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ON THE GEOMETRY OF FRAME BUNDLES

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Abstract. Let (M,g) be a Riemannian manifold, $L(M)$ its frame bundle. We construct new examples of Riemannian metrics, which are obtained from Riemannian metrics on the tangent bundle TM. We compute the Levi–Civita connection and curvatures of these metrics.

1. Introduction

Let (M,g) be a Riemannian manifold, $L(M)$ its frame bundle. The first example of a Riemannian metric on $L(M)$ was considered by Mok [12]. This metric, called the Sasaki–Mok metric or the diagonal lift g^d of g, was also investigated in [5] and [6]. It is very rigid, for example, $(L(M),g^d)$ is never locally symmetric unless (M,g) is locally Euclidean. Moreover, with respect to the Sasaki–Mok metric vertical and horizontal distributions are orthogonal. A wider and less rigid class of metrics $\bar{g}$, in which vertical and horizontal distributions are no longer orthogonal, has been recently considered by Kowalski and Sekizawa in the series of papers [9, 10, 11]. These metrics are defined with respect to the decomposition of the vertical distribution $\mathcal{V}$ into $n=\dim M$ subdistributions $\mathcal{V}^1,\dots,\mathcal{V}^n$.

In this short paper we introduce a new class of Riemannian metrics on the frame bundle. We identify distributions $\mathcal{V}^i$ with the vertical distribution in the second tangent bundle TTM. Namely, each map $R_i\colon L(M)\to TM$, $R_i(u_1,\dots,u_n)=u_i$ induces a linear isomorphism $R_{i*}\colon \mathcal{H}\oplus\mathcal{V}^i\to TTM$, where $\mathcal{H}$ is a horizontal distribution defined by the Levi–Civita connection ∇ on M. By this identification we pull–back the Riemannian metric from TM. We pull–back natural metrics, in the sense of Kowalski and Sekizawa [8], from TM and study the geometry of such Riemannian manifolds. We compute the Levi–Civita connection, the curvature tensor, sectional and scalar curvature.

2. Riemannian metrics on frame bundles

Let (M,g) be a Riemannian manifold. Its frame bundle $L(M)$ consists of pairs (x,u) where $x=\pi_{L(M)}(u)\in M$ and $u=(u_1,\dots,u_n)$ is a basis of a tangent space

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T_xM. We will write u instead of (x,u). Let $(x_1,\dots,x_n)$ be a local coordinate system on M. Then, for every $i=1,\dots,n$, we have

$$u_i=\sum_j u_i^j\frac{\partial}{\partial x_j}$$

for some smooth functions u_i^j on $L(M)$. Putting $\alpha_i=x_i\circ\pi_{L(M)}$, (α_i,u_k^j) is a local coordinate system on $L(M)$. Let ω be a connection form of $L(M)$ corresponding to Levi–Civita connection ∇ on M. We have a decomposition of the tangent bundle $TL(M)$ into the *horizontal* and *vertical* distribution:

$$T_uL(M)=\mathcal{H}_u^{L(M)}\oplus\mathcal{V}_u^{L(M)}\,,$$

where $\mathcal{H}^{L(M)}=\ker\omega$ and $\mathcal{V}^{L(M)}=\ker\pi_{L(M)*}$. Let X_u^h denotes the horizontal lift of a vector $X\in T_xM$, $\pi_{L(M)}(u)=x$, to $\mathcal{H}_u^{L(M)}$.

Let $L_u\colon GL(n)\to L(M)$, $L_u(a)=ua$, be a left multiplication of $a\in GL(n)$ by a basis $u\in L(M)$. Let $A_u^*=L_{u*}(A)$ be a fundamental vertical vector corresponding to a matrix $A\in\mathrm{gl}(n)$.

Denote by $\mathcal{V}^i$ a linear subspace of $\mathcal{V}^{L(M)}$ spanned by fundamental vertical vectors A^*, where the matrix $A\in\mathrm{gl}(n)$ has only nonzero i-th column.

For an index $i=1,\dots,n$ define a map $R_i\colon L(M)\to TM$ as follows

$$R_i(u)=u_i\,,\quad u=(u_1,\dots,u_n)\in L(M)\,.$$

R_i is the right multiplication by a i-th vector of a canonical basis in $\mathbb{R}^n$.

We will need some basic facts about the second tangent bundle TTM. There is a decomposition of TTM into horizontal and vertical part, $T_\zeta TM=\mathcal{H}_\zeta^{TM}\oplus\mathcal{V}_\zeta^{TM}$, with respect to the connection map $K\colon TTM\to TM$ and the projection in the tangent bundle $\pi_{TM}\colon TM\to M$, see for example [7]. Let $X_\zeta^{h,TM}$ and $X_\zeta^{v,TM}$ denote the horizontal and vertical lifts to $T_\zeta TM$, $\zeta\in T_xM$, of a vector $X\in T_xM$, respectively.

Proposition 2.1. *The operator R_i has the following properties.*

(1) *R_{i*} is a linear isomorphism of $\mathcal{H}^{L(M)}$ onto $\mathcal{H}^{TM}$. Moreover,*

$$R_{\xi*}X^h=X^{h,TM}\,.$$

(2) *R_{i*} is a linear isomorphism of $\mathcal{V}^i$ onto $\mathcal{V}^{TM}$ and R_{i*} is identically equal zero on $\mathcal{V}^j$ for $j\neq i$.*

(3) *There is a decomposition*

$$\mathcal{V}^{L(M)}=\mathcal{V}^1\oplus\ldots\oplus\mathcal{V}^n\,.$$

Proof. Easy computations left to the reader. □

By Proposition 2.1, we have natural identifications

$$\begin{array}{ccccc}\mathcal{H}^{L(M)} & \longleftrightarrow & \mathcal{H}^{TM} & \longleftrightarrow & TM\\ X^h & \longleftrightarrow & X^{h,TM} & \longleftrightarrow & X\end{array}\tag{2.1}$$

and

$$\begin{array}{ccccc} \mathcal{V}^i & \longleftrightarrow & \mathcal{V}^{TM} & \longleftrightarrow & TM \\ X^{v,i} & \longleftrightarrow & X^{v,TM} & \longleftrightarrow & X \end{array} \tag{2.2}$$

Hence, we have defined the vertical lift $X^{v,i}_u \in \mathcal{V}^i_u$, $u \in L(M)$, of the vector $X \in T_xM$, $\pi_{L(M)}(u) = x$, satisfying the property

$$R_{i*}X^{v,i}_u = X^{v,TM}_{u_i}\,.$$

Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ and $C = (c_{ij})$ be $n \times n$ matrix. We assume that the $(n+1) \times (n+1)$ matrix

$$\bar{C} = \begin{pmatrix} 1 & c \\ c^\top & C \end{pmatrix}$$

is symmetric and positive definite. Let g_{TM} be a Riemannian metric on TM.

Now, we are able to define a new class of Riemannian metrics $\bar{g} = \bar{g}_{\bar{C}}$ on $L(M)$. Let $F\colon L(M) \to TM$ be any smooth function. Put

$$\begin{aligned} \bar{g}(X^h, Y^h)_u &= g_{TM}(X^{h,TM}, Y^{h,TM})_{F(u)}\,, \\ \bar{g}(X^h, Y^{v,i})_u &= c_i g_{TM}(X^{h,TM}, Y^{v,TM})_{F(u)}\,, \\ \bar{g}(X^{v,i}, Y^{v,j})_u &= c_{ij} g_{TM}(X^{v,TM}, Y^{v,TM})_{F(u)}\,. \end{aligned}$$

Fix $u \in L(M)$. Let $e_1, \dots, e_n$ be a basis in T_xM, $\pi_{L(M)}(u) = x$, such that $(e_1)^{h,TM}_{F(u)}, \dots, (e_1)^{h,TM}_{F(u)}$ is an orthonormal basis in $\mathcal{H}^{TM}_{F(u)}$. Then

$$e^h_1, \dots, e^h_n, e^{v,1}_1, \dots, e^{v,1}_n, \dots, e^{v,n}_1, \dots, e^{v,n}_n \tag{2.3}$$

is a basis in $T_uL(M)$. Let G be a matrix of the Riemannian metric g_{TM} with respect to the basis $e^{h,TM}_1, \dots, e^{h,TM}_n, e^{v,TM}_1, \dots, e^{v,TM}_n$. The fact that $\bar{g}$ is positive definite follows from the following lemma.

Lemma 2.2. *Let*

$$G = \begin{pmatrix} I & g^{hv} \\ g^{vh} & \hat{g} \end{pmatrix}$$

be a positive definite symmetric $2n \times 2n$ block matrix. Then the matrix

$$\bar{G} = \begin{pmatrix} I & c \otimes g^{vh} \\ c^\top \otimes g^{hv} & C \otimes \hat{g} \end{pmatrix}$$

is positive definite.

Proof. It suffices to show that each principal minor $\bar{G}_k$, $k = 1, \dots, n + n^2$, of $\bar{G}$ is positive. Obviously $\det \bar{G}_k = 1 > 0$ for $k = 1, \dots, n$. Hence we assume $k > n$. Then each minor $\bar{G}_k$ is of the same form as the whole matrix $\bar{G}$, thus we will make calculations using matrix $\bar{G}$. Computing the determinant of the block matrix we get

$$\begin{aligned} \det \bar{G} &= \det\left(C \otimes \hat{g} - (c^\top \otimes g^{vh})(c \otimes g^{hv})\right) \\ &= \det\left(C \otimes \hat{g} - (c^\top c) \otimes (g^{vh} g^{hv})\right) \\ &= \det\left((C - c^\top c) \otimes \hat{g} + (c^\top c) \otimes (\hat{g} - g^{vh} g^{hv})\right). \end{aligned}$$

Since

$$\begin{aligned}
&\det\big(C - c^\top c\big) = \det \bar{C} > 0\,,\\
&\det \hat{g} > 0\,,\\
&\det\big(c^\top c\big) \geq 0\,,\\
&\det\big(\hat{g} - g^{vh} g^{hv}\big) = \det G > 0\,,
\end{aligned}$$

it follows that matrices $(C - c^\top c) \otimes \hat{g}$ and $(c^\top c) \otimes (\hat{g} - g^{vh} g^{hv})$ are positive definite. Hence theirs sum is positive definite. □

If $\bar{C} = I$ and g_{TM} is the Sasaki metric, then we get Sasaki–Mok metric.

Assume now $\bar{C} = I$ and g_{TM} is a natural Riemannian metric on TM [8, 1] such that $g_{TM}(X^h, Y^h) = g(X, Y)$ and distributions $\mathcal{H}^{TM}$, $\mathcal{V}^{TM}$ are orthogonal. Hence, there are two smooth real functions $\alpha, \beta : [0, \infty) \to \mathbb{R}$ such that

$$\begin{aligned}
\bar{g}(X^h, Y^h)_u &= g(X, Y)\,,\\
\bar{g}(X^h, Y^{v,i})_u &= 0\,,\\
\bar{g}(X^{v,i}, Y^{v,j})_u &= 0\,, \qquad i \neq j\,,\\
\bar{g}(X^{v,i}, Y^{v,i})_u &= \alpha\big(|F(u)|^2\big) g(X, Y)\\
&\quad + \beta\big(|F(u)|^2\big) g\big(X, F(u)\big) g\big(Y, F(u)\big)\,.
\end{aligned} \tag{2.4}$$

The above Riemannian metric does not "see" the index i of the distribution $\mathcal{V}^i$. Since all distributions $\mathcal{H}^{L(M)}, \mathcal{V}^1, \ldots, \mathcal{V}^n$ are orthogonal, it follows that we may put $F_i(u) = u_i$ in the last condition, that is consider a family of maps $F_1, \ldots, F_n$ rather than one map F. Then we obtain the positive definite bilinear form, hence the Riemannian metric, of the form

$$\begin{aligned}
\bar{g}(X^h, Y^h)_u &= g(X, Y)\,,\\
\bar{g}(X^h, Y^{v,i})_u &= 0\,,\\
\bar{g}(X^{v,i}, Y^{v,j})_u &= 0\,, \qquad i \neq j\,,\\
\bar{g}(X^{v,i}, Y^{v,i})_u &= \alpha(|u_i|^2) g(X, Y) + \beta(|u_i|^2) g(X, u_i) g(Y, u_i)\,.
\end{aligned} \tag{2.5}$$

Now, we define functions $\alpha_i, \alpha_i' \colon L(M) \to \mathbb{R}$ etc. as follows

$$\alpha_i(u) = \alpha(|u_i|^2)\,, \qquad \alpha_i'(u) = \alpha'(|u_i|^2)\,, \quad \text{etc.}$$

To make the formulas in the next section more concise, we will write α_i, α_i' etc. instead of $\alpha(|u_i|^2)$, $\alpha'(|u_i|^2)$ etc.

3. Geometry of $\bar{g}$

Let (M, g) be a Riemannian manifold, $(L(M), \bar{g})$ its frame bundle equipped with the metric $\bar{g}$ of the form (2.5). Let $\bar{\nabla}$ and $\bar{R}$ denote the Levi–Civita connection and the curvature tensor of $\bar{g}$, respectively.

We recall the identities concerning Lie bracket of horizontal and vertical vector fields [9]

$$
\begin{aligned}
[X^h, Y^h]_u &= [X,Y]^h_u - \sum_i (R(X,Y)u_i)^{v,i}\ ,\\
[X^h, Y^{v,i}]_u &= (\nabla_X Y)^{v,i}_u\ ,\\
[X^{v,i}, Y^{v,j}]_u &= 0\,.
\end{aligned}
\tag{3.1}
$$

Moreover, in the local coordinates, for $X = \sum_i \xi_i \frac{\partial}{\partial x_i}$ we have

$$X^h(u_i^j) = -\sum_{a,b} \Gamma^j_{ab} u_i^a \xi_b \tag{3.2}$$

$$X^{v,k}(u_i^j) = \xi_j \delta_{ik} \tag{3.3}$$

where Γ^j_{ab} are Christoffel's symbols [9].

Proposition 3.1. *Connection $\bar\nabla$ satisfies the following relations*

$$
\begin{aligned}
\left(\bar\nabla_{X^h} Y^h\right)_u &= (\nabla_X Y)^h_u - \frac{1}{2}\sum_i (R(X,Y)u_i)^{v,i}_u\\
\left(\bar\nabla_{X^h} Y^{v,i}\right)_u &= \frac{\alpha_i}{2}\,(R(u_i,Y)X)^h_u + (\nabla_X Y)^{v,i}_u\\
\left(\bar\nabla_{X^{v,i}} Y^h\right)_u &= \frac{\alpha_i}{2}\,(R(u_i,X)Y)^h_u\\
\left(\bar\nabla_{X^{v,i}} Y^{v,j}\right)_u &= 0 \quad i \neq j,\\
\left(\bar\nabla_{X^{v,i}} Y^{v,i}\right)_u &= \frac{\alpha_i'}{\alpha_i}\left(g(X,u_i)Y^{v,i}_u + g(Y,u_i)X^{v,i}_u\right)\\
&\quad + \Big(\frac{\beta_i'\alpha_i - 2\alpha_i'\beta_i}{\alpha_i(\alpha_i + |u_i|^2\beta_i)} g(X,u_i)g(Y,u_i) + \frac{\beta_i - \alpha_i'}{\alpha_i + |u_i|^2\beta_i} g(X,Y)\Big) U^i_u,
\end{aligned}
$$

where $U^i_u = u_i^{v,i}$.

Proof. Follows from the formula for the Levi–Civita connection

$$
\begin{aligned}
2\bar g(\bar\nabla_A B, C) &= A\bar g(B,C) + B\bar g(A,C) - C\bar g(A,B)\\
&\quad + \bar g([A,C],B) + \bar g([B,C],A) + \bar g([A,B],C)
\end{aligned}
$$

relations (3.1) and the following equalities

$$
\begin{aligned}
X^{v,i}_u(g(u_i,Y)) &= g(X,Y),\\
X^{v,i}_u(|u_i|^2) &= 2g(X,u_i),\\
X^h_u(g(u_i,Y)) &= g(u_i,\nabla_X Y)\,.
\end{aligned}
$$

□

Before we compute the curvature tensor, we will need some formulas concerning the Levi–Civita connection $\bar\nabla$ of certain vector fields.

Lemma 3.2. *The following equalities hold*

$$
\begin{aligned}
\big(\bar\nabla_{X^h}U^i\big)_u &= 0\,,\\
\big(\bar\nabla_{X^{v,i}}U^j\big)_u &= 0 \qquad i\neq j\,,\\
\big(\bar\nabla_{X^{v,i}}U^i\big)_u &= \frac{\alpha_i+|u_i|^2\alpha_i'}{\alpha_i}X_u^{v,i}+\frac{|u_i|^2(\alpha_i\beta_i'-\alpha_i'\beta_i)+\alpha_i\beta_i}{\alpha_i(\alpha_i+|u_i|^2\beta_i)}g(X,u_i)U_u^i\,.
\end{aligned}
$$

and

$$
\big(\bar\nabla_W(R(u_i,X)Y)^Q\big)_u=\sum_j W(u_i^j)(R(u_i,X)Y)_u^Q+\sum_j u_i^j\Big(\bar\nabla_W(R(\frac{\partial}{\partial x_j},X)Y)^Q\Big)_u
$$

for any $W\in TL(M)$, where Q denotes the horizontal or vertical lift.

Proof. Follows by standard computations in local coordinates. □

Proposition 3.3. *The curvature tensor $\bar R$ at $u\in L(M)$ satisfies the following relations*

$$
\begin{aligned}
\bar R(X^h,Y^h)Z^h = \big(R(X,Y)Z\big)^h &+ \frac12\sum_i\big((\nabla_Z R)(X,Y)u_i\big)^{v,i}\\
&-\frac14\sum_i\alpha_i\,\big(R(u_i,R(Y,Z)u_i)X-R(u_i,R(X,Z)u_i)Y\\
&-2R(u_i,R(X,Y)u_i)Z\big)^h\,,
\end{aligned}
$$

$$
\begin{aligned}
\bar R(X^h,Y^h)Z^{v,i} = (R(X,Y)Z)^{v,i} &+ \frac{\alpha_i}{2}\big((\nabla_X R)(u_i,Z)Y-(\nabla_Y R)(u_i,Z)X\big)^h\\
&-\frac{\alpha_i}{4}\sum_j\big(R(X,R(u_i,Z)Y)u_j-R(Y,R(u_i,Z)X)u_j\big)^{v,j}\\
&+\frac{\alpha_i'}{\alpha_i}g(Z,u_i)\big(R(X,Y)u_i\big)^{v,i}-\frac{\beta_i-\alpha_i'}{\alpha_i+|u_i|^2\beta_i}g\big(R(X,Y)Z,u_i\big)U^i\,,
\end{aligned}
$$

$$
\begin{aligned}
\bar R(X^h,Y^{v,i})Z^h = \frac{\alpha_i}{2}\big((\nabla_X R)(u_i,Y)Z\big)^h &- \frac12\big(R(Z,X)Y\big)^{v,i}\\
&+\frac{\alpha_i'}{2\alpha_i}g(Y,u_i)\big(R(X,Z)u_i\big)^{v,i}-\frac{\alpha_i}{4}\sum_j\big(R(X,R(u_i,Y)Z)u_j\big)^{v,j}\\
&-\frac{\beta_i-\alpha_i'}{2(\alpha_i+|u_i|^2\beta_i)}g\big(R(X,Z)Y,u_i\big)U^i\,,
\end{aligned}
$$

$$
\begin{aligned}
\bar{R}(X^h, Y^{v,i})Z^{v,j} &= -\frac{\alpha_i\alpha_j}{4}\big(R(u_i,Y)R(u_j,Z)X\big)^h\\
\bar{R}(X^h, Y^{v,i})Z^{v,i} &= \frac{\alpha_i'}{2}\big(g(Z,u_i)R(u_i,Y)X - g(Y,u_i)R(u_i,Z)X\big)^h\\
&\quad - \frac{\alpha_i^2}{4}\big(R(u_i,Y)R(u_i,Z)X\big)^h - \frac{\alpha_i}{2}\big(R(Y,Z)X\big)^h\\
\bar{R}(X^{v,i}, Y^{v,i})Z^h &= \alpha_i\big(R(X,Y)Z\big)^h\\
&\quad + \frac{\alpha_i^2}{4}\big(R(u_i,X)R(u_i,Y)Z - R(u_i,Y)R(u_i,X)Z\big)^h\\
&\quad + \alpha_i'\big(g(X,u_i)(R(u_i,Y)Z)^h - g(Y,u_i)(R(u_i,X)Z)^h\big)\\
\bar{R}(X^{v,i}, Y^{v,j})Z^h &= \frac{\alpha_i\alpha_j}{4}\big(R(u_i,X)R(u_j,Y)Z - R(u_j,Y)R(u_i,X)Z\big)^h\\
\bar{R}(X^{v,i}, Y^{v,i})Z^{v,i}) &= C_i\big(g(X,u_i)g(Y,Z) - g(Y,u_i)g(X,Z)\big)U^i\\
&\quad + \big(A_i g(Y,u_i)g(Z,u_i) + B_i g(Y,Z)\big)X^{v,i}\\
&\quad - \big(A_i g(X,u_i)g(Z,u_i) + B_i g(X,Z)\big)Y^{v,i}\\
\bar{R}(X^{v,i}, Y^{v,j})Z^{v,k} &= 0 \qquad \textit{if} \ \ \sharp\{i,j,k\} > 1
\end{aligned}
$$

where

$$
\begin{aligned}
A_i &= \frac{3(\alpha_i')^2 - 2\alpha_i\alpha_i''}{\alpha_i^2} + \frac{(\alpha_i\beta_i' - 2\alpha_i'\beta_i)(\alpha_i + |u_i|^2\alpha_i')}{\alpha_i^2(\alpha_i + |u_i|^2\beta_i)},\\
B_i &= \frac{\alpha_i\beta_i - 2\alpha_i\alpha_i' - (\alpha_i')^2|u_i|^2}{\alpha_i(\alpha_i + |u_i|^2\beta_i)},\\
C_i &= -\frac{2\alpha_i''}{\alpha_i + |u_i|^2\beta_i}\\
&\quad + \frac{3\alpha_i(\alpha_i')^2 + 2(\alpha_i')^2\beta_i|u_i|^2 + \alpha_i^2\beta_i' - \alpha_i\beta_i^2 + \alpha_i\alpha_i'\beta_i'|u_i|^2}{\alpha_i(\alpha_i + |u_i|^2\beta_i)^2)}
\end{aligned}
$$

Proof. Follows from the characterization of the Levi–Civita connection $\bar{\nabla}$ and Lemma 3.2. □

Remark 3.4. Notice that

$$A_i\alpha_i - B\beta_i = C_i(\alpha_i + |u_i|^2\beta_i)\,,$$

which is equivalent to the condition

$$\bar{g}(\bar{R}(X^{v,i}, Y^{v,i})Z^{v,i}, W^{v,i}) = \bar{g}\big(\bar{R}(Z^{v,i}, W^{v,i})X^{v,i}, Y^{v,i}\big)\,.$$

Corollary 3.5. *Let X, Y be two orthonormal vectors in the tangent space T_xM. Then the scalar curvature $\bar{K}$ of $(L(M),\bar{g})$ at $u \in L(M)$, $\pi_{L(M)}(u) = x$, and K of*

(M,g) at $x \in M$ are related as follows

$$\begin{aligned}
\bar{K}(X^h, Y^h) &= K(X,Y) - \frac{3}{4}\sum_i \alpha_i |R(X,Y)u_i|^2\,,\\
\bar{K}(X^h, Y^{v,i}) &= \frac{\alpha_i^2}{4(\alpha_i + \beta_i g(Y,u_i)^2)}|R(u_i,Y)X|^2\,,\\
\bar{K}(X^{v,i}, Y^{v,i}) &= \frac{A_i(g(X,u_i)^2 + g(Y,u_i)^2) + B_i}{\alpha_i + \beta_i(g(X,u_i)^2 + g(Y,u_i)^2)}\,,\\
\bar{K}(X^{v,i}, Y^{v,j}) &= 0 \qquad i \neq j\,.
\end{aligned}$$

Corollary 3.6. *If (M,g) is of constant sectional curvature κ, then*

$$\begin{aligned}
\bar{K}(X^h, Y^h) &= \kappa - \frac{3}{4}\kappa^2 \sum_i \alpha_i\big(g(X,u_i)^2 + g(Y,u_i)^2\big)\,,\\
\bar{K}(X^h, Y^{v,i}) &= \frac{\kappa^2\alpha_i^2 g(X,u_i)^2}{4(\alpha_i + \beta_i g(Y,u_i))} \geq 0\,.
\end{aligned}$$

If, in addition, $\sum_i \alpha(t_i)t_i < \frac{4}{3\kappa}$ for all $t_i > 0$, then $\bar{K}(X^h,Y^h) > 0$.

Proof of Corollary 3.5 and 3.6. The formula for $\bar{K}$ follows by Proposition 3.3. Assume now (M,g) is of constant sectional curvature κ and $\sum_i \alpha(t_i)t_i < \frac{4}{3\kappa}$ for all $t_i > 0$. Since $g(X,u_i)^2 + g(Y,u_i)^2 \leq |u_i|^2$, then

$$\bar{K}(X^h, Y^h) \geq \kappa - \frac{3}{4}\kappa^2 \sum_i \alpha_i |u_i|^2 > 0\,. \qquad \square$$

Corollary 3.7. *The scalar curvature $\bar{s}$ of $(L(M),\bar{g})$ at $u \in L(M)$ is of the form*

$$\bar{s} = s - \frac{1}{4}\sum_{i,j,k}\alpha_k |R(e_i,e_j)u_k|^2 + (n-1)\sum_k \frac{2|u_k|^2 C_k + nB_k}{\alpha_k}\,.$$

where s is the scalar curvature of (M,g) at $x \in M$ and $e_1,\ldots,e_n$ is an orthonormal basis in T_xM, $\pi_{L(M)}(u) = x$.

Proof. Fix $u \in L(M)$ and let $e_1,\ldots,e_n$ be an orthonormal basis in T_xM, $\pi_{L(M)}(u) = x$. Consider a basis of $T_uL(M)$ of the form (2.3). Put

$$\bar{g}^k_{ij} = \bar{g}(e_i^{v,k}, e_j^{v,k}) = \alpha_k\delta_{ij} + \beta_k g(e_i,u_k)g(e_j,u_k)\,.$$

The inverse matrix $(\bar{g}^{ij}_k)$ to $(\bar{g}^k_{ij})$ is the following

$$\bar{g}^{ij}_k = \frac{1}{\alpha_k}\delta_{ij} - \frac{\beta_k}{\alpha_k(\alpha_k + |u_k|^2\beta_k)}g(e_i,u_k)g(e_j,u_k)\,.$$

Hence

$$\begin{aligned}\bar{s} = \sum_{i,j} \bar{g}(\bar{R}(e_i^h, e_j^h)e_j^h, e_i^h) + 2 \sum_{i,j,l,k} \bar{g}_k^{jl} \bar{g}(\bar{R}(e_i^h, e_j^{v,k})e_l^{v,k}, e_i^h) \\ + \sum_{i,j,k,l,p} \bar{g}_k^{ip} \bar{g}_k^{jl} \bar{g}(\bar{R}(e_i^{v,k}, e_j^{v,k})e_l^{v,k}, e_p^{v,k})\end{aligned}$$

The formula for $\bar{s}$ follows now by Proposition 3.3, Remark 3.4 and the equality

$$\sum_{i,j} |R(e_i, e_j)u_k|^2 = \sum_{i,j} |R(u_k, e_j)e_i)|^2 \,. \qquad \square$$

In the end, we show that, in the case of a Cheeger–Gromoll type metric over the manifold of constant sectional curvature, the sectional curvature of $L(M)$ is nonnegative.

Corollary 3.8. *Assume*

$$\alpha(t) = \beta(t) = \frac{1}{1+t}\,, \qquad t > 0\,.$$

Then

$$\bar{K}(X^{v,i}, Y^{v,i}) = \frac{-|u_i|^2(g(X,u_i)^2 + g(Y,u_i)^2) + |u_i|^4 + 3|u_i|^2 + 3}{(1+|u_i|^2)^2(1 + g(X,u_i)^2 + g(Y,u_i)^2)}\,.$$

In particular, if (M,g) is of constant sectional curvature $0 < \kappa < \frac{4}{3n}$, then the sectional curvature $\bar{K}$ is nonnegative.

Proof. We have

$$\sum_i \alpha(t_i)t_i = \sum_i \frac{t_i}{1+t_i} < \frac{4}{3\kappa} \quad \text{for all } t_i > 0$$

if and only if $0 < \kappa < \frac{4}{3n}$. Hence, by Corollary 3.6, $\bar{K}(X^h, Y^h) \geq 0$ for $X, Y \in T_xM$ unit and orthogonal. Moreover, $g(X,u_i)^2 + (Y,u_i)^2 \leq |u_i|^2$. Thus

$$\bar{K}(X^{v,i}, Y^{v,i}) \geq \frac{-|u_i|^4 + |u_i|^4 + 3|u_i|^2 + 3}{|u_i|^2(1+|u_i|^2)^2} = \frac{3}{|u_i|^2(1+|u_i|^2)} > 0. \qquad \square$$

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