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SOME PROPERTIES OF TANGENT DIRAC STRUCTURES OF HIGHER ORDER

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Abstract. Let M be a smooth manifold. The tangent lift of Dirac structure on M was originally studied by T. Courant in [3]. The tangent lift of higher order of Dirac structure L on M has been studied in [10], where tangent Dirac structure of higher order are described locally. In this paper we give an intrinsic construction of tangent Dirac structure of higher order denoted by L^r and we study some properties of this Dirac structure. In particular, we study the Lie algebroid and the presymplectic foliation induced by L^r.

Introduction

Let M be a differential manifold of dimension $m > 0$, in this paper, we denote by $\langle \cdot,\cdot\rangle_M \colon TM \times_M T^*M \to \mathbb{R}$ the usual canonical pairing. In [2], is defined the natural symmetric and skew-symmetric pairings on $TM \oplus T^*M$ by:

$$\langle X \oplus \omega, Y \oplus \mu\rangle_+ = \frac{1}{2}\big(\omega(Y) + \mu(X)\big)$$

$$\langle X \oplus \omega, Y \oplus \mu\rangle_- = \frac{1}{2}\big(\omega(Y) - \mu(X)\big)\,.$$

An almost-Dirac structure, or a Dirac bundle, on a manifold M is a subbundle L of vector bundle $TM \oplus T^*M$ which is maximally isotropic under the symmetric pairing $\langle \cdot \mid \cdot\rangle_+$. We denote by ρ_M and ρ_M^* the natural projection of $TM \oplus T^*M$ onto TM and T^*M respectively. Clearly, $\rho_M(L)$ is a generalized distribution on M. We set

$$\rho_M(L)^* = \bigcup_{x\in M} \big(\rho_M(L_x)\big)^*\,.$$

In [2], is defined a 2-form $\Omega_L \colon \rho_M(L) \to \rho_M(L)^*$ such that:

$$\Omega_L\big(\rho_M(X,\omega)\big)\big(\rho_M(Y,\mu)\big) = \langle X \oplus \omega, Y \oplus \mu\rangle_- = \omega(Y)\,,$$

and the bilinear bracket operation on the sections of $(TM \oplus T^*M \to M)$ by:

$$[X \oplus \omega, Y \oplus \mu] = [X,Y] \oplus \big(\mathcal{L}_X\mu - \mathcal{L}_Y\omega + d(\langle X \oplus \omega, Y \oplus \mu\rangle_-)\big)\,.$$

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If $\Gamma(L)$ is closed under this bracket, the author of [2] has said that the almost-Dirac structure L is integrable or L is a Dirac structure on M. This condition is equivalent to $\mathbb{T}_L = 0$, where $\mathbb{T}_L$ is the restriction on L of 3-tensor $\mathbb{T}$ defined on $TM \oplus T^*M$ by:

$$\mathbb{T}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) = \langle [\mathbf{s}_1, \mathbf{s}_2], \mathbf{s}_3 \rangle_+ \,.$$

Where $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \in \Gamma(TM \oplus T^*M)$.

Theorem 1. *An almost-Dirac structure L is integrable if and only if $(L, [\cdot,\cdot], \rho_M|_L)$ is a Lie algebroid.*

By this theorem, T. Courant in [2] has shown that, if L is an integrable Dirac structure, then the generalized distribution $\rho_M(L)$ generates a generalized foliation on M and by the same way, we have:

Theorem 2. *An integrable Dirac structure has a foliation by presymplectic leaves.*

For the proof of these theorems, see [2].

In [10], we have defined the tangent lift of higher order L^r $(r \geq 1)$ of an almost-Dirac structure L on a manifold M, and we have shown that this lifting is an almost-Dirac structure on a manifold T^rM. We have shown that, L is integrable if and only if L^r is integrable. In this paper we study some properties of L^r namely the structures of Lie algebroid and generalized foliation induced by L^r. The main results of this paper are Theorems 3, 4, 5, 6 and Proposition 3.

All manifolds and maps are assumed to be infinitely differentiable. r will be a natural integer $(r \geq 1)$.

1. Tangent lifts of higher order of some tensor fields revisited

1.1. Prolongations of sections of vector bundle.

For all $\alpha \in \{0, \dots, r\}$, we denote by $\chi^{(\alpha)} \colon T^r \to T^r$ the natural transformation defined for all vector bundle (E, M, π) and $\Psi \in C^\infty(\mathbb{R}, E)$ by:

$$\chi_E^{(\alpha)}(j_0^r \Psi) = j_0^r(t^\alpha \Psi) \,.$$

Where $t^\alpha \Psi$ is the smooth map defined for all $t \in \mathbb{R}$ by: $(t^\alpha \Psi)(t) = t^\alpha \Psi(t)$.

Let $S \colon M \to E$ be a smooth section on E, we define the section $\overline{S}^{(\alpha)}$ of $(T^rE, T^rM, T^r\pi)$ by:

$$\overline{S}^{(\alpha)} = \chi_E^{(\alpha)} \circ T^rS \,, \qquad 0 \leq \alpha \leq r \,.$$

For the sake convenience we define $\overline{S}^{(\alpha)} = 0$ for all $\alpha > r$ or $\alpha < 0$.

Definition 1. This section $\overline{S}^{(\alpha)}$ of T^rE is called α-prolongation of order r of S.

Remark 1. Let (E, M, π) be a vector bundle and $\varphi \colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ a local trivialization of E over an open $U \subset M$. For $j = 1, \dots, n$, we put:

$$\varepsilon_j(x) = \varphi^{-1}(x, e_j) \quad \text{where} \quad x \in U \quad \text{and} \quad (e_j)_{j=1,\dots,n} \quad \text{is the usual basis of } \mathbb{R}^n.$$

$(\varepsilon_j)_{j=1,\dots,n}$ is a basis of sections of E over U associated to φ. Using the identification $T^r(U \times \mathbb{R}^n) = T^rU \times \mathbb{R}^{n(r+1)}$, we define a family of sections

$$(\varepsilon_j^\alpha), \qquad 1 \le j \le n, \quad 0 \le \alpha \le r$$

of T^rE over T^rU by:

$$\varepsilon_j^\alpha(\widetilde{x}) = T^r\varphi^{-1}(\widetilde{x}, e_j^\alpha)$$

where $\widetilde{x} \in T^rU$ and (e_j^α) the usual basis of $T^r\mathbb{R}^n = \mathbb{R}^{n(r+1)}$.
We have:

$$\varepsilon_j^\alpha = \overline{\varepsilon_j}^{(\alpha)}, \qquad \text{for all} \quad j = 1, \dots, n \quad \text{and} \quad \alpha = 0, \dots, r\,. \tag{1}$$

Proposition 1. *Let (E, M, π) be a vector bundle. If Ψ, Ψ' are two tensor fields of type $(0,p)$ on the vector bundle $(T^rE, T^rM, T^r\pi)$ such that for all smooth sections $S_1, \dots, S_p$ on E and $\alpha_1, \dots, \alpha_p \in \{0, 1, \dots, r\}$ the equality*

$$\Psi\big(\overline{S_1}^{(\alpha_1)}, \dots, \overline{S_p}^{(\alpha_p)}\big) = \Psi'\big(\overline{S_1}^{(\alpha_1)}, \dots, \overline{S_p}^{(\alpha_p)}\big)$$

holds, then $\Psi = \Psi'$.

Proof. See [5]. $\square$

For the prolongations of functions, vector fields and differential form of manifold M to manifold T^rM and related properties, see [5] or [11]. From now, we adopt the notations of [11].

1.2. **Prolongations of tensor fields of type (0, p).** Let (E, M, π) be a vector bundle and φ a tensor field of type $(0,p)$ on E. We interpret a tensor φ on E as a p-linear mapping $\varphi\colon E \times_M \cdots \times_M E \to \mathbb{R}$ of the bundle product over M of p-copies of E. For all $\alpha \in \{0, 1, \dots, r\}$, we denote by τ_α the linear form on $J_0^r(\mathbb{R}, \mathbb{R})$ defined by:

$$\tau_\alpha(j_0^r g) = \frac{1}{\alpha!}\frac{d^\alpha}{dt^\alpha}\big(g(t)\big)|_{t=0}\,.$$

We set:

$$\overline{\varphi}^{(\alpha)} = \tau_\alpha \circ T^r\varphi\,; \tag{2}$$

$\overline{\varphi}^{(\alpha)}$ is a tensor field of type $(0,p)$ on $(T^rE, T^rM, T^r\pi)$ called α-prolongation of φ from E to T^rE. When $\alpha = r$, it is denoted by $\overline{\varphi}^{(c)}$ called complete lift of φ from E to T^rE.

Proposition 2. *$\overline{\varphi}^{(\alpha)}$, $0 \le \alpha \le r$, is the only tensor field of type $(0,p)$ on T^rE satisfying:*

$$\overline{\varphi}^{(\alpha)}\big(\overline{S_1}^{(\alpha_1)}, \dots, \overline{S_p}^{(\alpha_p)}\big) = \big(\varphi(S_1, \dots, S_p)\big)^{(\alpha - \sum_{i=1}^{p} \alpha_i)}\,. \tag{3}$$

for all $S_1, \dots, S_p \in \Gamma(E)$ and $\alpha_1, \dots, \alpha_p \in \{0, 1, \dots, r\}$,

Proof. Let $j_0^r\eta \in T^rM$, we have:

$$\begin{aligned}
\overline{\varphi}^{(\alpha)}\big(\overline{S_1}^{(\alpha_1)}, \ldots, \overline{S_p}^{(\alpha_p)}\big)(j_0^r\eta) &= \overline{\varphi}^{(\alpha)}\big(\chi_E^{(\alpha_1)} \circ T^rS_1(j_0^r\eta), \ldots, \chi_E^{(\alpha_p)} \circ T^rS_p(j_0^r\eta)\big) \\
&= \overline{\varphi}^{(\alpha)}\big(j_0^r(t^{\alpha_1}S_1 \circ \eta), \ldots, j_0^r(t^{\alpha_p}S_p \circ \eta)\big) \\
&= \tau_\alpha\big(j_0^r\varphi(t^{\alpha_1}S_1 \circ \eta, \ldots, t^{\alpha_p}S_p \circ \eta)\big) \\
&= \tau_\alpha\big(j_0^r t^{\alpha_1+\cdots+\alpha_p}\varphi(S_1, \ldots, S_p) \circ \eta\big) \\
&= \big(t^{\alpha_1+\cdots+\alpha_p}\varphi(S_1, \ldots, S_p)\big)^{(\alpha)}(j_0^r\eta) \\
&= \big(\varphi(S_1, \ldots, S_p)\big)^{(\alpha - \sum_{i=1}^{p}\alpha_i)}(j_0^r\eta)
\end{aligned}$$

The unicity comes from the equation (1) and Proposition 1. □

2. Tangent Dirac structure of higher order

2.1. **Almost-Dirac structure of higher order.** We denote by $\alpha^r \colon T^* \circ T^r \to T^r \circ T^*$ and $\kappa^r \colon T^r \circ T \to T \circ T^r$ the natural transformations defined in [1] and [5], such that, for all manifold M, we have:

$$\langle \kappa_M^r(u), v^* \rangle_{T^rM} = \langle u, \alpha_M^r(v^*) \rangle'_{T^rM}\,, \qquad (u, v^*) \in T^rTM \oplus T^*T^rM\,,$$

where $\langle \cdot \mid \cdot \rangle'_{T^rM} = \tau_r \circ T^r\langle \cdot \mid \cdot \rangle_M$. Let L be an almost-Dirac structure on m-dimensional manifold defined locally by the bundle morphisms $a \colon U \times \mathbb{R}^m \to TM$ and $b \colon U \times \mathbb{R}^m \to T^*M$. (e_j) denote the canonical basis of $\mathbb{R}^m$. We set:

$$S_i \colon U \to L\,, \quad x \mapsto a(x, e_i) \oplus b(x, e_i)\,,$$

$(S_i)_{1 \le i \le n}$ is a basis of sections of L over U. In [10], we have showed that: the almost Dirac structure of order r L^r is determined by the maps a^r and b^r such that:

$$a^r = \kappa_M^r \circ T^r a \quad \text{and} \quad b^r = \varepsilon_M^r \circ T^r b\,;$$

where ε_M^r is the inverse map of α_M^r. The matrix form of a^r and b^r is given by:

$$a^r = \begin{pmatrix} a_j^i & \ldots & 0 \\ \vdots & \ddots & \vdots \\ (a_j^i)^{(r)} & \ldots & a_j^i \end{pmatrix} \qquad \text{and} \qquad b^r = \begin{pmatrix} (b_{ij})^{(r)} & \ldots & b_{ij} \\ \vdots & \iddots & \vdots \\ b_{ij} & \ldots & 0 \end{pmatrix}$$

So that,

$$L^r = (\kappa_M^r \oplus \varepsilon_M^r)(T^rL) \subset TT^rM \oplus T^*T^rM\,.$$

Theorem 3. *Let $X \oplus \omega \in \Gamma(L)$, for all $\alpha \in \{0, ..., r\}$, we have $X^{(\alpha)} \oplus \omega^{(r-\alpha)} \in \Gamma(L^r)$.*

Proof. If $(X, \omega) \in \Gamma(L)$ then, they are the maps $\gamma_1, \ldots, \gamma_m \in C^\infty(U)$ such that:

$$X \oplus \omega = \sum_{i=1}^{m} \gamma^i S_i.$$

In this case,

$$\begin{cases} X|_U = \gamma^i a_i^j \frac{\partial}{\partial x^j} \\ \omega|_U = \gamma^i b_{ij} dx^j \end{cases}$$

$$X^{(\alpha)} = (\gamma^i)^{(\nu)} (a_i^j)^{(\beta-\alpha-\nu)} \frac{\partial}{\partial x_\beta^j}\,.$$

We deduce that:

$$X^{(\alpha)} = \begin{pmatrix} a_i^j & 0 & \dots & 0 \\ \dot{a}_i^j & a_i^j & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ (a_i^j)^{(r)} & (a_i^j)^{(r-1)} & \dots & a_i^j \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \gamma^i \\ \vdots \\ (\gamma^i)^{(r-\alpha)} \end{pmatrix}$$

$$\omega^{(r-\alpha)} = (\gamma^i b_{ij})^{(r-\alpha-\beta)} dx_\beta^j = (\gamma^i)^{(r-\nu)} (b_{ij})^{(\nu-\alpha-\beta)} dx_\beta^j\,.$$

In the same way, we have:

$$\omega^{(r-\alpha)} = \begin{pmatrix} (b_{ij})^{(r)} & (b_{ij})^{(r-1)} & \dots & b_{ij} \\ (b_{ij})^{(r-1)} & (b_{ij})^{(r-2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{ij} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \gamma^i \\ \vdots \\ (\gamma^i)^{(r-\alpha)} \end{pmatrix}\,.$$

Thus that $(X^{(\alpha)}, \omega^{(r-\alpha)}) \in \Gamma(L^r)$. □

For all $X \oplus \omega \in \Gamma(TM \oplus T^*M) = \mathfrak{X}(M) \oplus \Omega^1(M)$, we set:

$$(X \oplus \omega)^{(\alpha)} = X^{(\alpha)} \oplus \omega^{(r-\alpha)}\,.$$

Corollary 1. *Let L be an almost-Dirac structure on M.*

(1) *For all $X \oplus \omega, Y \oplus \mu \in \mathfrak{X}(M) \oplus \Omega^1(M)$ and $\alpha, \beta = 0, \dots, r$, we have:*

$$\big[(X \oplus \omega)^{(\alpha)}, (Y \oplus \mu)^{(\beta)}\big] = [X \oplus \omega, Y \oplus \mu]^{(\alpha+\beta)}\,.$$

(2) *For all $f \in C^\infty(M)$ and $X \oplus \omega \in \mathfrak{X}(M) \oplus \Omega^1(M)$, we have:*

$$\big(f \cdot (X \oplus \omega)\big)^{(\alpha)} = \sum_{\beta=0}^{r-\alpha} f^{(\beta)} \cdot (X \oplus \omega)^{(\alpha+\beta)}\,.$$

(3) *For all $X \oplus \omega, Y \oplus \mu, Z \oplus \nu \in \Gamma(L)$, we have:*

$$\mathbb{T}_{L^r}\big((X \oplus \omega)^{(\alpha)}, (Y \oplus \mu)^{(\beta)}, (Z \oplus \nu)^{(\gamma)}\big) = \big(\mathbb{T}_L(X \oplus \omega, Y \oplus \mu, Z \oplus \nu)\big)^{(r-\alpha-\beta-\gamma)}\,,$$

for all $\alpha, \beta, \gamma \in \{0, 1, \dots, r\}$.

Proof. The proof comes of some properties of tangent lift of higher order of functions, vector fields and differential forms.

□

For all $S \in \Gamma(L)$ and $\alpha \in \{0,1,\dots,r\}$, we have:

$$(\kappa^r_M \oplus \varepsilon^r_M)(\overline{S}^{(\alpha)}) = S^{(\alpha)}\,.$$

Theorem 4. *$\overline{\mathbb{T}_L}^{(c)}$ is a complete lift of $\mathbb{T}_L$ from L to T^rL. We denote by η^r_M the inverse map of κ^r_M. We have:*

$$\mathbb{T}_{L^r} = \overline{\mathbb{T}_L}^{(c)} \circ \Big(\bigoplus^3 (\eta^r_M \oplus \alpha^r_M)\Big)\,. \tag{4}$$

Proof. $\mathbb{T}_{L^r} \circ \big(\bigoplus^3(\kappa^r_M \oplus \varepsilon^r_M)\big)$ is a tensor field of type $(0,3)$ on T^rL. Let $S_1, S_2, S_3 \in \Gamma(L)$ and $\alpha_1,\alpha_2,\alpha_3 \in \{0,1,\dots,r\}$, we have:

$$\begin{aligned}
\mathbb{T}_{L^r} \circ \bigoplus^3 (\kappa^r_M \oplus \varepsilon^r_M)\big(\overline{S_1}^{(\alpha_1)}, \overline{S_2}^{(\alpha_2)}, \overline{S_3}^{(\alpha_3)}\big) &= \mathbb{T}_{L^r}\big(S_1^{(\alpha_1)}, S_2^{(\alpha_2)}, S_3^{(\alpha_3)}\big)\\
&= \big(\mathbb{T}_L(S_1,S_2,S_3)\big)^{(r-\alpha_1-\alpha_2-\alpha_3)}\\
&= \overline{\mathbb{T}_L}^{(c)}\big(\overline{S_1}^{(\alpha_1)}, \overline{S_2}^{(\alpha_2)}, \overline{S_3}^{(\alpha_3)}\big)\,.
\end{aligned}$$

We have the result by the Proposition 2. □

Remark 2. The equation (4) shows that L is integrable if and only if L^r is integrable. Thus, we have given an intrinsic construction of tangent lift of higher order of an almost-Dirac structure, and we have shown independent of any local coordinates system that: this lifting is integrable if and only if the initial almost-Dirac structure is integrable.

Let $X \oplus \omega$, $Y \oplus \mu$ be sections of an almost-Dirac structure L. Define

$$X \bullet (Y \oplus \mu) = [X,Y] \oplus \mathcal{L}_X \mu\,.$$

Definition 2. L is said invariance under $X \oplus \omega \in \Gamma(L)$ if and only if $X \bullet L \subset L$. When L is integrable this is equivalent to say $d\omega|\rho_M(L) = 0$.

Corollary 2. *If L is an integrable Dirac structure invariant under $X \in \rho_M(\Gamma(L))$, then L^r is invariant under $X^{(\alpha)}$ for all $\alpha = 0,\dots,r$*

Proof. Let $X \oplus \omega \in \Gamma(L)$, we have $X^{(\alpha)} \oplus \omega^{(r-\alpha)} \in \Gamma(L^r)$ by the equality

$$d\omega^{(r-\alpha)} = (d\omega)^{(r-\alpha)} \quad \text{(see [11])},$$

we deduce that $d\omega^{(r-\alpha)}|\rho(L^r) = 0$. □

2.2. **Admissible functions of L^r.** Let L be an integrable Dirac structure over M. A function f is an admissible relatively to L, if there is vector field X_f such that $(X_f, df) \in \Gamma(L)$. If f and g are two admissible functions, T. Courant defines in [2] their bracket by:

$$\{f,g\} = X_f(g)\,.$$

Proposition 3. (1) *If f is an admissible function relatively to L, then $f^{(\alpha)}$ is an admissible function relatively to L^r and we have:*

$$X_{f^{(\alpha)}} = (X_f)^{(r-\alpha)}\,. \tag{5}$$

(2) *For all f, g two admissible functions, $\alpha, \beta = 0, \ldots, r$, we have:*

$$\{f^{(\alpha)}, g^{(\beta)}\} = \{f, g\}^{(\alpha+\beta-r)} . \tag{6}$$

Proof. (1) If f is an admissible function, then $(X_f, df) \in \Gamma(L)$. For all α,

$$\big((X_f)^{(r-\alpha)}, (df)^{(\alpha)}\big) \in \Gamma(L^r) .$$

Since $(df)^{(\alpha)} = df^{(\alpha)}$, it follows that $((X_f)^{(r-\alpha)}, df^{(\alpha)}) \in \Gamma(L^r)$. Thus, $f^{(\alpha)}$ is an admissible function relatively to L^r and $X_{f^{(\alpha)}} = (X_f)^{(r-\alpha)}$.

(2) For $\alpha, \beta = 0, \ldots, r$, we have:

$$\begin{aligned}\{f^{(\alpha)}, g^{(\beta)}\} &= X_{f^{(\alpha)}}(g^{(\beta)}) \\ &= (X_f)^{(r-\alpha)}(g^{(\beta)}) \\ &= \{f, g\}^{(\alpha+\beta-r)}\end{aligned}$$

□

2.3. **The Lie algebroid** $(L^r, [\cdot,\cdot], \rho_{T^rM}|_{L^r})$**.** For all $\alpha \in \{0, 1, \ldots, r\}$, consider the map

$$\chi^{(\alpha)}_{TM\oplus T^*M} \colon T^r(TM \oplus T^*M) \to T^r(TM \oplus T^*M) ,$$

we have:

$$\chi^{(\alpha)}_{TM\oplus T^*M} = \chi^{(\alpha)}_{TM} \oplus \chi^{(\alpha)}_{T^*M} .$$

In this case, $\chi^{(\alpha)}_L = \chi^{(\alpha)}_{TM} \oplus \chi^{(\alpha)}_{T^*M}|_{T^rL}$.

Proposition 4. *Let $(E, [\cdot,\cdot], \rho)$ be a Lie algebroid. There is one and only one Lie algebroid structure on T^rE such that: For all $S_1, S_2 \in \Gamma(E)$ and $\alpha, \beta \in \{0, 1, ..., r\}$*

$$\big[\overline{S_1}^{(\alpha)}, \overline{S_2}^{(\beta)}\big] = \overline{[S_1, S_2]}^{(\alpha+\beta)} .$$

The anchor map $\rho^{(r)}$ is given by:

$$\rho^{(r)} = \kappa^r_M \circ T^r\rho .$$

This Lie algebroid structure is called tangent lift of order r of Lie algebroid $(E, [\cdot,\cdot], \rho)$.

Proof. See [9]. □

Theorem 5. *Let L be an integrable Dirac structure on M. The tangent Lie algebroid of order r T^rL, is isomorphic to the Lie algebroid $(L^r, [\cdot], \rho_{T^rM}|_{L^r})$ over T^rM induced by the integrable Dirac structure L^r.*

Proof. Let (S_i) be a basis of sections of L over U.

$$S_i(x) = a(x, e_i) \oplus b(x, e_i) , \quad \forall i = 1, \ldots, m$$

we have $\kappa^r_M \oplus \varepsilon^r_M(\overline{S_i}^{(\alpha)}) = S_i^{(\alpha)}$.
The tangent Lie algebroid of order r T^rL is given by:

$$[\overline{S_i}^{(\alpha)}, \overline{S_j}^{(\beta)}] = \overline{[S_i, S_j]}^{(\alpha+\beta)}$$

$$\begin{aligned}[\kappa^r_M \oplus \varepsilon^r_M(\overline{S_i}^{(\alpha)}), \kappa^r_M \oplus \varepsilon^r_M(\overline{S_j}^{(\beta)})] &= [S_i^{(\alpha)}, S_j^{(\beta)}] = [S_i, S_j]^{(\alpha+\beta)} \\ &= \kappa^r_M \oplus \varepsilon^r_M\big(\overline{[S_i, S_j]}^{(\alpha+\beta)}\big).\end{aligned}$$

It follows that,

$$\kappa^r_M \oplus \varepsilon^r_M|_{T^rL} \colon T^rL \to L^r$$

is a Lie algebroids isomorphism. □

2.4. **Symplectic foliation induced by L^r.** For the tangent lift of higher order of singular foliation of manifold M to T^rM we can see [9]. However, let E be a smooth generalized distribution on M, we denote by $\mathfrak{X}_E$ the set of all local vector fields such that: for all $x \in M$, $X(x) \in E_x$. Let us notice that for a completely integrable distribution E, the family $\mathfrak{X}_E$ is a Lie subalgebra of the Lie algebra of vector fields on M.

Proposition 5. *Let E be a completely integrable generalized distribution on M. Then the distribution E^r generated by the family $\{X^{(\alpha)},\ X \in \mathfrak{X}_E,\ 0 \le \alpha \le r\}$ of vector fields on T^rM is completely integrable.*

Proof. See [9]. □

Let $\mathcal{F}$ be a generalized foliation defined by E, the tangent lift of order r of $\mathcal{F}$ denoted by $T^r\mathcal{F}$ is defined by E^r.

Proposition 6. *If a submanifold $F \subset M$ is a leaf of generalized foliation $\mathcal{F}$, then T^rF is a leaf of generalized foliation $T^r\mathcal{F}$.*

Proof. See [9]. □

By the Propositions 5 and 6, we deduce this result.

Theorem 6. *Let L be an integrable Dirac structure, $\mathcal{F}$ the generalized foliation induced by L and F a leaf of $\mathcal{F}$.*

1. *The generalized foliation induced by L^r is the tangent lift of order r of generalized foliation $\mathcal{F}$.*
2. *If Ω_F is a presymplectic form on F then $\Omega_F^{(c)}$ is a presymplectic form on the leaf T^rF. Where $\Omega_F^{(c)}$ is a complete lift of differential form Ω_F.*

Proof. Let $X, Y \in \rho_M(\Gamma(L))$ tangent to F, we have:

$$\begin{aligned}\Omega_{T^rF}(X^{(\alpha)}, Y^{(\beta)}) &= \omega^{(r-\alpha)}(Y^{(\beta)}) \\ &= \big(\omega(Y)\big)^{(r-\alpha-\beta)} \\ &= \big(\Omega_F(X, Y)\big)^{(r-\alpha-\beta)} \\ &= \Omega_F^{(c)}(X^{(\alpha)}, Y^{(\beta)})\end{aligned}$$

Thus $\Omega_{T^r F} = \Omega_F^{(c)}$. □

These results generalize the properties of tangent lifting of higher order of Poisson manifold.

References

[1] Cantrijn, F., Crampin, M., Sarlet, W., Saunders, D., *The canonical isomorphism between T^kT^* and T^*T^k*, C. R. Acad. Sci. Paris Sér. II **309** (1989), 1509–1514.

[2] Courant, T., *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (2) (1990), 631–661.

[3] Courant, T., *Tangent Dirac Structures*, J. Phys. A: Math. Gen. **23** (22) (1990), 5153–5168.

[4] Courant, T., *Tangent Lie Algebroids*, J. Phys. A: Math. Gen. **27** (13) (1994), 4527–4536.

[5] Gancarzewicz, J., Mikulski, W., Pogoda, Z., *Lifts of some tensor fields and connections to product preserving functors*, Nagoya Math. J. **135** (1994), 1–41.

[6] Grabowski, J., Urbanski, P., *Tangent lifts of poisson and related structure*, J. Phys. A: Math. Gen. **28** (23) (1995), 6743–6777.

[7] Kolář, I., *Functorial prolongations of Lie algebroids*, Proceedings of the 9th International Conference on Differential Geometry and its Applications, DGA 2004, Prague, Czech Republic, 2005, pp. 301–309.

[8] Kolář, I., Michor, P., Slovák, J., *Natural operations in differential geometry*, Springer–Verlag, 1993.

[9] Kouotchop Wamba, P. M., Ntyam, A., Wouafo Kamga, J., *Tangent lift of higher order of multivector fields and applications*, to appear.

[10] Kouotchop Wamba, P. M., Ntyam, A., Wouafo Kamga, J., *Tangent Dirac structures of higher order*, Arch. Math. (Brno) **47** (2011), 17–22.

[11] Morimoto, A., *Lifting of some type of tensors fields and connections to tangent bundles of p^r-velocities*, Nagoya Math. J. **40** (1970), 13–31.

[12] Ntyam, A., Wouafo Kamga, J., *New versions of curvatures and torsion formulas of complete lifting of a linear connection to Weil bundles*, Ann. Polon. Math. **82** (3) (2003), 233–240.

[13] Ntyam, A., Mba, A., *On natural vector bundle morphisms $T^A \circ \bigotimes_s^q \to \bigotimes_s^q \circ T^A$ over id_{T^A}*, Ann. Polon. Math. **96** (3) (2009), 295–301.

[14] Wouafo Kamga, J., *Global prolongation of geometric objets to some jet spaces*, International Centre for Theoretical Physics, Trieste, Italy, November 1997.

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