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SOME LOGARITHMIC FUNCTIONAL EQUATIONS

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Abstract. The functional equation $f(y-x) - g(xy) = h\,(1/x - 1/y)$ is solved for general solution. The result is then applied to show that the three functional equations $f(xy) = f(x) + f(y)$, $f(y-x) - f(xy) = f(1/x - 1/y)$ and $f(y-x) - f(x) - f(y) = f(1/x - 1/y)$ are equivalent. Finally, twice differentiable solution functions of the functional equation $f(y-x) - g_1(x) - g_2(y) = h\,(1/x - 1/y)$ are determined.

1. Introduction

The functional equation

$$f(xy) = f(x) + f(y) \tag{1.1}$$

with $f\colon \mathbb{R}^+ (:= (0,\infty)) \to \mathbb{R}$, whose best-known solution is the natural logarithmic function $\log x$, is usually called the *classical logarithmic functional equation* or *the Cauchy logarithmic functional equation*. Its solution is generically written as $L_+(x)$ and referred to as a *logarithmic function*. Although the classical logarithmic function $\log x$ is one particular solution to (1.1) which is continuous, there are, however, uncountably many non-continuous logarithmic functions, which are constructed from the solutions, referred to as *additive functions* and generically written as $A(x)$, of the Cauchy functional equation,

$$f(x+y) = f(x) + f(y) \tag{1.2}$$

with $f\colon \mathbb{R} \to \mathbb{R}$. The Cauchy functional equation (1.2) possesses uncountably many solutions $f\colon \mathbb{R} \to \mathbb{R}$, the fact established by Hamel in 1905 using the notion of a basis of $\mathbb{R}$, which bears his name. The number of elements in any Hamel basis (of the reals over the rationals) is uncountable. More precisely, recall that a subset $H \subset \mathbb{R}$ is a Hamel basis for $\mathbb{R}$ if every $x \in \mathbb{R}$ can be written uniquely as $x = \sum_{i=1}^{n} r_i h_i$, for some $n \in \mathbb{N}$, $r_i \in \mathbb{Q}$ and $h_i \in H$ $(i = 1, \dots, n)$. Consider the class of functions given by $f_g(x) = r_1 g(h_1) + \dots + r_n g(h_n)$, where $g\colon H \to \mathbb{R}$. Clearly, each f_g is a solution of (1.2) over $\mathbb{R}$ and since the choice of g is arbitrary, we obtain uncountably many

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non-continuous solutions to (1.2). Uncountably many solutions to the logarithmic functional equation (1.1) are immediately obtained from the following connection between logarithmic functions and additive functions proved in [7] (see also [6, Theorem 13.1.2, p. 344] and [5, Theoremm 1.42, p. 29]).

Proposition 1.1 ([7, Theorem 1(a)]). *The logarithmic function* $f\colon \mathbb{R}^+ \to \mathbb{R}$ *and the additive function* $g\colon \mathbb{R} \to \mathbb{R}$ *correspond 1-1 by virtue of* $f = g \circ \log$, *respectively,* $g = f \circ \log^{-1}$, *where* $\circ$ *denotes the composite symbol,* $\log$ *the natural logarithm function and* $\log^{-1}$ *its inverse function.*

There have appeared several works on functional equations satisfied by the logarithmic function, referred to as logarithmic functional equations. Of interest to us are the three papers [3], [4] and [1]. In [3], for $f\colon \mathbb{R}^+ \to \mathbb{R}$, it is proved that the two functional equations

$$f(x+y) - f(x) - f(y) = f(1/x + 1/y) \tag{1.3}$$

$$f(xy) = f(x) + f(y)\,. \tag{1.1}$$

are equivalent in the sense that each solution of one equation is also a solution of the other equation. The proof of the part that a solution of (1.1) is also a solution of (1.3) is correct, but unfortunately, the proof that a solution of (1.3) is a solution of (1.1) has a gap which occurs in the change of variables that is not 1-1 at the bottom half of page 262.

In [4], the authors add the following functional equation

$$f(x+y) - f(xy) = f(1/x + 1/y) \tag{1.4}$$

to the above list of equivalent equations by proving that (1.1) and (1.4) are equivalent. In addition, they considered the Pexider generalizations of (1.4) and (1.3), namely,

$$f(x+y) - g(xy) = h(1/x + 1/y) \tag{1.5}$$

$$f(x+y) - g(x) - h(y) = k(1/x + 1/y)\,. \tag{1.6}$$

For (1.5), they proved that its general solution when f, g, $h\colon \mathbb{R}^+ \to \mathbb{R}$ is given by

$$f(x) = a + L(x)\,, \quad g(x) = b + L(x)\,, \quad h(x) = a - b + L(x)\,, \tag{1.7}$$

where a and b are arbitrary constants and L satisfies (1.1).

For (1.6), they proved that its general twice differentiable solution functions f, g, h, k sending $\mathbb{R}^+$ into $\mathbb{R}$ are given by

$$f(x) = -a\log x + bx + c_1\,, \qquad g(x) = -a\log x + bx - d/x + c_1 + c_3\,, \tag{1.8}$$

$$h(x) = -a\log x + bx - d/x - c_2 - c_3\,, \qquad k(x) = -a\log x + dx + c_2\,, \tag{1.9}$$

where a, b, c_1, c_2, c_3 and d are arbitrary constants.

Recently, in [1], Chung gave a beautifully simple proof of the general solution of (1.5). In addition, Chung proved, using the concept of Schwartz distribution, that general locally integrable solution functions of (1.5) f, g, $h\colon \mathbb{R}^+ \to \mathbb{C}$ are given by

$$f(x) = c_1 + c_2 + a\log x\,, \quad g(x) = c_1 + a\log x\,, \quad h(x) = c_2 + a\log x\,, \tag{1.10}$$

where $c_1, c_2, a \in \mathbb{C}$.

Here, we complement the works of Heuvers-Kannappan and Chung mentioned above by solving a few other logarithmic functional equations in the Pexider form. Our first main result is as follows:

Theorem 1.2. *Let F, G, $H\colon \mathbb{R}^*(:= \mathbb{R}\setminus\{0\}) \to \mathbb{C}$. If F, G and H satisfy the functional equation*

$$F(y-x) - G(xy) = H\left(1/x - 1/y\right) \tag{1.11}$$

whenever x, $y \in \mathbb{R}^*$ are subject to the condition $y - x \neq 0$, then

$$F(x) = L(x) + a + b\,, \tag{1.12}$$

$$G(x) = L(x) + a\,, \tag{1.13}$$

$$H(x) = L(x) + b\,, \tag{1.14}$$

where $L\colon \mathbb{R}^* \to \mathbb{C}$ is a logarithmic function (i.e., satisfying (1.1)) and a, b are complex constants.

As an application, we use Theorem 1.2 to establish, in Section 3, the equivalence of three logarithmic functional equations.

Theorem 1.3. *Let $f\colon \mathbb{R}^* \to \mathbb{C}$. For x, $y \in \mathbb{R}^*$ with $y - x \neq 0$, the three functional equations*

$$f(xy) = f(x) + f(y)\,, \tag{1.1}$$

$$f(y-x) - f(xy) = f(1/x - 1/y)\,, \tag{1.15}$$

$$f(y-x) - f(x) - f(y) = f(1/x - 1/y) \tag{1.16}$$

are equivalent in the sense that a solution of any one equation is also a solution of the other two.

In Section 4, we solve for twice differentiable solution functions another functional equation which is a Pexider form of the third equation in Theorem 1.3.

Theorem 1.4. *Let f, g_1, g_2, $h\colon \mathbb{R}^* \to \mathbb{C}$. If f, g_1, g_2 and h are twice differentiable and satisfy the functional equation*

$$f(y-x) - g_1(x) - g_2(y) = h\left(1/x - 1/y\right) \qquad (x, y \in \mathbb{R}^*) \tag{1.17}$$

whenever $y - x \neq 0$, then over the domain $\mathbb{R}^*$ we have

$$f(x) = c\log(|x|) + dx + p\,, \qquad h(x) = c\log(|x|) + mx + r\,,$$

$$g_1(x) = c\log(|x|) - dx - m/x + p + q\,, \qquad g_2(x) = c\log(|x|) + dx + m/x - q - r\,,$$

where c, d, m, p, q and r are complex constants.

All the main results proved here also hold for functions defined over the positive real numbers and we list them in the last section.

2. Proof of Theorem 1.2

For $x,\, y \in \mathbb{R}^*$, put

$$t = xy \neq 0\,, \qquad s = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} \neq 0\,.$$

To each pair $(x, y) \in \mathbb{R}^* \times \mathbb{R}^*$, there clearly corresponds exactly one pair $(s, t) \in \mathbb{R}^* \times \mathbb{R}^*$. Yet, to each pair $(s, t) \in \mathbb{R}^* \times \mathbb{R}^*$, solving for x, y, we get

$$x = \frac{-st \pm \sqrt{s^2t^2 + 4t}}{2}\,, \qquad y = x + st = \frac{st \pm \sqrt{s^2t^2+4t}}{2}\,.$$

Thus, there are two corresponding pairs $(x, y) \in \mathbb{R}^* \times \mathbb{R}^*$ provided $t(ts^2 + 4) \geq 0$, with the two pairs being distinct whenever $t(ts^2 + 4) > 0$. In order to make this change of variables invertible, for each $(s, t) \in \mathbb{R}^* \times \mathbb{R}^*$, we first restrict ourselves the case where

$$x = \frac{-st + \sqrt{s^2t^2 + 4t}}{2} \neq 0\,, \qquad y = \frac{st + \sqrt{s^2t^2+4t}}{2} \neq 0\,. \tag{2.1}$$

The functional equation (1.11) becomes

$$F(ts) = G(t) + H(s) \qquad (s, t \in \mathbb{R}^*)\,, \tag{2.2}$$

subject to the condition

$$t(ts^2 + 4) \geq 0\,.$$

To circumvent this condition, we use an ingenious idea of Chung in [1]. For each $(s, t) \in \mathbb{R}^* \times \mathbb{R}^*$, choose $u \in \mathbb{R}^*$ so that the following two inequalities hold simultaneously

$$t\left\{t(su)^2 + 4\right\} \geq 0\,, \qquad ts\left\{tsu^2 + 4\right\} \geq 0\,. \tag{2.3}$$

To confirm the existence of such u, we consider the two possibilities $t > 0$ and $t < 0$.

Case $t > 0$. If $s > 0$, then any $u \in \mathbb{R}^*$ satisfies the two inequalities in (2.3). If $s < 0$, then the first inequality in (2.3) holds for each $u \in \mathbb{R}^*$, while the second inequality holds only when $u^2 \geq -4/ts$.

Case $t < 0$. If $s < 0$, the second inequality in (2.3) holds for any $u \in \mathbb{R}^*$, while the first inequality holds only when $u^2 \geq -4/ts^2$. If $s > 0$, the first inequality in (2.3) holds when $u^2 \geq -4/ts^2$, while the second inequality holds when $u^2 \geq -4/ts$.

To fulfil all the requirements, it thus suffices to choose $u \in \mathbb{R}^*$ satisfying

$$u^2 \geq \max\{|4/ts|, |4/ts^2|\} > 0 \tag{2.4}$$

confirming its existence.

Taking appropriate pairs in (2.2) through the use of each inequality in (2.3) successively, we get

$$F(tsu) = G(t) + H(su)\,,$$

$$F(tsu) = G(ts) + H(u)\,.$$

These last two equations yield

$$G(ts) - G(t) = H(su) - H(u) =: \alpha(s) \qquad (t, s \in \mathbb{R}^*)\,, \tag{2.5}$$

i.e.,

$$G(ts) = G(t) + \alpha(s)\,,$$

which is a Pexider form of the logarithmic equation. Its solution is of the form ([5, p. 40])

$$G(x) = L(x) + a\,, \qquad \alpha(x) = L(x)\,, \tag{2.6}$$

where $L\colon \mathbb{R}^* \to \mathbb{C}$ is a logarithmic function and $a \in \mathbb{C}$. Substituting these functions back into (2.5), we deduce that

$$H(su) = H(u) + \alpha(s) = H(u) + L(s)\,, \tag{2.7}$$

for all $s \in \mathbb{R}^*$ and for those $u \in \mathbb{R}^*$ satisfying (2.4). Since (2.7) is free of t, it thus holds for all s, $u \in \mathbb{R}^*$, which in turn yields

$$H(x) = L(x) + b \qquad (x \in \mathbb{R}^*)\,,$$

for some $b \in \mathbb{C}$ and consequently,

$$F(x) = L(x) + a + b \qquad (x \in \mathbb{R}^*)\,.$$

Since the above arguments do not involve the choice of the square root determining the values of x, y in (2.1), the same set of solution functions to (1.11) is identical in this case and the theorem is proved.

3. Proof of Theorem 1.3

If $f\colon \mathbb{R}^* \to \mathbb{C}$ satisfies (1.1), then it is easily checked that it is also a solution of (1.15) and (1.16). If $f\colon \mathbb{R}^* \to \mathbb{C}$ satisfies (1.15), then Theorem 1.2 shows that $f(x) = L(x)$, is a logarithmic function which then satisfies (1.1) and (1.16).

To finish the proof, it suffices to show that a solution of (1.16) is a solution of (1.1). Suppose f satisfies (1.16). Replace x by $-t$ to get

$$f(y + t) - f(-1/t - 1/y) = f(-t) + f(y)\,, \tag{3.1}$$

valid for $yt(y + t) \neq 0$. The left hand side of the equation is symmetric in y and t, hence

$$f(-t) + f(y) = f(-y) + f(t)\,,$$

or

$$f(-t) - f(t) = f(-y) - f(t) = c\,,$$

for some constant c. But it then follows that $c = 0$, since

$$f(t) = f(-(-t)) = f(-t) + c = f(t) + 2c\,.$$

Therefore $f(-t) = f(t)$ and equation (1) can be rewritten as

$$f(y + t) - f(y) - f(t) = f(1/y + 1/t),$$

which is the equation (1.3), first studied by Heuvers, [3], and its solutions is also given in Ebanks, [2]. Thus f is a logarithmic function $L(x)$.

4. Proof of Theorem 1.4

Differentiating (1.17) with respect to x, we get

$$-f'(y-x) - g_1'(x) = -\frac{1}{x^2} h'\left(\frac{1}{x} - \frac{1}{y}\right) \quad (x, y \in \mathbb{R}^*,\ y \neq x)$$

and differentiating this last expression with respect to y, we get

$$f''(y-x) = \left(\frac{1}{xy}\right)^2 h''\left(\frac{1}{x} - \frac{1}{y}\right) \quad (x, y \in \mathbb{R}^*,\ y \neq x). \tag{4.1}$$

For each pair $(x, y) \in \mathbb{R}^* \times \mathbb{R}^*$ with $x \neq y$, let $u = y - x \neq 0$, $v = \frac{1}{x} - \frac{1}{y} \neq 0$. This change of variables is not invertible because to each pair $(u, v) \in \mathbb{R}^* \times \mathbb{R}^*$, solving for x, y, we get

$$x = \frac{-uv \pm \sqrt{u^2v^2 + 4uv}}{2v} \neq 0\,, \qquad y = x + u = \frac{uv \pm \sqrt{u^2v^2 + 4uv}}{2v} \neq 0\,.$$

Thus, there are two corresponding pairs $(x, y) \in \mathbb{R}^* \times \mathbb{R}^*$ provided $uv(uv + 4) \geq 0$, with the two pairs being distinct whenever $uv(uv + 4) > 0$. In order to make this change of variables invertible, for each $(u, v) \in \mathbb{R}^* \times \mathbb{R}^*$, we first restrict ourselves to the case where

$$x = \frac{-uv + \sqrt{u^2v^2 + 4uv}}{2v} \neq 0\,, \qquad y = \frac{uv + \sqrt{u^2v^2 + 4uv}}{2v} \neq 0\,. \tag{4.2}$$

The functional equation (4.1) becomes

$$u^2 f''(u) = v^2 h''(v) \qquad (u, v \in \mathbb{R}^*) \tag{4.3}$$

subject to the condition

$$uv(uv + 4) \geq 0\,. \tag{4.4}$$

Next, we will show that the restriction (4.4) can be removed, i.e., the equation (4.3) holds for all $u,\ v \in \mathbb{R} \setminus \{0\}$.

If $uv \geq 0$, then (4.4) holds for all $u,\ v \in \mathbb{R}^*$.

For the case $uv < 0$, the restriction (4.4) holds whenever $uv \leq -4$. In this case, we claim that

I) (4.3) holds with both sides being constant for $(u, v) \in \mathbb{R}^- \times \mathbb{R}^+$ $(\mathbb{R}^- := (-\infty, 0))$ and

II) (4.3) holds with both sides being constant for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^-$.

To show I), we set $u = -4$ in (4.3) to get

$$v^2 h''(v) = 16 f''(-4) =: K_1 \text{ (constant)} \qquad (v \geq 1)\,.$$

Next, substituting $v = 4$ into (4.3) and equating the result with what we have just found, we obtain

$$u^2 f''(u) = 16 h''(4) = K_1 \qquad (u \leq -1) \tag{4.5}$$

showing that I) holds for $u \le -1$, $v \ge 1$. There are two remaining intervals to check: $0 < v < 1$ and $-1 < u < 0$. For $0 < v < 1$, we simply substitute $u \le \frac{-4}{v}$ in (4.3) and using (4.5) to deduce the result. For $-1 < u < 0$, take $v \ge \frac{-4}{u}$ in (4.3) to complete the proof of I).

To prove II), first substitute $u = 4$ into (4.3) to get

$$v^2 h''(v) = 16 f''(4) = K_2\,, \quad \text{a constant} \qquad (v \le -1)\,.$$

Next, substituting $v = -4$ into (4.3) and equating the result with what we have just found, we obtain

$$u^2 f''(u) = 16 h''(-4) = K_2 \qquad (u \ge 1)\,,$$

so that II) holds for $u \ge 1, v \le -1$. The remaining intervals are taken care of in the manner similar to that in the proof of I).

The results of the case $uv \ge 0$ together with I) and II) show that we need solve the equation (4.3) for u, $v \in \mathbb{R}^*$. Since the variables in (4.3) are separable, we deduce that

$$x^2 f''(x) = -c \text{ (a constant)} = x^2 h''(x) \qquad (x \in \mathbb{R}^*)\,.$$

Thus,

$$f(x) = c\log(|x|) + dx + p\,, \quad h(x) = c\log(|x|) + mx + r \qquad (x \in \mathbb{R}^*)\,, \tag{4.6}$$

where d, m, p, r are complex constants. Substituting the two functions from (4.6) into (1.17), we obtain

$$g_1(x) + g_2(y) = c\log(|x|) + c\log(|y|) + dy - dx + p - \frac{m}{x} + \frac{m}{y} - r \quad (x, y \in \mathbb{R}^*)\,.$$

Separating the variables x and y, we get

$$\begin{aligned} g_1(x) - c\log(|x|) + dx + \frac{m}{x} - p &= -g_2(y) + c\log(|y|) + dy + \frac{m}{y} - r \\ &=: q \quad \text{(a constant)} \qquad (x, y \in \mathbb{R}^*) \end{aligned}$$

so that

$$g_1(x) = c\log(|x|) - dx - \frac{m}{x} + p + q\,, \quad g_2(x) = c\log(|x|) + dx + \frac{m}{x} - q - r \qquad (x \in \mathbb{R}^*)\,.$$

To complete the proof, we simply note that the other choice of making the change of variables invertible as mentioned prior to (4.2) proceeds exactly as above.

5. The case with domain $\mathbb{R}^+$

Since the domain of definition of all functions treated above is the nonzero real numbers $\mathbb{R}^*$, it is natural to ask whether all the results proved in Theorems 1.2–1.4 continue to hold if the domain is the positive real numbers $\mathbb{R}^+$. The answer is affirmative in case of Theorems 1.2 and 1.4. Since the proofs are easier, we merely state the two results without proof.

Theorem 5.1. *Let f, g, $h\colon \mathbb{R}^+ \to \mathbb{C}$. If f, g and h satisfy the functional equation*

$$f(y-x) - g(xy) = h\,(1/x - 1/y) \tag{5.1}$$

whenever x, $y \in \mathbb{R}^+$ are subject to the condition $y - x > 0$, then

$$f(x) = L_+(x) + c_1 + c_2\,, \quad g(x) = L_+(x) + c_1\,, \quad h(x) = L_+(x) + c_2 \qquad (x \in \mathbb{R}^+)\,,$$

where $L_+\colon \mathbb{R}^+ \to \mathbb{C}$ is a logarithmic function and c_1, c_2 are complex constants.

Theorem 5.2. *Let f, g_1, g_2, $h\colon \mathbb{R}^+ \to \mathbb{C}$. If f, g_1, g_2 and h are twice differentiable and satisfy the functional equation*

$$f(y-x) - g_1(x) - g_2(y) = h\,(1/x - 1/y) \qquad (x, y \in \mathbb{R}^+)$$

whenever $y - x > 0$, then over the domain $\mathbb{R}^+$ we have

$$f(x) = c\log(x) + dx + p\,, \qquad h(x) = c\log(x) + mx + r\,,$$

$$g_1(x) = c\log(x) - dx - m/x + p + q\,, \qquad g_2(x) = c\log(x) + dx + m/x - q - r\,,$$

where c, d, m, p, q and r are complex constants.

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