \{a \in C \mid na \in \langle S\rangle\,;\ \text{for some}\ \ 0 \neq n \in \mathbb{Z}\}$$

is the pure subgroup of C generated by S. Moreover, we set $\langle S\rangle_* = \langle S\rangle_*^A$. A torsion-free group A is completely decomposable if A is the direct sum of rank one groups. A proper subgroup C of A is called strongly nil if for any ring $(A, \cdot)$ over A we have $a \cdot c = c \cdot a = 0$ for all $a \in A$ and for all $c \in C$. If C is not strongly nil then it said to be a strongly non-nil subgroup of A. If $A = \oplus_{i\in I} A_i$ is a completely decomposable group and $S = \{x_i \mid x_i \in A_i\}_{i\in I}$ is a maximal independent set of A, then for any subgroup C of A we define $U_i^C := \{\beta_i \in \mathbb{Q} \mid \beta_i x_i \in C\}$ and $U_i := U_i^{A_i}$.

We end this section with the following proposition which is used later.

Proposition 2.1. *Let $A = \oplus_{i\in I} A_i$ be a completely decomposable group which supports a non-zero ring $R = (A, \cdot)$. Let $S = \{x_i \mid x_i \in A_i\}_{i\in I}$ be a maximal independent set of A and $U_i = \{u_i \in \mathbb{Q} \mid u_i x_i \in A_i\}$ for all $i \in I$. Then*

(1) *If there exist r, $s \in I$ such that $x_r \cdot x_s = \sum_{i\in I} \alpha_{r,s,i} x_i$ then $\alpha_{r,s,i} U_r U_s \subseteq U_i$ for all $i \in I$.*

(2) *In (1) if $\alpha_{r,s,t} \neq 0$ for some $t \in I$, then the multiplication $*$ defined as*

$$x_i * x_j = \begin{cases} \alpha_{r,s,t} x_t & (i,j) = (r,s)\,; \\ 0 & \textit{otherwise}, \end{cases}$$

yields a non-zero ring over A.

Proof. Straightforward. □

3. Completely decomposable homogeneous groups

Theorem 3.1. *Let $A = \oplus_{i=i}^n A_i$ be a completely decomposable and homogeneous group of rank n. If A is non-nil, then A contains no non-trivial subgroup of rank less than n which is an ideal in every ring on A.*

Proof. Let $t(A) = t$. Then $t^2 = t$ since A is non-nil. Now suppose that $x_i \in A_i$ and $\{x_1, x_2, \ldots, x_n\}$ be a maximal independent set of A. Hence $t(U_i) = t = t(U_iU_j)$ for all $i, j \in \{1, 2, \ldots, n\}$.

By the way of contradiction, suppose that C is a non-trivial subgroup of A with $r(C) = k \leq n - 1$ such that C is an ideal in every ring on A. Let $0 \neq c = \sum_{i=1}^{n} \alpha_i x_i$ be an element of C. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $\alpha_i \neq 0$. Without loss of generality suppose that $\alpha_1 \neq 0$. On the other hand $t(U_1^2) = t(U_1) = t(U_2) = \cdots = t(U_n)$, hence there exist some non-zero integer numbers $m_1, m_2, \ldots, m_n, k_1, k_2, \ldots, k_n$ such that:

$$m_1 U_1^2 = k_1 U_1, m_2 U_1^2 = k_2 U_2, \ldots, m_n U_1^2 = k_n U_n\,.$$

Now we define $*_1$ as follows

$$x_i *_1 x_j = \begin{cases} m_1 x_1 & \text{if} \quad i = j = 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $*_1$ yields a ring on A such that $c *_1 x_1 = m_1\alpha_1 x_1 \in C$. In fact if $u_1 = \beta_1 x_1 + \cdots + \beta_n x_n$, $u_2 = \gamma_1 x_1 + \cdots + \gamma_n x_n$ are two arbitrary elements of A, then $u_1 *_1 u_2 = m_1\beta_1\gamma_1 x_1$. But $m_1\beta_1\gamma_1 \in m_1 U_1^2 = k_1 U_1 \subseteq U_1$, hence $*_1$ is actually a ring on A. Similarly we may define multiplications $*_2, *_3, \ldots, *_n$ such that $(A, *_2), \ldots, (A, *_n)$ are rings on A and for all $l = 2, 3, \ldots, n$ we have

$$0 \neq c *_l x_1 = m_l \alpha_1 x_l \in C\,.$$

This implies that $r(C) = n$, a contradiction. $\square$

4. Completely decomposable groups of rank n whose typesets contains n maximal elements

Theorem 4.1. *Let $A = \oplus_{i=1}^{n} A_i$ be a completely decomposable group of rank n. Let $S = \{x_i \mid x_i \in A_i,\ i = 1, 2, \ldots, n\}$ be a maximal independent set of A such that $t(x_i) = t_i$ are maximal elements in $T(A)$ for all $i = 1, 2, \ldots, n$. Then*

(1) *Any rank one subgroup C which is an ideal in every ring on A is of the form $C = U_i^C(mx_i)$ with $t_i^2 = t_i$, $m \in \mathbb{Z} \setminus \{0\}$ or C is generated by a rational combination of some elements in S with non-idempotent types. Moreover, C in the first case is strongly non-nil and in the second case is strongly nil.*

(2) *Any subgroup C of rank $k < n$ which is an ideal in every ring on A is generated by $l(\leq k)$ rational multiples of some elements in S with idempotent types and $k - l$ combinations with rational coefficients of some elements in S with non-idempotent types. Moreover, if $l \neq 0$ then C is strongly non-nil.*

Proof. 1) Let C be any rank one subgroup of A which is an ideal in every ring on A and let $c = \sum_{i=1}^{n} \alpha_i x_i$ be a non-zero element of C. We consider two cases. First suppose that $\alpha_i \neq 0$, for some $i \in \{1, 2, \ldots, n\}$ with $t^2(x_i) = t(x_i)$. For example let

$\alpha_1 \neq 0$ and $t^2(x_1) = t(x_1)$. This implies $t(U_1^2) = t(U_1)$. Hence, as in the proof of Theorem 3.1, there exists a non-zero integer m such that

$$x_i * x_j = \begin{cases} mx_1 & \text{if} \quad i = j = 1\,, \\ 0 & \text{otherwise,} \end{cases}$$

is a ring on A. Clearly, $0 \neq c * x_1 = \alpha_1 m x_1 \in C$. Now if $\alpha_j \neq 0$ for some $j \neq 1$, then $r(C) \geq 2$, which is a contradiction. Consequently, $C = U_1^C(mx_1)$ for some $m \in \mathbb{Z} \setminus \{0\}$ and clearly C is strongly non-nil. In the second case let $c = \sum_{i=1}^n \beta_i x_i, t^2(x_i) \neq t(x_i), \beta_i \in \mathbb{Q}$. Now $t(x_i)$ is maximal and non-idempotent, hence any ring on A satisfies: $x_i x_i = x_i x_j = 0$ which yields C is strongly nil.

2) Let C be a rank k subgroup of A which is an ideal in every ring on A and let $\{c_1 = \alpha_{11}x_1 + \cdots + \alpha_{1n}x_n, c_2 = \alpha_{21}x_1 + \cdots + \alpha_{2n}x_n, \ldots, c_k = \alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n\}$ be a maximal independent set of C. If there exist $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, n\}$ such that $\alpha_{ij} \neq 0$ and $t^2(x_j) = t(x_j)$, then as in case (i) there exist a non-zero integer m and a ring on A with $0 \neq c_i x_j = \alpha_{ij} m x_j \in C$. Let $\alpha_{ij} m = \beta_j$, hence there exist $c'_2, \ldots, c'_k \in C$ such that $\{\beta_j x_j, c'_2, \ldots, c'_k\}$ is an independent set in C and for all $i = 1, 2, \ldots, k$,

$$c'_i = \alpha'_{i1}x_1 + \cdots + \alpha'_{ij-1}x_{j-1} + \alpha'_{ij+1}x_{j+1} + \cdots + \alpha'_{in}x_n\,.$$

Repeating this procedure we get a maximal independent set in C

$$\{\beta_{j_1}x_{j_1}, \ldots, \beta_{j_l}x_{j_l}, c''_1, \ldots, c''_{k-l}\}\,,$$

such that $t^2(x_{j_1}) = t(x_{j_1}), \ldots, t^2(x_{j_l}) = t(x_{j_l})$ and $c''_1, \ldots, c''_{k-l}$ are rational combinations of some elements in S with non-idempotent types. Now the Final claim is obvious. ☐

5. Completely decomposable group of rank n whose typesets contains less than n maximal elements

Theorem 5.1. *Let $A = \oplus_{i=1}^n A_i$ be a completely decomposable group of rank n such that $T(A)$ contains $k < n$ maximal elements. Suppose that $S = \{x_i \mid x_i \in A, i = 1, 2, \ldots, n\}$ be a maximal independent set of A. Then*

(1) *For any rank one subgroup C which is an ideal in every ring on A we have one of the following three cases:*

 (a) *$C = U_i^C(mx_i)$ for some non-zero integer m and $t(x_i)$ is idempotent. Moreover, such a subgroup is strongly non-nil.*

 (b) *$C = \langle \alpha x_i \rangle_*^C$ with $t^2(x_i) \neq t(x_i)$. Moreover, if C is strongly non-nil then there exists $(x_i \neq) x_k \in S$ such that $t(x_i)t(x_k) = t(x_i)$.*

 (c) *$C = \langle \sum_{j \in J \subseteq \{1,2,\ldots,n\}} \alpha_j x_j \rangle_*^C, |J| \geq 2$, such that every $t(x_j)$ is of non-idempotent type. Moreover, such a subgroup is strongly nil.*

(2) *Any rank $m(< n)$ subgroup C of A which is an ideal in every ring on A is $C = \langle H_1', H_2', H_3' \mid H_i' \subseteq H_i, i = 1,2,3\rangle_*^C$, in which:*

$$H_1 = \{c_i = \alpha_i x_i \mid \alpha_i \in \mathbb{Q},\ t(x_i) \text{ is maximal and idempotent}\},$$

$$H_2 = \{c_i' = \sum_{j\in J\subseteq\{1,2,\ldots,n\}} \alpha_{ij} x_j \mid \alpha_i \in \mathbb{Q},\ t(x_i) \text{ is not idempotent}\},$$

$$H_3 = \{c_k'' = \alpha_k x_k \mid \alpha_i \in \mathbb{Q},\ t^2(x_k) = t(x_k),\ t(x_k) \text{ is not maximal}\}.$$

Moreover, in this case if $c = \sum_{j\in J\subseteq\{1,2,\ldots,n\}} \alpha_j x_j \in H_2 \cap C$ and $\alpha_j x_j \neq 0$ with $t(x_j)$ maximal and $x_j x_k \neq 0$ for some $x_k \in S$, then $t(x_k)t(x_j) = t(x_j)$.

Proof. 1-a) Let C be any rank one subgroup of A which is an ideal in every ring over A and $c = \sum_{i=1}^n \alpha_i x_i \in C$. If $\alpha_i \neq 0, t^2(x_i) = t(x_i)$, then by the proof of Theorem 3.1, there exists a non-zero integer m such that

$$x_r * x_s = \begin{cases} m x_i & \text{if} \quad r = s = i\,, \\ 0 & \text{otherwise.} \end{cases}$$

yields a ring on A such that $c \cdot x_i = m\alpha_i x_i \in C$. Moreover, by $r(C) = 1$ we obtain $\alpha_j = 0$ for all $j \neq i$. Consequently, $C = U_i^C(m x_i)$ which clearly is strongly non-nil.

b, c) Suppose that C is strongly non-nil and any arbitrary element of C is of the form αc with $\alpha \in \mathbb{Q}$ and $c = \sum_{j\in J\subseteq\{1,2,\ldots,n\}} \alpha_j x_j$ such that $t(x_j)$ is not idempotent. If there exists exactly one index $i \in J$ such that $\alpha_i \neq 0$, then $\alpha_i x_i x_k \in C$ for any $x_k \in S$ such that $t(x_k) > t(x_i)$. But $t(\alpha_i x_i x_k) > t(x_i)$ hence $x_i x_k = 0$. If $t(x_k) = t(x_i)$ then $x_i x_k = 0$, because $t(x_i)$ is not idempotent. On the other hand C is strongly non-nil and therefore $0 \neq \alpha_i x_i x_k \in C$ for some $x_k \in S$, hence $t(\alpha_i x_i x_k) = t(x_i)$. But $t(\alpha_i x_i x_k) \geq t(x_i)t(x_k) \geq t(x_i)$ which yields the result. For the last case if at least two coefficients α_j are non-zero and $c \cdot x_k \neq 0$, for some $x_k \in S$ then there exists $j \in J$ such that $x_j x_k \neq 0$. Now by Proposition 2.1 there exists a ring $R = (A, *)$ on A such that $c * x_k$ is a non-zero rational multiple of an element in S which means $r(C) \geq 2$, a desired contradiction.

2)Let C be any rank m subgroup which is an ideal in every ring on A. Similarly as previous part, if $c = \sum_{i=1}^n \alpha_i x_i \in C$ and $\alpha_i \neq 0$ for some i, with $t^2(x_i) = t(x_i)$ then a non-zero multiple of x_i is in C, i.e., there is $\beta_i \in \mathbb{Q}$ with $\beta_i x_i \in C$. Hence such a generator of C must be in H_1 or H_3. Consequently, as Theorem 4.1, any generator of C is in H_1, H_2 or H_3. Moreover, if there exist $c = \sum_{j\in J\subseteq\{1,2,\cdots,n\}} \alpha_j x_j \in H_2 \cap C$ and $\alpha_j x_j \neq 0$ with $t(x_j)$ maximal and $x_j x_k \neq 0$ for some $x_k \in S$ with $t(x_k) < t(x_j)$, then $t(x_k x_j) \geq t(x_k)t(x_j) \geq t(x_j)$. But $t(x_j)$ is maximal, hence we must have $t(x_k x_j) = t(x_j)$ and this completes the proof. $\square$

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