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A CHARACTERIZATION OF HARMONIC SECTIONS AND A LIOUVILLE THEOREM

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Abstract. Let $P(M,G)$ be a principal fiber bundle and $E(M,N,G,P)$ an associated fiber bundle. Our interest is to study the harmonic sections of the projection π_E of E into M. Our first purpose is give a characterization of harmonic sections of M into E regarding its equivariant lift. The second purpose is to show a version of a Liouville theorem for harmonic sections of π_E.

1. Introduction

Let $\pi_E\colon (E,k)\to (M,g)$ be a Riemannian submersion and σ a section of π_E, that is, $\pi_E\circ\sigma = \mathrm{Id}_M$. It is known that $TE = VE\oplus HE$ where $VE=\ker(\pi_{E*})$ and HE is the horizontal bundle orthogonal to VE. C. Wood has studied the harmonic sections in many contexts, see [22], [24], [23], [25] and [2]. To recall, a harmonic section is a minimal solution for the vertical energy functional

$$E(\sigma)=\frac{1}{2}\int_M \|\mathbf{v}\sigma_*\|^2\,\mathrm{vol}(g)\,,$$

where $\mathbf{v}\sigma_*$ is the vertical component of σ_*. Furthermore, in [22], Wood showed that if σ is a minimizer of the vertical energy functional, then

$$\tau^v_\sigma = \mathrm{tr}\nabla^{\mathrm{v}}\mathbf{v}\sigma_* = 0\,,$$

where ∇^v is the vertical part of the Levi-Civita connection on E, since π_E has totally geodesics fibers. Wood called σ a harmonic section if $\tau^v_\sigma=0$.

In this work, the Riemannian submersion condition of π_E will be replaced by another submersion condition of π_E. Thus, equip E, which is not necessarily a Riemannian manifold, with a symmetric connection ∇^E. Let $\pi_E\colon (E,\nabla^E)\to (M,g)$ be a submersion with totally geodesic fibers.

With these conditions we can define harmonic sections in the same way as Wood [22], only observing that ∇^v is the vertical connection induced by ∇^E. There is not, necessarily, a compatibility between ∇^E and the Levi-Civita connection on M.

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Furthermore, the context of our study will be restricted. Let $P(M,G)$ be a Riemannian G-principal fiber bundle over a Riemannian manifold M such that the projection π of P into M is a Riemannian submersion. Suppose that P has a connection form ω. Let $E(M,N,G,P)$ be an associated fiber bundle of P with fiber N, where N is a differential manifold (see for example [19, ch.1]). It is well known that ω yields horizontal spaces on E. Our goal is to study the harmonic sections of the projection $\pi_E \colon E \to M$.

Let $F\colon P \to N$ be a differential map. We call F a horizontally harmonic map if $\tau_F \circ (H \otimes H) = 0$, where H is the horizontal lift of M into P associated to ω.

Let σ be a section of π_E. There exists a unique equivariant lift $F_\sigma \colon P \to N$ associated to σ. Our first purpose is to give a stochastic characterization horizontally harmonic map F_σ, σ a section of π_E. Furthermore, we extent Theorem 1 in [22], namely, a section of π_E is harmonic section if and only if F_σ is horizontally harmonic. C. Wood considers π_E as Riemannian submersions and we deal with π_E as submersions with totally geodesic fibers.

Our second purpose is to show our main theorem. For this, we consider $P(M,G)$ endowed with the Kaluza-Klein metric, M and G with the Brownian coupling property and N with the nonconfluence property of martingales. With these conditions we show a version of a Liouville Theorem, namely, being σ a section of π_E, if σ is a harmonic section then its equivariant lift F_σ is a constant map. Further, we show a version of the result to harmonic sections due to T. Ishihara in [11].

Aiming applications of our Liouville Theorem, suppose that M is a complete Riemannian manifold with nonnegative Ricci curvature or a compact Riemannian manifold. If its tangent bundle TM is endowed with a complete lift connection or the Sasaky metric, then the harmonic sections σ of π_{TM} are the 0-section. In the same way we can establish conditions for Hopf fibrations, with a Riemannian structure, such that the harmonic sections are the 0-section.

2. Preliminaries

In this work we use freely the concepts and notations of P. Protter [20], E. Hsu [9], P. Meyer [17], M. Emery [7] and [8], W. Kendall [14] and S. Kobayashi and N. Nomizu [15]. We suggest the reading of [3] for a complete survey about the objects of this section.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypotheses (see for example [7]). Our basic assumption is that every stochastic process is continuous.

Definition 2.1. Let M be a differential manifold. Let X be a stochastic process with value in M. We call X a semimartingale if, for all f smooth functions on M, $f(X)$ is a real semimartingale.

Let M be a differential manifold endowed with a symmetric connection ∇^M. Let X be a semimartingale in M and θ a 1-form on M defined along X. Let $(x_1, \ldots, x_n)$ be a local coordinate system on M. We define the Itô integral of θ along X, locally, by

$$\int_0^t \theta d^{\nabla^M} X_s = \int_0^t \theta_i(X_s) dX_s^i + \frac{1}{2}\int_0^t \Gamma^i_{jk}(X_s)\theta_i(X_s) d[X^j, X^k]_s\,,$$

where $\theta = \theta_i dx^i$ with θ_i smooth functions and Γ^i_{jk} are the Christoffel symbols of the connection ∇^M. Let $b \in T^{(2,0)}M$ be defined along X. We define the quadratic integral on M along X, locally, by

$$\int_0^t b(dX, dX)_s = \int_0^t b_{ij}(X_s) d[X^i, X^j]_s\,,$$

where $b = b_{ij} dx^i \otimes dx^j$ with b_{ij} smooth functions.

Let M and N be differential manifolds endowed with the symmetric connections ∇^M and ∇^N, respectively. Let $F\colon M \to N$ be a differential map and θ a section of TN^*. We have the following geometric Itô formula:

$$\int_0^t \theta d^{\nabla^N} F(X_s) = \int_0^t F^*\theta d^{\nabla^M} X_s + \frac{1}{2}\int_0^t \beta_F^*\theta(dX, dX)_s\,, \tag{1}$$

where β_F is the second fundamental form of F (see [3] or [12] for the definition of β_F). It is well known that F is an affine map if $\beta_F \equiv 0$. Here, the second fundamental form is important because based on this we can define harmonic maps. Another notation for β_F is ∇dF.

On the next section we will characterize harmonic maps in the stochastic way. For that we need a concept of martingales on manifolds. Following, we define martingales and Brownian motions in smooth manifolds. Furthermore, we will define two important properties of both. First, we will define martingales. In stochastic calculus the Itô integral of a real martingale is also a real martingale. In an analogous way, martingales in manifolds are defined from the Itô integral along a semimartingale(see for example [17] or [8]).

Definition 2.2. Let M be a differential manifold endowed with a symmetric connection ∇^M. A semimartingale X with values in M is called a ∇^M-martingale if $\int_0^t \theta\, d^M X_s$ is a real local martingale for all $\theta \in \Gamma(TM^*)$.

The most relevant stochastic process in stochastic calculus is the Brownian motion. Further, in our work, the Brownian coupling property is fundamental to show Theorem 4.2, which gives a strong result about harmonic sections.

Definition 2.3. Let (M, g) be a Riemannian manifold. Let B be a semimartingale with values in M. We say that B is a g-Brownian motion in M if B is a ∇^g-martingale, where ∇^g is the Levi-Civita connection of g, and for any section b of $T^{(2,0)}M$ we have that

$$\int_0^t b(dB, dB)_s = \int_0^t \operatorname{tr}\, b_{B_s}\, ds\,. \tag{2}$$

From (1) and (2) we deduce the useful formula:

$$\int_0^t \theta d^{\nabla^N} F(B_s) = \int_0^t F^*\theta d^{\nabla^g} B_s + \frac{1}{2}\int_0^t \tau_F^*\theta_{B_s}\, ds\,, \tag{3}$$

where τ_F is the tension field of F.

From formula (2) and the Doob-Meyer decomposition it follows that F is an harmonic map if and only if F sends g-Brownian motions to ∇^N-martingales.

We now introduce a necessary material about the nonconfluence property of martingales and the Brownian coupling property. Both will be used in Section 4.

Definition 2.4. Let M be a differential manifold endowed with a symmetric connection ∇^M. M has the nonconfluence property of martingales if for every filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, every ∇^M-martingales X and Y defined over Ω and every finite stopping time T such that

$$X_T = Y_T \quad \text{a.s. we have} \quad X = Y \quad \text{over} \quad [0, T]\,.$$

Example 2.1. Let $M = V$ be a n-dimensional vector space with a flat connection ∇^V. Let X and Y be ∇^V-martingales. Suppose that there is a stopping time τ with respect to $(\mathcal{F}_t)_{t\geq 0}$, $K > 0$ such that $\tau \leq K < \infty$ and $X_\tau = Y_\tau$. Then straightforward calculus shows that $X_t = Y_t$ for $t \in [0, \tau]$.

Definition 2.5. A Riemannian manifold M has the Brownian coupling property if for all $x_0, y_0 \in M$ we can construct a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t; t \geq 0)$ and two Brownian motions X and Y, not necessarily independent, but both adapted to the filtration such that

$$X_0 = x_0, Y_0 = y_0$$

and

$$\mathbb{P}(X_t = Y_t \quad \text{for some} \quad t \geq 0) = 1\,.$$

The stopping time $T(X, Y) = \inf\{t > 0; X_t = Y_t\}$ is called coupling time.

Example 2.2. Let M be a complete Riemannian manifold. In [13], W. Kendall has showed that if M is compact or M has nonnegative Ricci curvature then M has the Brownian coupling property.

Our next step is to construct a useful result about Brownian coupling, which is the key to prove Theorem 4.2. Let M be a Riemmanian manifold with metric g. Consider X and Y two g-Brownian motions in M which satisfy the Brownian coupling property and $X_0 = x$, $Y_0 = y$, where $x, y \in M$. Denote by $T(X, Y)$ their coupling time. The process $\bar{Y}$ is defined by

$$\bar{Y}_t = \begin{cases} Y_t\,, & t \leq T(X, Y) \\ X_t\,, & t \geq T(X, Y)\,. \end{cases} \tag{4}$$

It follows immediately that $\bar{Y}_0 = y$.

Proposition 2.1. *Let M be a Riemannian manifold with metric g. Suppose that M has the Brownian coupling property. Let X, Y be two g-Brownian motions in M which satisfy the Brownian coupling property. Then the process $\bar{Y}$ is a g-Brownian motion in M.*

Proof. It is a straightforward proof from the definition of Brownian motion. □

In the sequel we explain the idea we use to prove Theorem 4.2, a theorem type Liouville. With this we expect to show the role of the Brownian coupling property, the nonconfluence property of martingales and the Brownian motion $\bar{Y}$.

Let (M, g) be a Riemmanian manifold with the Brownian coupling property, N a differential manifold with a connection ∇^N and the nonconfluence property of martingales and $F\colon M \to N$ a harmonic map. Given x, $y \in M$ distinct points. Then, by Brownian coupling property in M, there exist two Brownian motion X and Y in M such that $X_0 = x$, $Y_0 = y$ and the coupling time $T(X, Y) > 0$. By definition (4) and Proposition 2.1, we have the Brownian motion $\bar{Y}$. Applying F in X and $\bar{Y}$ we see that, for $t \geq T(X, Y)$,

$$F(X) = F(\bar{Y})\,. \tag{5}$$

Since X and $\bar{Y}$ are Brownian motions and F is a harmonic map, $F(X)$ and $F(\bar{Y})$ are ∇^N-martingales. From nonconfluence property of martingale in N we obtain $F(X_0) = F(Y_0)$. Thus we get $F(x) = F(y)$. Because x, y are arbitrary points, F is a constant map.

The Brownian motion $\bar{Y}$ has a fundamental role in the argument above. First, its geometric nature gives equality (5). Second, its stochastic nature turns $F(\bar{Y})$ into a ∇^N-martingale, since F is a harmonic map. Then, by nonconfluence property of martingale, we obtain the constancy of F.

3. Harmonic sections

In this section we work to obtain a characterization of harmonic sections. Our line of work is: first, we introduce an appropriated geometric context; second, we define harmonic sections, horizontally harmonic maps and vertical martingales; third, we characterize, stochastically, horizontally harmonic maps; finally we show an equivalence between harmonic maps and horizontally harmonic maps.

Let $P(M, G)$ be a principal fiber bundle over M and $E(M, N, G, P)$ an associated fiber bundle to $P(M, G)$, where the differential manifold N is known as fiber of E (see for example [19, ch.1]). We denote the canonical projection from $P \times N$ into E by μ, namely, $\mu(p, \xi) = p \cdot \xi$. For each $p \in P$, we have the map $\mu_p\colon N \to E$ defined by $\mu_p(\xi) = \mu(p, \xi)$. Let $\sigma\colon M \to E$ be a section of the projection $\pi_E\colon E \to M$, that is, $\pi_E \circ \sigma = \mathrm{Id}_M$. There exists a unique equivariant lift $F_\sigma\colon P \to N$ associated to σ, which is defined by

$$F_\sigma(p) = \mu_p^{-1} \circ \sigma \circ \pi(p)\,. \tag{6}$$

The equivariance property of F_σ is given by

$$F_\sigma(p \cdot g) = g^{-1} \cdot F_\sigma(p)\,, \qquad g \in G\,.$$

Let us endow P and M with Riemmanian metrics k and g, respectively, such that $\pi\colon (P, k) \to (M, g)$ is a Riemmanian submersion. Let ω be a connection form on P. We observe that the connection form ω yields a horizontal structure on E, that is, for each $b \in E$, $T_bE = V_bE \oplus H_bE$, where $V_bE := \mathrm{Ker}(\pi_{Eb*})$ and H_bE is the horizontal subspace yielded by ω on E (see for example [15, pp.87]). We denote by $\mathbf{v}\colon TE \to VE$ and $\mathbf{h}\colon TE \to HE$ the vertical and horizontal projection, respectively.

Let ∇^M denote the Levi-Civita connection on M and let ∇^E be a connection on E. We are interested in connections ∇^E such that the projection π_E of E into M has totally geodesic fibers.

We denote by ∇^v the vertical connection associated to ∇^E on TE, that is, ∇^v is the vertical projection of ∇^E. In other words, since that π_E has totally geodesic fibers, for U, V vertical vector fields we have $\nabla^v_U V = \mathbf{v}(\nabla^E_U V)$. The ∇^v is usually founded in study of Riemanian submanifolds as the vertical projection of the Levi-Civita connection.

We endow N with a connection ∇^N such that, for each $p \in P$, μ_p is an affine map over its image, the fiber $\pi_E^{-1}(x)$ with $\pi(p) = x$.

Let σ be a section of π_E. Write $\sigma_* = \mathbf{v}\sigma_* + \mathbf{h}\sigma_*$, where $\mathbf{v}\sigma_*$ and $\mathbf{h}\sigma_*$ are the vertical and the horizontal components of σ_*, respectively. The second fundamental form for $\mathbf{v}\sigma_*$ is defined by

$$\beta^v_\sigma = \bar{\nabla}^v \circ \mathbf{v}\sigma_* - \mathbf{v}\sigma_* \circ \nabla^M\,,$$

where $\bar{\nabla}^v$ is the induced connection on $\sigma^{-1}VE$. The vertical tension field is given by

$$\tau^v_\sigma = \mathrm{tr}\beta^{\mathrm{v}}_\sigma\,.$$

Following, we extend the definition given by C. M. Wood [24] for harmonic sections.

Definition 3.1.
1. A section σ of π_E is called a harmonic section if $\tau^v_\sigma = 0$;
2. A differential map $F\colon P \to N$ is called horizontally harmonic if $\tau_F \circ (H \otimes H) = 0$, where H is the horizontal lift of M into P associated to ω.

We will now give a stochastic way to characterize horizontally harmonic maps. In the sequel, a 1-form θ on E will be called a vertical form if $\theta \in VE^*$, the adjoint of the vertical bundle VE.

Proposition 3.1. *Let $P(M,G)$ be a Riemannian principal fiber bundle endowed with a connection form ω and M a Riemannian manifold such that the projection π of P into M is a Riemannian submersion. Let $E(M,N,G,P)$ be an associated fiber bundle to P and suppose that N has a symmetric connection ∇^N. Then the equivariant lift F_σ associated to σ, σ a section of π_E, is a horizontally harmonic map if and only if, for every horizontal Brownian motion B^h in P, $F_\sigma(B^h)$ is a ∇^N-martingale.*

Proof. Let B be a g-Brownian motion in M and B^h a horizontal Brownian motion in P, that is, B^h is a solution of the stochastic differential equation

$$d^{\nabla^P} B^h = H_B d^{\nabla^M} B\,, \tag{7}$$

where H is the horizontal lift of M into P associated to ω. Set $\theta \in \Gamma(TN^*)$. By geometric Itô formula (1),

$$\int_0^t \theta \; d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t F_\sigma^* \theta \; d^{\nabla^P} B_s^h + \frac{1}{2}\int_0^t \beta_{F_\sigma}^* \theta (dB^h, dB^h)_s \, .$$

From (7) we see that

$$\int_0^t \theta \; d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t H^* F_\sigma^* \theta \; d^{\nabla^M} B_s + \frac{1}{2}\int_0^t \beta_{F_\sigma}^* \theta (H_B d^{\nabla^M} B, H_B d^{\nabla^M} B)_s \, .$$

As B is a Brownian motion we have

$$\int_0^t \theta \; d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t H^* F_\sigma^* \theta \; d^{\nabla^M} B_s + \frac{1}{2}\int_0^t (\tau_{F_\sigma}^H)^* \theta (B_s) \, ds \, ,$$

where $\tau_{F_\sigma}^H = \tau_{F_\sigma} \circ (H \otimes H)$. Since θ and B are arbitrary, the Doob-Meyer decomposition shows that $\int_0^t \theta d^{\nabla^N} F_\sigma(B_s^h)$ is a real local martingale if and only if $\tau_{F_\sigma}^H$ vanishes. From the definitions of martingales and horizontally harmonic maps we conclude the proof. □

Remark 1. In equation (7) we can see that the hypothesis of Riemannian submersion over $\pi : P \to M$ is necessary. In fact, the horizontal Brownian motion is defined as a solution of the Stratonovich stochastic equation, see for example [21]. However, Corollary 16 in [6] shows that Stratonovich and Itô differential equations are equivalent because the horizontal lift of a geodesic in M is a geodesic in P, since π is a Riemannian submersion.

Now we will give an extension of the harmonic sections characterization obtained by C.M. Wood (see Theorem 1 in [24]). The key of this proof is Lemma 3 in [24] showed by Wood. The reader can see that the application of this Lemma can be more general than Wood used in his paper [24]. In fact, it is possible to use the same Lemma in our context. For the convenience of the reader we repeat this Lemma without proof.

Lemma 3.2. *Let $P(M,G)$ be a Riemannian principal fiber bundle endowed with a connection form ω and M a Riemannian manifold such that the projection π of P into M is a Riemannian submersion. Let $E(M,N,G,P)$ be an associated fiber to P endowed with a symmetric connection ∇^E such that the projection π_E has totally geodesic fibers. Moreover, suppose that N has a symmetric connection ∇^N such that μ_p is an affine map for each $p \in P$. For any $X, Y \in T_pP$ we have that*

$$\mu_{p*}\beta_{F_\sigma}(\mathbf{h}X, \mathbf{h}Y) = \beta_\sigma^v(\pi_* X, \pi_* Y) \, ,$$

where $\mathbf{h}X, \mathbf{h}Y$ are the horizontal components of X, Y.

Following, we state the main theorem of this section.

Theorem 3.3. *Under hypothesis of Lemma 3.2, a section σ of π_E is a harmonic section if and only if F_σ is a horizontally harmonic map.*

Proof. Let σ be a section of π_E and F_σ its equivariant lift. Let B_t be a Brownian motion and θ a vertical form on E. From Lemma 3.2 we see that $\beta^v_\sigma(x)(X,Y) = \mu_{q*}\beta_{F_\sigma}(X^h, Y^h)$, where $\pi(q) = x$ and $X, Y \in T_xM$. It follows that

$$\int_0^t \beta^{v*}_\sigma\theta(dB,dB)_s = \int_0^t \theta\beta^v_\sigma(dB,dB)_s = \int_0^t \theta\mu_{B^h*}\beta_{F_\sigma}(dB^h,dB^h)_s$$
$$= \int_0^t \psi\beta_{F_\sigma}(dB^h,dB^h)_s = \int_0^t \beta^*_{F_\sigma}\psi(dB^h,dB^h)_s,$$

where $\psi = \mu^*_{B^h}\theta$ is a 1-form on N. As $dB^h = H_{B_t}d^{\nabla^M}B$ we have

$$\int_0^t \beta^{v*}_\sigma\theta(dB,dB) = \int_0^t \beta^*_{F_\sigma}\psi(H_Bd^{\nabla^M}B, H_Bd^{\nabla^M}B)_s\,.$$

Since B is a Brownian motion, it follows that

$$\int_0^t \tau^{v*}_\sigma\theta(B_s)ds = \int_0^t \tau^{H*}_{F_\sigma}\psi(B_s)\,ds\,,$$

where $\tau^H_{F_\sigma} = \tau_{F_\sigma}\circ(H\otimes H)$. Being B an arbitrary Brownian motion and θ an arbitrary vertical form, we conclude that

$$\tau^{v*}_\sigma = \tau^{H*}_{F_\sigma}\,.$$

Therefore, σ is a harmonic section if and only if F_σ is a horizontally harmonic map. □

Remark 2. One can think that to use of the stochastic tools is not necessary in the proof of Theorem 3.3, because it could be just a geometric computation from Lemma 3.2. It is not the case, because the vertical fundamental form β^v_σ is not symmetric. C. Wood, in a Riemmanian context [24], worked with this problem. He used the properties of the metric to identify the symmetric and skew-symmetric components of the vertical fundamental form β^v_σ, for a section σ of π_E (see Proposition 2 and Remark 2 in [24]). Here, the quadratic integral has an advantage, because it only computes the symmetric part of any bilinear form on a differential manifold, since for any skew-symmetric bilinear form the quadratic integral vanishes (see 3.14 in [7]).

4. A Liouville theorem for harmonic sections

We begin this section with the definition of the Kaluza-Klein metric on $P(M,G)$. Let $P(M,G)$ be a principal fiber bundle such that the differential Lie group G has a bi-invariant metric h, ω a connection form on P and M a Riemannian manifold with a metric g. The Kaluza-Klein metric is defined by

$$k = \pi^*g + \omega^*h\,. \tag{8}$$

From now on $P(M,G)$ is endowed with the Kaluza-Klein metric.

Here, the principal fiber bundle $P(M,G)$ with Kaluza-Klein metric can be view as particular example of the study done by D. Elworhty and W. Kendal in [5] with respect to Brownian motions in $P(M,G)$. In the proof of main Theorem we will use this remark.

Lemma 4.1. *Let $P(M,G)$ be a principal fiber bundle with a Kaluza-Klein metric k, where g is the Riemannian metric on M and h is the bi-invariant metric on G associated to k. The following assertions are true:*

(i) *Let $\tau\colon [0,1] \to P$ be a differential curve such that $\tau(t) = u \cdot \mu(t)$ with $\tau(0) = u$ and $\mu(t) \in G$, then*
$$\int_0^1 k(\dot\tau(t),\dot\tau(t))^{\frac{1}{2}}\,dt = \int_0^1 h(\dot\mu(t),\dot\mu(t))^{\frac{1}{2}}\,dt\,.$$

(ii) *Let $\tau\colon [0,1] \to P$ be a differential curve. If γ is a curve in M and if μ is a curve in G such that $\tau = \gamma(t)^h \cdot \mu(t)$, then*
$$\int_0^1 k(\dot\tau(t),\dot\tau(t))^{\frac{1}{2}}\,dt \le \int_0^1 g(\dot\gamma(t),\dot\gamma(t))^{\frac{1}{2}}\,dt + \int_0^1 h(\dot\mu(t),\dot\mu(t))^{\frac{1}{2}}\,dt\,.$$

(iii) *Let $x \in M$ and $u, v, w \in \pi^{-1}(x)$. If a and b are points in G such that $v = u \cdot a$ and $w = u \cdot b$, then*
$$d_P(v,w) = d_G(a,b)\,.$$

Proof. (i) and (ii) The proofs are straightforward.
(iii) Let $\tau\colon [0,1] \to P$ be a differential curve such that $\tau(0) = v$ and $\tau(1) = w$. Consider a curve γ in M such that $\pi(\tau) = \gamma$. There exists a differential curve μ in G such that $\mu(0) = a$, $\mu(1) = b$ and $\tau = \gamma^h \cdot \mu$. We observe that $\gamma(0) = x$ and $\gamma(1) = x$. This gives $\int_0^1 g(\dot\gamma(t),\dot\gamma(t))^{\frac{1}{2}}dt = 0$. Thus, from item (i) and item (ii) we conclude that
$$\int_0^1 k(\dot\tau(t),\dot\tau(t))^{\frac{1}{2}}\,dt = \int_0^1 h(\dot\mu(t),\dot\mu(t))^{\frac{1}{2}}\,dt\,.$$
Therefore, it is only necessary to consider vertical curves. It follows that $d_P(v,w) = d_P(u\cdot a, u\cdot b) = d_G(a,b)$, by the definition of Riemmanian distances. ☐

Finally, we can show our main theorem. Liouville type theorems say, under some conditions, that if a map is harmonic then it is a constant map. In our case, it is not possible that the sections are constant maps because of their definition. In fact, we will prove that if σ is a harmonic section then the equivariant map F_σ is constant. In this sense we have a Liouville type Theorem.

Theorem 4.2. *Let $P(M,G)$ be a principal fiber bundle equipped with a Kaluza-Klein metric and $E(M,N,G,P)$ an associated fiber bundle to P. Let ∇^E and ∇^N be symmetric connections on E and N, respectively, such that the projection π_E has totally geodesic fibers and μ_p is an affine map for each $p \in P$. Moreover, if N has the nonconfluence property of martingales and if M and G have the Brownian coupling property, then*

(i) *a section σ of π_E is a harmonic section if and only if F_σ is a constant map;*

(ii) *the left action of G into N has a fixed point if there exists a harmonic section σ of π_E;*

(iii) *a section σ of π_E is a harmonic section if and only if σ is parallel.*

Proof. (i) We first suppose that F_σ is a constant map. Then it is clear that $\tau_\sigma^v = 0$, so σ is a harmonic section.

Conversely, the proof will be divided into two parts. First, we find a suitable stopping time τ. After, we use τ to prove that F_σ is constant over P.

Set u, $v \in P$ such that $\pi(u) = x$ and $\pi(v) = x$. Thus $v = u \cdot a$. Since G has the Brownian coupling property, we have two h-Brownian motions μ and ν in G such that $\mu_0 = e$, $\nu_0 = a$. Moreover, there is a finite coupling time $T(\mu,\nu) > 0$. Proposition 2.1 now assures that the process

$$\bar{\nu_t} = \begin{cases} \nu_t\,, & t \leq T(\mu,\nu) \\ \mu_t\,, & t \geq T(\mu,\nu) \end{cases} \tag{9}$$

is a h-Brownian motion in G. Take a Brownian motion X in M such that it is independent of μ and ν and defining $X_t^h \cdot \mu(t)$ and $X_t^h \cdot \bar{\nu}(t)$. More exactly, Elworthy and Kendall in [5] showed that $X_t^h \cdot \mu(t)$ and $X_t^h \cdot \bar{\nu}(t)$ are Brownian motion in P. Taking $t > T(\mu,\nu) > 0$ we see that

$$d_P(X_t^h \cdot \mu_t, X_t^h \cdot \bar{\nu_t}) = d_G(\mu_t, \bar{\nu_t})\,,$$

which follows from Lemma 4.1, item (iii).

Setting $t \geq T(\mu,\nu)$ we obtain $F_\sigma(X_t^h \cdot \mu_t) = F_\sigma(X_t^h \cdot \bar{\nu_t})$. Since $X_t^h \cdot \mu_t$ and $X_t^h \cdot \bar{\nu_t}$ are Brownian motions, it follows that $F_\sigma(X_t^h \cdot \mu_t)$ and $F_\sigma(X_t^h \cdot \bar{\nu_t})$ are N-martingales. Using the nonconfluence property of martingale in N, we conclude that

$$F_\sigma(X_0^h \cdot \mu_0) = F_\sigma(X_0^h \cdot \bar{\nu_0})\,.$$

It follows immediately that $F_\sigma(u) = F_\sigma(v)$. Consequently, F_σ is a constant map over fibers.

As F_σ is a constant map over fibers we have a good definition for the map $\tilde{F}_\sigma$ from M into N defined by $\tilde{F}_\sigma(x) = F_\sigma(p)$ such that $\pi(p) = x$. Let x, y be distinct points in M. The Brownian coupling property yields two Brownian motion X, Y in M such that $X_0 = x$, $Y_0 = y$ and the coupling time $T(X,Y)$ is finite and positive. Proposition 2.1 now assures that the process

$$\bar{Y_t} = \begin{cases} Y_t\,, & t \leq T(X,Y) \\ X_t\,, & t \geq T(X,Y) \end{cases} \tag{10}$$

is a g-Brownian motion in M. Applying $\tilde{F}$ at X and $\bar{Y}$ we obtain for $t > T(X,Y)$ that

$$\tilde{F}_\sigma(X_t) = \tilde{F}_\sigma(\bar{Y_t})\,. \tag{11}$$

Let X_t^h and $\bar{Y}_t^h$ be two horizontal Brownian motions in P associated to X and $\bar{Y}$, respectively, such that $X_0^h = u$ and $\bar{Y}_0^h = v$. From (11) we see for $t > T(X,Y)$ that

$$F_\sigma(X_t^h) = F_\sigma(\bar{Y}_t^h)\,. \tag{12}$$

Since σ is a harmonic section, from Theorem 3.3 we see that F_σ is a horizontally harmonic map. Proposition 3.1 now shows that $F_\sigma(X_t^h)$ and $F_\sigma(\bar{Y}_t^h)$ are ∇^N-martingales in N. Since N has the nonconfluence property of martingales,

$$F_\sigma(X_0^h) = F_\sigma(\bar{Y}_0^h)\,.$$

It follows immediately that $F_\sigma(u) = F_\sigma(v)$. Consequently, F_σ is a constant map.
(ii) Let σ be a harmonic section of π_E. From item (i) there exists $\xi \in N$ such that $F_\sigma(p) = \xi$ for all $p \in P$. We claim that ξ is a fixed point. In fact, set $a \in G$. From the equivariant property of F_σ we deduce that

$$a \cdot \xi = a \cdot F_\sigma(p) = F_\sigma(p \cdot a^{-1}) = \xi\,.$$

(iii) Let σ be a section of π_E. Suppose that σ is parallel. Then $\sigma_*(X)$ is horizontal for all $X \in TM$ (see for example [15], pp.114). This gives $\mathbf{v}\sigma_*(X) = 0$. Then it is clear, by definition, that σ is a harmonic section.

Suppose that σ is a harmonic section. From item (i) it follows that there is $\xi \in N$ such that $F_\sigma(p) = \xi$ for all $p \in P$. By the definition of equivariant lift,

$$\sigma(x) = \sigma \circ \pi(p) = \mu(p, \xi) = \mu_\xi(p)\,, \qquad \pi(p) = x\,,$$

where μ_ξ is an application from P into E. Take $v \in T_xM$ and let $\gamma(t)$ be a curve in M such that $\gamma(0) = x$ and $\dot\gamma(0) = v$. Then,

$$\sigma_*(v) = \left.\frac{d}{dt}\right|_0 \sigma \circ \gamma(t) = \left.\frac{d}{dt}\right|_0 \mu_\xi \circ \gamma^h(t) = \mu_{\xi *}(\dot\gamma^h(0))\,,$$

where γ^h is the horizontal lift of γ into P. Since $\dot\gamma^h(0)$ is a horizontal vector in P, so $\mu_{\xi*}(\dot\gamma^h(0))$ is also horizontal in E (see for example [15], pp.87). Therefore $\sigma_*(v)$ is a horizontal vector. So we conclude that σ is parallel. □

Remark 3. Item i) of Theorem 4.2 can be weakened in the following way. Under the same hypothesis of Theorem 4.2, but without assumption that M has the Brownian coupling property, if σ is a harmonic section, then F_σ is constant over the fibers of P.

5. Examples

In this section, we will give three applications. First, using Theorem 4.2 we show that the unique harmonic section in the tangent bundle with complete lift is the 0-section. Also, from Theorem 4.2 we will show that under geometric conditions the unique harmonic section in the Tangent bundle with Sasaki metric is null. Finally, we will work with Hopf fibrations and harmonic sections.

Tangent bundle with complete lift

Let (M, g) be a complete Riemannian manifold which is compact or has nonnegative Ricci curvature. Let us denote by TM the tangent bundle associated to M. It is clear that TM is an associated fiber bundle to the orthonormal frame bundles OM, with fiber $\mathbb{R}^n$. It is possible to introduce a Kaluza-Klein metric on OM. In fact, the Lie group $O(n)$ is the group acting on OM, and $O(n)$ is a compact group. Therefore, there exists a bi-invariant metric h on $O(n)$ (see Theorem 3.8 in [1]). Thus, we define the Kaluza-Klein metric on OM in the same way that (8).

To study harmonic sections of π_E we need to introduce a connection on TM. Given a symmetric connection on M, we can prolong ∇ to a connection on TM. A well known way to prolong it is the complete lift ∇^c (see [10] for the definition of ∇^c). Let X, Y be vector fields on M, so ∇^c satisfies the following equations:

$$
\begin{aligned}
&\nabla^c_{X^V} Y^V = 0\\
&\nabla^c_{X^V} Y^H = 0\,,\\
&\nabla^c_{X^H} Y^V = (\nabla_X Y)^V\,,\\
&\nabla^c_{X^H} Y^H = (\nabla_X Y)^H + \gamma(R(-,X)Y\,,
\end{aligned}
\tag{13}
$$

where $R(-,X)Y$ denotes a tensor field W of type (1,1) on M such that $W(Z) = R(Z,X)Y$ for any $Z \in T^{(1,0)}M$, and γ is a lift of tensors, which is defined at page 12 in [10].

We claim that the applications $\mu_p \colon \mathbb{R}^n \to TM$, for all $p \in OM$, are harmonic maps over their images. Since the second fundamental form is a tensor, it is sufficient to show that $\beta_{\mu_p} = 0$ for local coordinates. In fact, let $(\mathbb{R}^n, \nu^\alpha)$ be a coordinate system in $\mathbb{R}^n$. Let us denote by ∂_α the coordinate vector fields on $(\mathbb{R}^n, \nu^\alpha)$. Applying β_{μ_p} on the coordinate vector fields we deduce that

$$\beta^x_{\mu_p}(\partial_\alpha, \partial_\beta) = \nabla^x_{\partial_\alpha} \mu_{p*}\partial_\beta - \mu_{p*}(\nabla^{\mathbb{R}^n}_{\partial_\alpha}\partial_\beta)\,.$$

Observing that $\nabla^x \equiv 0$ we get $\beta^x_{\mu_p}(\partial_\alpha, \partial_\beta) = 0$. So $\beta^x_{\mu_p} \equiv 0$. It follows that μ_p, for all $p \in P$, are affine maps. Furthermore, ∇^c is a symmetric connection because ∇ is also a symmetric connection (see Proposition 6.1 in [10, Ch.1]), and π_{TM} is a submersion with totally geodesic fibers. Besides, it follows from examples 2.1 and 2.2 that M and $O(n)$ have the Brownian coupling property, and $\mathbb{R}^n$ has the nonconfluence property of martingales.

Proposition 5.1. *Let M be a Riemannian manifold, ∇ a symmetric connection on M and TM its tangent bundle. Suppose that TM is endowed with the complete lift connection ∇^c and $\mathbb{R}^n$ is endowed with the euclidian metric. If σ is a section of π_{TM}, then σ is the 0-section.*

Proof. Let σ be a harmonic section of π_{TM}. By Theorem 4.2, item (i), there exists $\xi \in N$ such that $F_\sigma(u) = \xi$ for all $u \in P$. Moreover, by item (ii) of Theorem 4.2, ξ is a fixed point of the left action of $O(n)$ into $\mathbb{R}^n$. We observe that $0 \in \mathbb{R}^n$ is the unique fixed point of the left action. We thus get $F_\sigma(u) = 0$. Therefore σ is the 0-section. $\square$

Tangent bundle with Sasaki metric

Let M be a complete Riemannian manifold which is compact or has non-negative Ricci curvature. Let OM be the orthonormal frame bundle endowed with the Kaluza-Klein metric. See the first paragraph at the example above to the construction of the Kaluza-Klein metric on OM. Let TM be the tangent bundle equipped with the Sasaki metric g_s. Thus π_E is a Riemannian submersion with totally geodesic fibers and, for each $p \in P$, μ_p is an isometric map (see for example [18]). From these assumptions and Examples 2.1 and 2.2 it follows that the hypotheses of Theorem 4.2 are satisfied.

Proposition 5.2. *Under the conditions stated above, if σ is a harmonic section of π_{TM}, then σ is the 0-section.*

Proof. The proof is analogous to the proof of Proposition 5.1. $\square$

Hopf fibration

Let $S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1}$ be a Hopf fibration. It is well known that $S^{2n-1}(\mathbb{CP}^{n-1}, S^1)$ is a principal fiber bundle. We recall that $U(1) \cong S^1$. Let ϕ be the application of $U(1) \times \mathbb{C}^m$ into $\mathbb{C}^m$ given by

$$(g, (z_1, \dots, z_m)) \to g \cdot (z_1, \dots, z_m) = (gz_1, \dots, gz_m)\,. \tag{14}$$

Clearly, ϕ is a left action of $U(1)$ into $\mathbb{C}^m$. Thus, we can consider $\mathbb{C}^m$ as the fiber of the associated fiber bundle $E(\mathbb{CP}^{n-1}, \mathbb{C}^m, S^1, S^{2n-1})$, where $E = S^{2n-1} \times_{U(1)} \mathbb{C}^m$. We are considering the canonical scalar product $\langle , \rangle$ on $\mathbb{C}^m$ and the induced Riemannian metric g on $\mathbb{CP}^{n-1}$. Since $U(1)$ is invariant by $\langle , \rangle$, there exists one and only one Riemannian metric $\hat{g}$ on E such that π_E is a Riemannian submersion of $(E, \hat{g})$ into $(\mathbb{CP}^{n-1}, g)$ with totally geodesic fibers isometrics to $(\mathbb{C}^m, \langle , \rangle)$ (see for example [12]). From these assumptions and examples 2.1 and 2.2 wee see that the hypotheses of Theorem 4.2 hold.

Proposition 5.3. *Under the conditions stated above, if σ is a harmonic section of π_E, then σ is the 0-section.*

Proof. We first observe that $(0, \dots, 0)$ is the unique fixed point of the left action (14). Since σ is a harmonic section, from Theorem 4.2 we see that F_σ is a constant map and $F_\sigma(p) = (0, \dots, 0)$ for all $p \in S^{2n-1}$. Therefore σ is the 0-section. $\square$

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