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ON μ-SINGULAR AND μ-EXTENDING MODULES

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Abstract. Let M be a module and μ be a class of modules in $\mathrm{Mod}-R$ which is closed under isomorphisms and submodules. As a generalization of essential submodules Özcan in [8] defines a μ-essential submodule provided it has a non-zero intersection with any non-zero submodule in μ. We define and investigate μ-singular modules. We also introduce μ-extending and weakly μ-extending modules and mainly study weakly μ-extending modules. We give some characterizations of μ-co-H-rings by weakly μ-extending modules. Let R be a right non-μ-singular ring such that all injective modules are non-μ-singular, then R is right μ-co-H-ring if and only if R is a QF-ring.

1. Introduction

Let R be a ring with identity. All modules we consider are unitary right R-modules and we denote the category of all such modules by $\mathrm{Mod}-R$.

Let μ be a class of modules. For any module M the *trace of μ in M* is denoted by $\mathrm{Tr}(\mu, M) = \sum\{\mathrm{Im}\, f\colon f \in \mathrm{Hom}(C, M), C \in \mu\}$. Dually the *reject of μ in M* is denoted by $\mathrm{Rej}(M, \mu) = \bigcap\{\mathrm{Ker}\, f : f \in \mathrm{Hom}(M, C), C \in \mu\}$.

Let N be a submodule of M ($N \leq M$). The notations $N \ll M$, $N \leq_e M$ and $N \leq_d M$ is used for a small submodule, an essential submodule and a direct summand of M, respectively. $\mathrm{Soc}(M)$ will denote the socle of M. An R-module M is said to be *small*, if $M \cong L \ll K$ for some R-modules L and K. Dually, M is called *singular* if $M \cong N/K$ such that $K \leq_e N$. Every module M contains a largest singular submodule which is denoted by $Z(M)$. Then $Z(M) = \mathrm{Tr}(\mathcal{U}, M)$ where $\mathcal{U}$ denotes the class of all singular modules.

Simple modules split into four disjoint classes by combining the exclusive choices [injective or small] and [projective or singular]. Also note that if a module M is singular and projective, then it is zero.

Talebi and Vanaja in [10], define cosingular modules as a dual of singular modules. Let M be a module and $\mathcal{M}$ denotes the class of all small modules. Then $\overline{Z}(M) = \bigcap\{\ker g \mid g \in \mathrm{Hom}(M, L), L \in \mathcal{M}\}$ is a submodule of M. Then M is called *cosingular* (*non-cosingular*) if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Every small module

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is cosingular. The class of all cosingular modules is closed under submodules, direct sums and direct products ([10, Corollary 2.2]).

In Section 2, we give the definition of μ-singular modules and discuss some properties of such modules. It is proved that R is a GCO-ring (i.e. every simple singular module is injective) if and only if for every $\mathcal{M}$-singular module N, $\overline{Z}(N) = N$ if and only if for every δ-singular module N, $\overline{Z}(N) = N$ (Corollary 2.14). When we consider the class of all finitely cogenerated modules $\mathcal{FC}$ we prove that every finitely cogenerated R-module is projective if and only if for every $\mathcal{FC}$-singular R-module N, $\mathrm{Rej}(N, \mathcal{FC}) = N$ if and only if R is semisimple Artinian (Corollary 2.16).

In Section 3, we define μ-extending and weakly μ-extending modules and show that any direct summand of a weakly μ-extending module and any homomorphic image of a weakly μ-extending module are weakly μ-extending modules (Proposition 3.12 and Corollary 3.13).

In Section 4, we discuss when a direct sum of weakly μ-extending modules is a weakly μ-extending module. We show that a direct sum of a μ-singular module and a semisimple module is weakly μ-extending (Theorem 4.2).

In Section 5, we study rings in which every projective module is μ-extending. We call such rings *μ-co-H-ring*. We show that a ring R is μ-co-H-ring if and only if every R-module is weakly μ-extending (Theorem 5.3). Let R be a right non-μ-singular ring such that all injective modules are non-μ-singular, then R is right μ-co-H-ring if and only if R is a QF-ring (Corollary 5.6).

In this paper μ will be a class in $\mathrm{Mod}-R$ which is closed under isomorphisms and submodules, unless otherwise stated. We shall call any member of μ, a μ-module. In this article we denote the following classes:

$\mathcal{S} = \{M \in \mathrm{Mod}-R, M \text{ is simple}\}$,
$\mathcal{M} = \{M \in \mathrm{Mod}-R, M \text{ is small}\}$,
$\delta = \{M \in \mathrm{Mod}-R, \overline{Z}(M) = 0\}$,
$\mu - \mathrm{Sing} = \{M \in \mathrm{Mod}-R, M \text{ is } \mu\text{-singular}\}$,
$\mathcal{FC} = \{M \in \mathrm{Mod}-R, M \text{ is finitely cogenerated}\}$.

2. μ-SINGULAR MODULES

Özcan in [8], investigate some properties of μ-essential submodules. Let M be a module and $N \leq M$. Then N is called a *μ-essential* submodule of M, denoted by $N \leq_{\mu e} M$, if $N \cap K \neq 0$ for any nonzero submodule K of M such that $K \in \mu$. Now we list the properties of μ-essential submodules. We omit the proofs because they are similar to those for essential submodules (see, [4]).

Lemma 2.1. *Let M be a module.*
a) *Let $N \leq L \leq M$. Then $N \leq_{\mu e} M$ if and only if $N \leq_{\mu e} L \leq_{\mu e} M$.*
b) *If $K_1 \leq_{\mu e} L_1 \leq M$, $K_2 \leq_{\mu e} L_2 \leq M$, then $K_1 \cap K_2 \leq_{\mu e} L_1 \cap L_2$.*
c) *If $f\colon N \to M$ is a homomorphism and $K \leq_{\mu e} M$, then $f^{-1}(K) \leq_{\mu e} N$.*
d) *If $N/L \leq_{\mu e} K/L \leq M/L$, then $N \leq_{\mu e} K$.*
e) *Let K_i $(i \in I)$ be an independent family of submodules of M. If $K_i \leq_{\mu e} L_i \leq M$ for all $i \in I$, then $\bigoplus_{i\in I} K_i \leq_{\mu e} \bigoplus_{i\in I} L_i$.*

Definition 2.2. Let M be a module. M is called *μ-singular* if $M \cong K/L$ such that $L \leq_{\mu e} K$.

Every module M contains a largest μ-singular submodule which we denote by $Z_\mu(M) = \mathrm{Tr}(\mu - \mathrm{Sing}, M)$ where $\mu - \mathrm{Sing}$ is the class of all μ-singular modules. Then $Z(M) \leq Z_\mu(M)$. If M is a μ-singular module (i.e. $Z_\mu(M) = M$) and a μ-module, then M is singular. For, let $M \in \mu$ and $M \cong K/L$ where $L \leq_{\mu e} K$. We claim that $L \leq_e K$. Let $0 \neq X \leq K$ and assume that $L \cap X = 0$. Then $X \cong (L \oplus X)/L \leq K/L$ and so $X \in \mu$. Since $L \leq_{\mu e} K$ we have a contradiction. This proves that M is singular. If $Z_\mu(M) = 0$, then M is called *non-μ-singular*.

Proposition 2.3. *Let M be a μ-singular module and $f \in \mathrm{Hom}_R(R, M)$. Then* $\mathrm{Ker}\, f \leq_{\mu e} R$.

Proof. By assumption, $f(R) \cong L/K$ where $K \leq_{\mu e} L$. Since R is projective, there exists a homomorphism $g\colon R \to L$ such that $\pi g = f$ where π is the natural epimorphism $L \to L/K$. Then $\ker f = g^{-1}(K) \leq_{\mu e} R$ by Lemma 2.1. □

Proposition 2.4. *Let P be a projective module and $X \leq P$. Then P/X is μ-singular if and only if $X \leq_{\mu e} P$.*

Proof. If $I \leq R_R$ and R/I is μ-singular, then $I \leq_{\mu e} R$ by Proposition 2.3. Now let P/X be μ-singular and assume $X \not\leq_{\mu e} P$. Let F be a free module such that $F = P \oplus P'$, $P' \leq F$. Then $F/(X \oplus P') \cong P/X$ is μ-singular and $X \oplus P' \not\leq_{\mu e} F$. So we may assume that P is free i.e. $P = \bigoplus R_i$, each R_i is a copy of R. Then $R_i/(R_i \cap X) \cong (R_i + X)/X \leq P/X$ is μ-singular. So $R_i \cap X \leq_{\mu e} R_i$. This implies that $(\bigoplus R_i) \cap X \leq_{\mu e} \bigoplus R_i = P$, i. e. $X \leq_{\mu e} P$. □

Lemma 2.5. *Let M be a module. Then $Z_\mu(M) = \{x \in M \mid xI = 0, I \leq_{\mu e} R\}$.*

Proof. Let $xI = 0$ for some $I \leq_{\mu e} R$. Then R/I is μ-singular. Define $f\colon R/I \to xR$ by $r + I \mapsto xr$. Hence, $x \in \mathrm{Tr}(\mu - \mathrm{Sing}, M)$. Conversely assume that $x = x_1 + \cdots + x_n = f_1(l_1) + \cdots + f_n(l_n)$ and $x_i \in \mathrm{Im}\, f_i$ where $f_i\colon L_i \to M$ such that L_i is μ-singular. For each i we have $l_i R \cong R/\mathrm{ann}(l_i)$ which implies that $I_i = \mathrm{ann}(l_i) \leq_{\mu e} R$ by Proposition 2.4. Take $I = \bigcap_{i=1}^n I_i$. Then $I \leq_{\mu e} R$ by Lemma 2.1 and $xI = 0$. This completes the proof. □

Proposition 2.6. *A module M is non-μ-singular if and only if* $\mathrm{Hom}_R(N, M) = 0$ *for all μ-singular modules N.*

Proof. See [4, Proposition 1.20]. □

Proposition 2.7. *Let M be a non-μ-singular module and $N \leq M$. Then M/N is μ-singular if and only if $N \leq_{\mu e} M$.*

Proof. If M/N is μ-singular and x is a nonzero element of M. Then $\overline{x}I = 0$ for some $I \leq_{\mu e} R$. So, $xI \leq N$. Since M is non-μ-singular, we have $xI \neq 0$ and thus $xR \cap N \neq 0$. Therefore, $N \leq_{\mu e} M$. □

Proposition 2.8. (1) *The class of all non-μ-singular modules is closed under submodules, direct products, μ-essential extension and module extension.*

(2) *The class of all μ-singular modules are closed under submodules, factor modules and direct sums.*

Proof. It follows from Lemma 2.5 and [4, Proposition 1.22]. □

Proposition 2.9. *Assume that R is a right non-μ-singular ring, then:*

(1) *$Z_\mu(M/Z_\mu(M)) = 0$ for any R-module M.*

(2) *An R-module M is μ-singular if and only if* $\mathrm{Hom}_R(M, N) = 0$ *for all non-μ-singular modules N.*

(3) *The class of all μ-singular modules is closed under module extension and μ-essential extension.*

(4) *The set of all μ-essential right ideals of R denoted by $\mathcal{P}(R)$ is closed under finite products.*

Proof. The proof is easy by [4, Proposition 1.23] and Lemma 2.5. □

Proposition 2.10. *Let M be a simple module. Then M is either μ-singular or projective, but not both.*

Proof. See [4, Proposition 1.24]. □

It is easy to see that a ring R is right non-μ-singular if and only if all projective right R-modules are non-μ-singular.

From the properties of μ-singular modules and Proposition 2.4 the following can be seen easily.

Proposition 2.11. *For an R-module M the following are equivalent:*

(1) *M is μ-singular:*

(2) *$M \cong F/K$ with F a projective (free) module and $K \leq_{\mu e} F$;*

(3) *For every $m \in M$, the right annihilator $ann_r(m)$ is μ-essential in R.*

Lemma 2.12. *Let M be a module. If $Z_\mu(M) = 0$ and $K \leq_c M$, then $Z_\mu(M/K) = 0$.*

Theorem 2.13. *Let μ be closed under factor modules. Then the following are equivalent:*

(1) *Every μ-module is projective;*

(2) *For every singular module N,* $\mathrm{Rej}(N, \mu) = N$*;*

(3) *For every μ-singular module N,* $\mathrm{Rej}(N, \mu) = N$*;*

(4) *For every simple singular module N,* $\mathrm{Rej}(N, \mu) = N$.

Proof. (1) $\Rightarrow$ (2) Let N be a singular module and $g \colon N \to L$ where $L \in \mu$. Then $N/\ker g \in \mu$. By (1), $N/\ker g$ is projective. Since N is singular, we have that $N = \ker g$. Hence $\mathrm{Rej}(N, \mu) = N$.

(2) $\Rightarrow$ (3) Let N be a μ-singular module and $g \colon N \to L$ a homomorphism where $L \in \mu$. Then $N/\ker g \in \mu$. This implies that $\mathrm{Rej}(N/\ker g, \mu) = 0$. Since $N/\ker g$ is μ-singular and a μ-module, it is singular. Then by (2), $N = \ker g$. Hence $\mathrm{Rej}(N, \mu) = N$.

(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Let N be a μ-module. We claim that N is semisimple. Let $x \in N$ and K a maximal submodule of xR. Then xR/K is a simple μ-module. By (4), it cannot be singular. Hence xR/K is projective. This implies that K is a direct summand of xR. Hence N is semisimple. By above process every simple submodule of N is projective. It follows that N is projective. □

If we consider the class $\mathcal{M}$ of all small modules, we have a characterization of GCO-rings. A ring R is called a *GCO-ring* if every simple singular module is injective.

Corollary 2.14. *The following are equivalent for a ring R:*
(1) *Every small module is projective;*
(2) *Every singular module is non-cosingular;*
(3) *Every $\mathcal{M}$-singular module is non-cosingular;*
(4) *R is a GCO-ring;*
(5) *Every δ-singular module is non-cosingular.*

Proof. (1) $\Leftrightarrow$ (4) is by [7, Theorem 1.5]. (2) $\Leftrightarrow$ (4) is by [9, Theorem 4.1].

Simple modules are either injective or small. Hence (1)–(4) are equivalent by Theorem 2.13.

(5) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (5) It is clear since $\mathcal{M} \subseteq \delta$, every δ-singular module is $\mathcal{M}$-singular. $\square$

For the class δ of all cosingular modules we have the following corollary.

Corollary 2.15. *If the class δ is closed under the factor modules the following are equivalent:*
(1) *Every cosingular module is projective;*
(2) *For every singular module N, $\mathrm{Rej}(N, \delta) = N$;*
(3) *For every δ-singular module N, $\mathrm{Rej}(N, \delta) = N$;*
(4) *R is a GCO-ring.*

Proof. See [9, Theorems 4.1 and 4.2] and Theorem 2.13. $\square$

A module M is called *finitely cogenerated* if $\mathrm{Soc}(M)$ is finitely generated and essential submodule of M. Let $\mathcal{FC}$ be the class of all finitely cogenerated R-modules. Note that $\mathcal{FC}$ is closed under submodules. We next give a characterization of semi-simple Artinian rings which is taken from [8]. We give the proof for completeness.

Corollary 2.16. *The following statements are equivalent for a ring R:*
(1) *Every finitely cogenerated R-module is projective;*
(2) *For every singular module N, $\mathrm{Rej}(N, \mathcal{FC}) = N$;*
(3) *For every $\mathcal{FC}$-singular module N, $\mathrm{Rej}(N, \mathcal{FC}) = N$;*
(4) *R is semisimple Artinian.*

Proof. (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) By Theorem 2.13.

(4) $\Rightarrow$ (1) is clear.

(2) $\Rightarrow$ (4) Let E be an essential right ideal of R. Suppose that a is an element of R but $a \notin E$. Let F be a right ideal of R maximal with respect to the properties that E is contained in F and $a \notin F$. Then $(aR + F)/F$ is simple singular. By (2), we have a contradiction. Hence R is semisimple Artinian. $\square$

A ring R is a *quasi-Frobenius* ring (briefly QF-ring) if and only if every right R-module is a direct sum of an injective module and a singular module. In this result we may take μ-singular modules instead of singular as the following result shows.

Theorem 2.17. *The following are equivalent for a ring R:*

(1) *R is a QF-ring;*

(2) *Every right R-module is a direct sum of an injective module and a μ-singular module.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let M be a projective R-module. Then M is a direct sum of an injective module and a μ-singular module. Since projective μ-singular modules are zero, M is injective. Then R is a QF-ring. $\square$

3. μ-EXTENDING MODULES

In this section μ-extending modules will be introduced. Then we define and study weakly μ-extending modules. It is proved that any factor module, any direct summand and any fully invariant submodule of a weakly μ-extending module are weakly μ-extending.

Definition 3.1. Let M be a module. Then M is called an *μ-extending* module if for every submodule N of M there exists a direct summand D of M such that $N \leq_{\mu e} D$.

Clearly every essential submodule is μ-essential. So μ-extending modules are a generalization of extending modules.

Note that by [5, Proposition 2.4], a module M is extending if and only if every closed submodule is direct summand. This may not be true for a μ-extending module.

Let M be a module and K a submodule of M. Then K is called a *μ-closed submodule*, denoted by $K \leq_{\mu c} M$, provided $K \leq_{\mu e} L \leq M$ implies $K = L$, i.e. K doesn't have any proper μ-essential extension. A μ-closed submodule is closed but the converse is true when M is a μ-module (see [8, Corollary 1.1]).

Proposition 3.2. *The following statements hold for a module M.*
(1) *If $K \leq_{\mu c} M$, then whenever $Q \leq_{\mu e} M$ such that $K \subseteq Q$, then $Q/K \leq_{\mu e} M/K$.*
(2) *If $L \leq_{\mu e} M$, then $L/K \leq_{\mu e} M/K$.*

Proof. (1) Suppose $K \leq_{\mu c} M$. Let $Q \leq_{\mu e} M$ such that $K \subseteq Q$. Let $P/K \leq M/K$ be a μ-module such that $(Q/K) \cap (P/K) = 0$. By Lemma 2.1(b), $K = Q \cap P \leq_{\mu e} P$ and hence $K = P$. Thus $Q/K \leq_{\mu e} M/K$.
(2) It is clear by Lemma 2.1(d). $\square$

The following proposition is clear by definitions.

Proposition 3.3. *Let M be a μ-extending module. Then every μ-closed submodule is a direct summand.*

We next give an equivalent condition for a μ-extending module.

Proposition 3.4. *Let M be a module. Then M is μ-extending if and only if for each submodule A of M there exists a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A + M_2 \leq_{\mu e} M_2$.*

Proof. Let M be μ-extending and $A \leq M$. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $A \leq_{\mu e} M_1$. Since $\{A, M_2\}$ is an independent family of submodules of M the result follows from Lemma 2.1.

The converse is clear. $\square$

A module M is called *μ-uniform* if every proper nonzero submodule is μ-essential in M.

Proposition 3.5. *An indecomposable module M is μ-extending if and only if M is μ-uniform.*

Definition 3.6. Let M be a module. Then M is called *weakly μ-extending* if for every submodule N of M there exists a direct summand K of M such that $N \leq K$ and K/N is μ-singular.

The definition shows that every μ-extending module is weakly μ-extending. Also any μ-singular module is weakly μ-extending.

Let M be a μ-singular module with unique composition series $M \supset U \supset V \supset 0$. By [2], $N = M \oplus (U/V)$ is not extending. But N is weakly μ-extending.

We next give some equivalent conditions for weakly μ-extending modules.

Proposition 3.7. *The following are equivalent for a module M:*

(1) *M is weakly μ-extending;*

(2) *For every $N \leq M$ there exists a decomposition $M = K \oplus K'$ such that $N \leq K$ and $M/(K' + N)$ is μ-singular;*

(3) *For every $N \leq M$ there exists a decomposition $M/N = K/N \oplus K'/N$ such that $K \leq_d M$ and M/K' is μ-singular;*

(4) *For every $N \leq M$, there exists a direct summand K of M such that $N \leq K$ and for any $x \in K$ there is a right ideal I with $I \leq_{\mu e} R$ such that $xI \leq N$.*

Proposition 3.8. *Let M be a non-μ-singular or projective module. Then, M is μ-extending if and only if M is weakly μ-extending.*

Proof. It is easy by Propositions 2.7 and 2.4. $\square$

Some special submodules of a weakly μ-extending module are weakly μ-extending. Recall that a submodule N of M is called *fully invariant* if $f(N) \subseteq N$ for each $f \in \mathrm{End}(M)$. A module M is called a *duo module*, if every submodule of M is fully invariant.

Proposition 3.9. *Let $N \leq M$ be fully invariant and M a weakly μ-extending module. Then N is weakly μ-extending.*

Proof. Let $L \leq N \leq M$. By assumption, there exists a decomposition $M = K \oplus K'$ such that $L \leq K$ and K/L is μ-singular. Since N is fully invariant, we have $N = (N \cap K) \oplus (N \cap K')$. Obviously, $L \leq N \cap K$ and $(N \cap K)/L \leq K/L$ is μ-singular. Hence N is weakly μ-extending. $\square$

The Proposition 3.9 shows that every submodule of a duo module or of a multiplication weakly μ-extending module is weakly μ-extending.

Proposition 3.10. *Let M be a module and N a submodule of M.*

(1) *If M is weakly μ-extending and the intersection of N with any direct summand of M is a direct summand of N, then N is weakly μ-extending.*

(2) *If N is weakly μ-extending and D a direct summand of M such that $(D+N)/D$ is non-μ-singular, then $D \cap N$ is a direct summand of N.*

(3) *If M is weakly μ-extending and $(D+N)/D$ is non-μ-singular for any direct summand D of M, then N is weakly μ-extending if and only if $D \cap N$ is a direct summand of N for any direct summand D of M.*

Proof. (1) It is similar to the proof of Proposition 3.9.

(2) Let $Y = D \cap N$. Since N is weakly μ-extending, there is a direct summand K of N such that K/Y is μ-singular. By assumption, $N/Y \cong (D+N)/D$ is non-μ-singular. Hence, $K/Y \leq N/Y$ is both μ-singular and non-μ-singular. It follows that $K = Y$ is a direct summand of N.

(3) It is a consequence of (1) and (2). □

The following proposition shows the equivalent condition of a cyclic submodule of a module to be weakly μ-extending over a right weakly μ-extending ring.

Proposition 3.11. *Let R be a right weakly μ-extending ring and M a cyclic right R-module such that every nonzero direct summand of M contains a nonzero μ-module. Then the following are equivalent:*

(1) *M is non-μ-singular;*

(2) *Every cyclic submodule of M is projective and weakly μ-extending;*

(3) *Every cyclic submodule of M is projective.*

Proof. (1) $\Rightarrow$ (2) Suppose that M is non-μ-singular and N a cyclic submodule of M. Then there is a right ideal I of R such that $N \cong R/I$. Since R is μ-extending and N is non-μ-singular, I is a μ-closed submodule of R_R, hence I is a direct summand of R_R. Thus N is isomorphic to a direct summand of R_R. Therefore, N is projective and weakly μ-extending.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) For any $m \in Z_\mu(M)$, mR is projective and is isomorphic to $R/\operatorname{ann}_r(m)$, where $\operatorname{ann}_r(m)$ is the right annihilator of m. Since R is right weakly μ-extending and mR is μ-singular, then $\operatorname{ann}_r(m) \leq_{\mu e} R$ is a direct summand of R. Then, $R = \operatorname{ann}_r(m) \oplus L$. By assumption, if $L \neq 0$ then it contains a nonzero μ-module. Hence, $\operatorname{ann}_r(m) = R$ and $m = 0$. Hence, $Z_\mu(M) = 0$. □

Any factor module of a μ-singular module is μ-singular and we show that any image of a weakly μ-extending module is weakly μ-extending. The direct summand of a μ-extending module may not be μ-extending. For weakly μ-extending modules, we first show the following proposition and then show that any direct summand of a weakly μ-extending module is weakly μ-extending.

Proposition 3.12. *Let M be a weakly μ-extending module. Then any homomorphic image of M is weakly μ-extending.*

Proof. Let $f\colon M \to N$ be an epimorphism and L a submodule of N. Then there is a submodule H of M such that $L \cong H/\operatorname{Ker} f$. Since M is weakly μ-extending, there are direct summands K, K' of M such that $M = K \oplus K'$, $H \leq K$ and that K/H is μ-singular. So $N \cong M/\operatorname{Ker} f = (K/\operatorname{Ker} f) \oplus (K' + \operatorname{Ker} f)/\operatorname{Ker} f$ and $L \cong H/\operatorname{Ker} f \leq K/\operatorname{Ker} f$. Since $(K/\operatorname{Ker} f)/(H/\operatorname{Ker} f) \cong K/H$ is μ-singular, N is weakly μ-extending. □

Corollary 3.13. (1) *Let M be a weakly μ-extending module. Then any direct summand of M is weakly μ-extending.*

(2) *Let M be a μ-extending module. Then any non-μ-singular homomorphic image of M is μ-extending.*

Corollary 3.14. *The following are equivalent:*

(1) *Every (resp., finitely generated) module is weakly μ-extending;*

(2) *Every (resp., finitely generated) projective module is weakly μ-extending.*

Proposition 3.15. *Let R be a right non-μ-singular ring and $f\colon M \to M'$ an epimorphism. Suppose that M' is weakly μ-extending and* $\operatorname{Ker} f$ *is μ-singular injective, then M is weakly μ-extending.*

Proof. Let N be a submodule of M. First, we assume that $\operatorname{Ker} f \subseteq N \leq M$, then $f(N) \leq M'$. Since M' is weakly μ-extending, there is a decomposition, $M' = K \oplus H$ such that $K/f(N)$ is μ-singular. So $M = f^{-1}(K) + f^{-1}(H)$. Since $\operatorname{Ker} f \leq f^{-1}(H)$ and $\operatorname{Ker} f$ is injective, then $f^{-1}(H) = T \oplus \operatorname{Ker} f$ for some submodule T of $f^{-1}(H)$. Thus $M = f^{-1}(K) + T$. Since $f^{-1}(K) \cap T \leq f^{-1}(K) \cap f^{-1}(H) = \operatorname{Ker} f$ and $f^{-1}(K) \cap T \leq \operatorname{Ker} f \cap T = 0$, we have $M = f^{-1}(K) \oplus T$ and $N \leq f^{-1}(K)$.

For any $x \in f^{-1}(K)$, $f(x) \in K$ and there is an μ-essential right ideal I of R such that $f(x)I \leq f(N)$. It is easy to see that $xI \leq N$ and that $f^{-1}(K)/N$ is μ-singular.

Now we assume that N does not contain $\operatorname{Ker} f$. Set $L = N + \operatorname{Ker} f$, then $f(L) = f(N)$. As the case above, there is a decomposition $M = f^{-1}(K) \oplus T$ such that $f^{-1}(K)/L$ is μ-singular. Since $\operatorname{Ker} f$ is μ-singular, we have that $(N + \operatorname{Ker} f)/N \cong \operatorname{Ker} f/(N \cap \operatorname{Ker} f)$ is μ-singular. Since R is right non-μ-singular, by Proposition 2.9 we have that $f^{-1}(K)/N$ is μ-singular. In either case, M is weakly μ-extending. □

Proposition 3.16. *Let R be a right non-μ-singular ring and M a weakly μ-extending module. Then $M = Z_\mu(M) \oplus T$ for some μ-extending submodule T of M and T is $Z_\mu(M)$-injective.*

Proof. If $Z_\mu(M) = 0$ or $Z_\mu(M) = M$, it is clear.

Suppose that $0 < Z_\mu(M) < M$. Since M is weakly μ-extending, there are direct summands K, T of M such that $M = K \oplus T$, $Z_\mu(M) \leq K$ and that $K/Z_\mu(M)$ is μ-singular. So K is μ-singular. Since $Z_\mu(M) = Z_\mu(K) \oplus Z_\mu(T) = K \oplus Z_\mu(T)$, so $Z_\mu(M) = K$ and T is non-μ-singular. By Proposition 3.12, T is μ-extending.

Since for any submodule N of $Z_\mu(M)$, $\operatorname{Hom}_R(N, T) = 0$, so T is $Z_\mu(M)$-injective, as required. □

Corollary 3.17. *Let R be a right non-μ-singular ring and M an injective module. Then $Z_\mu(M)$ is injective.*

Corollary 3.18. *Let R be a right non-μ-singular ring and M an indecomposable weakly μ-extending module. Then M is either a μ-singular module or a non-μ-singular μ-uniform module.*

Proposition 3.19. *Let M be a weakly μ-extending module which contains maximal submodules. Then for any maximal submodule N of M, either M/N is μ-singular or $M = N \oplus S$ for some simple submodule S of M.*

Proof. Let N be a maximal submodule of M and suppose that M/N is not μ-singular. Then N is a direct summand of M, i.e., $M = N \oplus S$ for some submodule S of M. Since $S \cong M/N$, so S is simple. □

A module M is called *local* if it has a largest submodule, i.e., a proper submodule which contains all other proper submodules. For a local module M, $\mathrm{Rad}(M)$, the Jacobson radical of M is small in M.

Corollary 3.20. *Let M be a local weakly μ-extending module. Then $M/\mathrm{Rad}(M)$ is μ-singular.*

Proposition 3.21. *Let R be a right hereditary ring and M an injective module. Then any factor module of M is a direct sum of an injective module and a μ-singular injective module.*

Proof. Let L be any factor module of M, then there is a submodule N of M such that $L \cong M/N$. Since any injective module is weakly μ-extending, there are direct summands K, K' of M such that $M = K \oplus K'$, $N \leq K$ and that K/N is μ-singular. So $L \cong M/N = K/N \oplus (K' + N)/N$. Since R is hereditary and M is injective, so M/N is injective. Thus K/N is a μ-singular injective module and $(K' + N)/N$ is injective. □

4. Direct sum of weakly μ-extending modules

A direct sum of μ-singular modules is also μ-singular. But a direct sum of μ-extending modules may not be μ-extending. Also a direct sum of weakly μ-extending modules need not be weakly μ-extending (see [1, Example 2.4]).

It may be interesting to see when a direct sum of weakly μ-extending modules is weakly μ-extending.

Proposition 4.1. *Let $M = \bigoplus_{i \in I} M_i$ be a distributive module. Then M is weakly μ-extending if and only if each M_i is weakly μ-extending for $i \in I$.*

Proof. Let N be any submodule of M, then $N = \bigoplus_{i \in I}(N \cap M_i)$. Since M_i is weakly μ-extending, there is direct summand $H_i \leq_d M_i$, such that $M_i = H_i \oplus H_i'$ and $(N \cap M_i) \leq H_i$ and that $H_i/(N \cap M_i)$ is μ-singular for $i \in I$. Hence $M = (\bigoplus_{i \in I} H_i) \oplus (\bigoplus_{i \in I} H_i')$ and $(N = \bigoplus_{i \in I}(N \cap M_i)) \leq (H = \bigoplus_{i \in I} H_i)$. Since $\frac{H}{N} = \frac{\bigoplus_{i \in I} H_i}{\bigoplus_{i \in I}(N \cap M_i)} \cong \bigoplus_{i \in I} \frac{H_i}{N \cap M_i}$ is μ-singular, so M is weakly μ-extending. □

Theorem 4.2. *Let $M = M_1 \oplus M_2$ with M_1 being μ-singular (μ-uniform) and M_2 semisimple. Then M is weakly μ-extending.*

Proof. Let N be any submodule of M. Then $N + M_1 = M_1 \oplus [(N + M_1) \cap M_2]$. Since M_2 is semisimple, then $(N + M_1) \cap M_2$ is a direct summand of M_2 and therefore $N+M_1$ is a direct summand of M. Note that $(N+M_1)/N \cong M_1/(N\cap M_1)$ is μ-singular, since M_1 is μ-singular (μ-uniform). So M is weakly μ-extending. $\square$

Proposition 4.3. *Let $M = M_1 \oplus M_2$ with M_1 being weakly μ-extending and M_2 semisimple. Suppose that for any submodule N of M, $N \cap M_1$ is a direct summand of N. Then M is weakly μ-extending.*

Proof. Let N be any submodule of M. As in Theorem 4.2, $N + M_1$ is a direct summand of M. By the hypothesis, $N = (N \cap M_1) \oplus K$ for some submodule K of N. Since M_1 is weakly μ-extending, there is a direct summand T of M_1 such that $T/(N \cap M_1)$ is μ-singular. But $N + M_1 = (N \cap M_1) + K + M_1 = M_1 \oplus K$, so $(T \oplus K)/N = (T \oplus K)/[(N \cap M_1) \oplus K] \cong T/(N \cap M_1) \oplus K/K$ is μ-singular. Since $T \oplus K$ is a direct summand of $N + M_1$ and hence a direct summand of M, then M is weakly μ-extending. $\square$

Proposition 4.4. *Let $M = M_1 \oplus M_2$ with M_1 being weakly μ-extending and M_2 injective. Suppose that for any submodule N of M, we have $N \cap M_2$ is a direct summand of N, then M is weakly μ-extending.*

Proof. Let $N \leq M$. By the hypothesis, there is a submodule N' of N such that $N = (N \cap M_2) \oplus N'$. Note that $N' \cap M_2 = 0$ and hence $(M_2 + N')/N' \cong M_2$ is an injective module, so there is a submodule M' of M containing N' such that $M/N' = [(M_2 + N')/N'] \oplus (M'/N')$. Thus it is easy to see that $M = M_2 \oplus M'$ and that $M' \cong M/M_2 \cong M_1$. Hence M' is weakly μ-extending. There are direct summands K, K' of M' such that $M' = K \oplus K'$ and that K/N' is μ-singular. Since $N \cap M_2$ is a submodule of an injective module M_2, so there is a direct summand H of M_2 such that $H/(N \cap M_2)$ is μ-singular. Following from the fact that $(H \oplus K)/[(N \cap M_2) \oplus N'] \cong [H/(N \cap M_2)] \oplus (K/N')$ and that $H \oplus K \leq_d M$, then M is weakly μ-extending. $\square$

Proposition 4.5. *Let $M = M_1 \oplus M_2$ such that M_1 is weakly μ-extending and M_2 is an injective module. Then M is weakly μ-extending if and only if for every submodule N of M such that $N \cap M_2 \neq 0$, there is a direct summand K of M such that K/N is μ-singular.*

Proof. Suppose that for every submodule N of M such that $N \cap M_2 \neq 0$, there is a direct summand K of M such that K/N is μ-singular. Let N be a submodule of M such that $N \cap M_2 = 0$. Then, since $(M_2 + N)/N \cong M_2$ is an injective module, there is a submodule M' of M containing N such that $M/N = (M'/N) \oplus ((M_2 + N)/N)$. It is easy to see that $M = M' \oplus M_2$. Since $M' \cong M/M_2 \cong M_1$ is weakly μ-extending, there is a direct summand K of M', hence of M, such that K/N is μ-singular. So M is weakly μ-extending. The converse is obvious. $\square$

5. Rings whose projective modules are μ-extending

In [6], a ring R is called a *right co-H-ring* if every projective right R-module is extending. It is known that a ring R is a right co-H-ring if and only if R is right

$\sum$-extending (i.e., any direct sum of R_R is extending). In this section we introduce rings in which all projective right modules are μ-extending. We call such rings *μ-co-H-rings*. It is easy to check that a ring R is μ-co-H-ring if and only if any direct sum of R_R is μ-extending.

Lemma 5.1. *Let R be a ring. A projective R-module M is weakly μ-extending if and only if every factor module of M is a direct sum of a μ-singular module and a projective module.*

Proof. Suppose that M is weakly μ-extending. Let M' be any factor module of M, then there is a submodule N of M such that $M/N \cong M'$. Since M is weakly μ-extending, then there are direct summands K, K' of M such that $M = K \oplus K'$ and K/N is μ-singular. Thus $M/N = (K/N) \oplus ((K' + N)/N)$. As M is projective, $K' \cong (K' + N)/N$ is projective. Conversely, let N be any submodule of M, then M/N is a direct sum of a μ-singular module and a projective module. We may assume that $M/N = S/N \oplus T/N$, where S/N is μ-singular and T/N is projective. Then $M = S + T$ and as $M/S \cong T/N$ is projective, S is a direct summand of M. Thus M is weakly μ-extending. □

Lemma 5.2. *Let R be any right non-μ-singular ring. Then the following are equivalent:*

(1) *All modules are weakly μ-extending;*
(2) *All projective modules are weakly μ-extending;*
(3) *All non-μ-singular modules are μ-extending.*

Proof. (1) ⇔ (2) By Corollary 3.14.

(1) ⇔ (3) This is a consequence of Propositions 3.8 and 3.12 and the fact that over a right non-μ-singular ring all projective modules are non-μ-singular. □

As an immediate consequence of Lemmas 5.1, 5.2 and Proposition 3.8, we have:

Theorem 5.3. *Let R be any ring, then the following are equivalent:*

(1) *R is a right μ-co-H-ring;*
(2) *All right R-modules are weakly μ-extending;*
(3) *All projective right R-modules are μ-extending;*
(4) *All projective right R-modules are weakly μ-extending;*
(5) Every factor module of any projective module is a direct sum of a μ-singular module and a projective module.

Theorem 5.4. *Let R be a right non-μ-singular ring, consider the following:*

(1) *R is a right μ-co-H-ring;*
(2) *Every non-μ-singular module is projective;*
(3) *Every module is weakly μ-extending;*
(4) *Every non-μ-singular module is μ-extending.*

Then (1) ⇔ (3) ⇔ (4) *and* (1) ⇒ (2).

Proof. (1) ⇒ (2) Suppose that R is a right μ-co-H-ring and M a non-μ-singular module. Then there is a projective module P and an epimorphism $f : P \to M$. Set $K = Ker f$, then K is a μ-closed submodule of P. Since P is μ-extending, then K

is a direct summand of P and hence M is isomorphic to a direct summand of P. Thus M is projective.

(1) $\Leftrightarrow$ (3) By Theorem 5.3.

(1) $\Rightarrow$ (4) It is clear by Propositions 3.8 and 3.12.

(4) $\Rightarrow$ (1) Since a ring R is right non-μ-singular if and only if all projective modules are non-μ-singular, by (4), all projective modules are μ-extending and R is a right μ-co-H-ring. □

Corollary 5.5. *Let R be a ring such that all μ-singular modules are projective, then R is a right μ-co-H-ring if and only if R is semisimple.*

Proof. Suppose that R is a right μ-co-H-ring. Let M be an R-module module and N a submodule of M, then by Theorem 5.3, M is weakly μ-extending, i.e., there is a direct summand K of M such that $N \leq K$ and K/N is μ-singular. By hypothesis, K/N is projective, so N is a direct summand of K and hence a direct summand of M. Thus M is semisimple and R is semisimple.

The converse is obvious. □

It is known from [3, Theorem 24.20] that a ring R is a QF-ring if and only if all projective modules are injective if and only if all injective modules are projective. Obviously every QF-ring R is a left and right μ-co-H-ring. As an immediate consequence of Theorem 5.4, we have:

Corollary 5.6. *Let R be a right non-μ-singular ring such that all injective modules are non-μ-singular. Then R is a right μ-co-H-ring if and only if R is a QF-ring.*

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