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EXISTENCE AND POSITIVITY OF SOLUTIONS FOR A NONLINEAR PERIODIC DIFFERENTIAL EQUATION

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Abstract. We study the existence and positivity of solutions of a highly nonlinear periodic differential equation. In the process we convert the differential equation into an equivalent integral equation after which appropriate mappings are constructed. We then employ a modification of Krasnoselskii's fixed point theorem introduced by T. A. Burton ([4], Theorem 3) to show the existence and positivity of solutions of the equation.

1. Introduction

Let $T > 0$ be fixed. We use a variant of Krasnoselskii's fixed theorem in [4] to prove the existence and positivity of solutions for the non-linear neutral periodic equation

$$\begin{aligned} x'(t) &= -a(t)x^3(t) + c(t)x'\big(g(t)\big)g'(t) + q\big(t, x^3(g(t))\big)\,, \\ x(t) &= x(t+T)\,. \end{aligned} \tag{1.1}$$

A number of authors in recent years have investigated the stability or periodicity of solutions for equations of forms similar to equation (1.1); see [3, 1, 11, 10, 13] and references therein. We are particularly motivated by the work in [11], where the non-linear term q and the function a are assumed to be continuous in all arguments. Our objective in this work is to impose much weaker conditions on the non-linear term q and the argument function a.

Equation (1.1) is clearly nonlinear so the variation of parameters formula cannot be applied directly. We therefore resort to the idea of adding and subtracting a linear term.

The map $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy Carathéodory conditions with respect to $L^1[0,T]$ if the following conditions hold.

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is Lebesgue measurable.

(ii) For almost all $t \in [0,T]$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0,T], R)$ such that for almost all $t \in [0,T]$ and for all z such that $|z| < r$, we have $|f(t,z)| \le \alpha_r(t)$.

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In Section 2 we present some preliminary material that we will employ to show the existence and positivity of solutions of (1.1). Also, we state a reformulated version of a fixed point theorem due to Krasnoselskii. We present our existence of periodic solutions results in Section 3. In Section 4, the results for the existence of positive solutions are presented.

2. Preliminaries

Define the set $P_T = \{\phi \in C(\mathbb{R},\mathbb{R}) : \phi(t+T) = \phi(t)\}$ and the norm $\|\phi\| = \sup_{t\in[0,T]} |\phi(t)|$, where C is the space of continuous real valued functions. Then $(P_T, \|\cdot\|)$ is a Banach space. In this paper we make the following assumptions.

(D1) $a \in L^1(\mathbb{R},\mathbb{R})$ is bounded and satisfies $a(t+T) = a(t)$ for all t and

$$1 - e^{-\int_{t-T}^{t} a(r)dr} \equiv \frac{1}{\rho} \neq 0\,.$$

(D2) $c \in C^1(\mathbb{R},\mathbb{R})$ satisfies $c(t+T) = c(t)$ for all t.

(D3) $g \in C^1(\mathbb{R},\mathbb{R})$ satisfies $g(t+T) = g(t)$ for all t.

(D4) q satisfies Carathéodory conditions with respect to $L^1[0,T]$, and

$$q(t+T,x) = q(t,x)\,.$$

Lemma 2.1. *Suppose that conditions* (D1), (D2), (D3), *and* (D4) *hold. Then $x \in P_T$ is a solution of equation* (1.1) *if and only if, $x \in P_T$ satisfies*

$$\begin{aligned} x(t) &= c(t)x(g(t)) + \rho\int_{t-T}^{t} a(u)[x(u) - x^3(u)]e^{-\int_u^t a(r)dr}du \\ &\quad + \rho\int_{t-T}^{t}\big[q\big(u, x^3(g(u))\big) - r(u)x\big(g(u)\big)\big]e^{-\int_u^t a(r)dr}du \end{aligned} \tag{2.1}$$

where $r(u) = a(u)c(u) + c'(u)$.

Proof. Let $x \in P_T$ be a solution of (1.1). We first rewrite (1.1) in the form

$$x'(t) + a(t)x(t) = a(t)x(t) - a(t)x^3(t) + c(t)x'\big(g(t)\big)g'(t) + q\big(t, x^3(g(t))\big)\,.$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)ds}$ and then integrate the resulting equation from $t-T$ to t. Thus we obtain,

$$\begin{aligned} &x(t)e^{\int_0^t a(s)ds} - x(t-T)e^{\int_0^{t-T} a(s)ds} \\ &= \int_{t-T}^{t}\big[a(u)\big(x(u) - x^3(u)\big) + c(u)x'(g(u))g'(u) + q(u, x^3(g(u)))\big]e^{\int_0^u a(s)ds}du\,. \end{aligned} \tag{2.2}$$

Dividing both sides of (2.2) by $e^{\int_0^t a(s)ds}$ and using the fact that $x \in P_T$ we obtain

$$\begin{aligned} &x(t)\frac{1}{\rho} \\ &= \int_{t-T}^{t}\big[a(u)\big(x(u) - x^3(u)\big) + c(u)x'(g(u))g'(u) + q\big(u, x^3(g(u))\big)\big]e^{-\int_u^t a(s)ds}du\,. \end{aligned} \tag{2.3}$$

Integrating the second term on the right hand side of (2.3) by parts gives

$$\begin{aligned}&\int_{t-T}^{t} c(u)x'(g(u))g'(u)e^{-\int_u^t a(s)ds}du\\ &= c(t)x\big(g(t)\big)-e^{-\int_{t-T}^t a(s)ds}c(t-T)x\big(g(t-T)\big)-\int_{t-T}^{t}\frac{d}{du}\big[c(u)e^{-\int_u^t a(s)ds}\big]x\big(g(u)\big)du\,.\end{aligned}$$

Since $c(t) = c(t - T)$, $g(t) = g(t - T)$, and $x \in P_T$, then

$$\begin{aligned}\int_{t-T}^{t} & c(u)x'\big(g(u)\big)g'(u)e^{-\int_u^t a(s)ds}du\\ &= \frac{1}{\rho}c(t)x(g(t)) - \int_{t-T}^{t}\frac{d}{du}\big[c(u)e^{-\int_u^t a(s)ds}\big]x\big(g(u)\big)du\,.\end{aligned} \tag{2.4}$$

Substituting the right hand side of (2.4) into (2.3) and simplifying gives the desired result.

The converse implication is easily obtained and the proof is complete. □

In this article, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions to prove our main results. Before stating this theorem we give the following definition and theorem which can be found in [4].

Definition 2.2. *Let* $(\mathbb{M}, d)$ *be a metric space and* $B\colon \mathbb{M} \to \mathbb{M}$. *B is said to be a large contraction if* $\psi, \varphi \in \mathbb{M}$, *with* $\psi \neq \varphi$ *then* $d(B\varphi, B\psi) < d(\varphi, \psi)$ *and if for all* $\epsilon > 0$ *there exists* $\delta < 1$ *such that*

$$[\psi, \varphi \in \mathbb{M}, d(\varphi, \psi) \geq \epsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi)\,.$$

Theorem 2.3. *Let* $(\mathbb{M}, d)$ *be a complete metric space and* B *a large contraction. Suppose there is an* $x \in \mathbb{M}$ *and an* $L > 0$, *such that* $d(x, B^n x) \leq L$ *for all* $n \geq 1$. *Then* B *has a unique fixed point in* $\mathbb{M}$.

The next theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem.

Theorem 2.4 ([4])**.** *Let* $\mathbb{M}$ *be a bounded convex non-empty subset of a Banach space* $(S, ||.||)$. *Suppose that* A, B *map* $\mathbb{M}$ *into* $\mathbb{M}$ *and that*

(i) *for all* x, $y \in \mathbb{M} \Rightarrow Ax + By \in \mathbb{M}$,

(ii) *A is continuous and AM is contained in a compact subset of M,*

(iii) *B is a large contraction.*

Then there is a $z \in \mathbb{M}$ *with* $z = Az + Bz$.

3. Existence of periodic solution

In this section we state and prove our existence results. In view of this we first define the operator H by

$$(H\varphi)(t) = c(t)\varphi(g(t)) + \rho\int_{t-T}^{t} a(u)[\varphi(u) - \varphi^3(u)]e^{-\int_u^t a(r)dr}du$$
$$+ \rho\int_{t-T}^{t}[q(u,\varphi^3(g(u))) - r(u)\varphi(g(u))]e^{-\int_u^t a(r)dr}du\,, \tag{3.1}$$

where r is given in Lemma 2.1. It therefore follows from Lemma 2.1 that fixed points of H are solutions of (1.1) and vice versa.

In order to employ Theorem 2.4 we need to express the operator H as a sum of two operators, one of which is completely continuous and the other is a large contraction. Let $(H\varphi)(t) = A\varphi(t) + B\varphi(t)$ where A, $B\colon P_T \to P_T$ are defined by

$$(B\varphi)(t) = \rho\int_{t-T}^{t} a(u)[\varphi(u) - \varphi^3(u)]e^{-\int_u^t a(r)dr}du, \tag{3.2}$$

and

$$(A\varphi)(t) = c(t)\varphi(g(t)) + \rho\int_{t-T}^{t}\big[q\big(u,\varphi^3(g(u))\big)$$
$$- r(u)\varphi\big(g(u)\big)\big]e^{-\int_u^t a(r)dr}du \tag{3.3}$$

respectively.

Lemma 3.1. *Suppose that conditions* (D1), (D2), (D3), *and* (D4) *hold. Then* $A\colon P_T \to P_T$ *is completely continuous.*

Proof. It follows from (3.3) and conditions (D1), (D2), that $r(\sigma + T) = r(\sigma)$ and $e^{-\int_{\sigma+T}^{t+T} a(r)dr} = e^{-\int_\sigma^t a(u)du}$. Consequently, we have that

$$(A\varphi)(t+T) = (A\varphi)(t)\,.$$

That is, if $\varphi \in P_T$ then $A\varphi$ is periodic with period T.

To see that A is continuous let $\{\varphi_i\} \subset P_T$ be such that $\varphi_i \to \varphi$. By the Dominated Convergence Theorem,

$$\begin{aligned}\lim_{i\to\infty}|A\varphi_i(t) - A\varphi(t)| \le \lim_{i\to\infty}\Big(&|c(t)|\,\big|\varphi_i\big(g(t)\big) - \varphi\big(g(t)\big)\big| \\ &+ \rho\int_{t-T}^{t}\big(\big|q(u,\varphi_i^3(g(u))) - q(u,\varphi^3(g(u)))\big| \\ &+ |r(u)|\big|\varphi_i(g(u)) - \varphi(g(u))\big|\big)e^{-\int_u^t a(r)dr}du\Big) \to 0\,.\end{aligned}$$

Hence $A\colon P_T \to P_T$.

We next show that A is completely continuous. Let $Q \subset P_T$ be a closed bounded subset and let μ be such that $\|\varphi\| \le \mu$ for all $\varphi \in Q$. Then

$$\begin{aligned}|A\varphi(t)| &\le \nu\mu + \rho\int_{t-T}^{t} \big(|q(u,\varphi^3(g(u)))| + |r(u)||\varphi(g(u))|\big)e^{-\int_u^t a(r)dr}du \\ &\le \nu\mu + \rho N\Big(\int_{t-T}^{t}\alpha_\mu(u)du + \mu\int_{t-T}^{t}|r(u)|du\Big) \equiv K\,,\end{aligned}$$

where $\nu = \max_{t\in[0,T]} c(t)$ and $N = \max_{u\in[t-T,t]} e^{-\int_u^t a(r)dr}$. And so the family of functions $A\varphi$ is uniformly bounded. Again, let $\varphi \in Q$. Without loss of generality, we can pick $\tau < t$ such that $t - \tau < T$. Then

$$\begin{aligned}\big|A\varphi(t) - A\varphi(\tau)\big| &= \Big|c(t)\varphi(t) + \rho\int_{t-T}^{t}\big(q(s,\varphi^3(g(s))) - r(s)\varphi(g(s))\big)e^{-\int_s^t a(r)dr}ds \\ &\quad - c(\tau)\varphi(\tau) - \rho\int_{\tau-T}^{\tau}\big(q(s,\varphi^3(g(s))) - r(s)\varphi(g(s))\big)e^{-\int_s^\tau a(r)dr}ds\Big| \\ &\le \big|c(t)\varphi(t) - c(\tau)\varphi(\tau)\big| + \rho\int_{\tau}^{t}\big(|q(s,\varphi^3(g(s)))| + |r(s)|\,|\varphi(g(s))|\big)e^{-\int_s^t a(r)dr}ds \\ &\quad + \rho\int_{\tau-T}^{\tau}\big(|q\big(s,\varphi^3(g(s))\big)| + |r(s)|\,|\varphi\big(g(s)\big)|\big)\big|e^{-\int_s^t a(r)dr}ds - e^{-\int_s^\tau a(r)dr}ds\big| \\ &\quad + \rho\int_{\tau-T}^{t-T}\big(|q(s,\varphi^3(g(s)))| + |r(s)|\,|\varphi\big(g(s)\big)|\big)e^{-\int_s^\tau a(r)dr}ds \\ &\le \big|c(t)\varphi(t) - c(\tau)\varphi(\tau)\big| + 2\rho N\Big(\int_\tau^t \alpha_\mu(s) + \mu|r(s)|\,ds\Big) \\ &\quad + \rho\int_{t-T}^{\tau}\big(\alpha_\mu(s) + \mu|r(s)|\big)\big|e^{-\int_s^t a(r)dr}ds - e^{-\int_s^\tau a(r)dr}\big|\,ds\,.\end{aligned}$$

Now $\big|c(t)\varphi(t) - c(\tau)\varphi(\tau)\big| \to 0$ and $\int_\tau^t \alpha_\mu(s) + \mu|r(s)|ds \to 0$ as $(t-\tau) \to 0$. Also, since

$$\begin{aligned}&\int_{t-T}^{\tau}\big(\alpha_\mu(s) + \mu|r(s)|\big)\big|e^{-\int_s^t a(r)dr}ds - e^{-\int_s^\tau a(r)dr}\big|\,ds \\ &\qquad \le \int_0^T \big(\alpha_\mu(s) + \mu|r(s)|\big)\big|e^{-\int_s^t a(r)dr}ds - e^{-\int_s^\tau a(r)dr}\big|\,ds\,,\end{aligned}$$

and $|e^{-\int_s^t a(r)dr}ds - e^{-\int_s^\tau a(r)dr}| \to 0$ as $(t-\tau) \to 0$, then by the Dominated Convergence Theorem,

$$\int_{t-T}^{\tau}\big(\alpha_\mu(s) + \mu|r(s)|\big)\big|e^{-\int_s^t a(r)dr}ds - e^{-\int_s^\tau a(r)dr}\big|\,ds \to 0$$

as $(t-\tau) \to 0$. Thus $|A\varphi(t) - A\varphi(\tau)| \to 0$ as as $(t-\tau) \to 0$ independently of $\varphi \in Q$. It therefore follows that the family of $A\varphi$ is equicontinuous on Q.

By the Arzelà-Ascoli Theorem, A is completely continuous and the proof is complete. □

Proposition 3.2. *Let $\|\cdot\|$ be the supremum norm, and*

$$\mathbb{M} = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi \in C, \|\varphi\| \le \sqrt{3}/3\}\,.$$

If $(F\varphi)(t) = \varphi(t) - \varphi^3(t)$. Then F is a large contraction of the set $\mathbb{M}$.

Proof. For each $t \in \mathbb{R}$ we have, for φ, ψ real functions,

$$\begin{aligned}|(F\varphi)(t) - (F\psi)(t)| &= |\varphi(t) - \varphi^3(t) - \psi(t) + \psi^3(t)|\\ &= |\varphi(t) - \psi(t)|\,\big|1 - \big(|\varphi^2(t) + \varphi(t)\psi(t) + \psi^2(t)\big)\big|\,.\end{aligned}$$

Then for

$$|\varphi(t) - \psi(t)|^2 = \varphi^2(t) - 2\varphi(t)\psi(t) + \psi^2(t) \le 2\big(\varphi^2(t) + \psi^2(t)\big)$$

and for $\varphi^2(t) + \psi^2(t) < 1$, we have

$$\begin{aligned}|(F\varphi)(t) - (F\psi)(t)| &= |\varphi(t) - \psi(t)|\big[1 - (\varphi^2(t) + \psi^2(t)) + |\varphi(t)\psi(t)|\big]\\ &\le |\varphi(t) - \psi(t)|\Big[1 - (\varphi^2(t) + \psi^2(t)) + \frac{\varphi^2(t) + \psi^2(t)}{2}\Big]\\ &\le |\varphi(t) - \psi(t)|\Big[1 - \frac{\varphi^2(t) + \psi^2(t)}{2}\Big]\,.\end{aligned}$$

Thus, we have shown that pointwise F is a large contraction. It is easy to see that this implies a large contraction in the supremum norm.

For a given $\epsilon \in (0,1)$, let φ, $\psi \in \mathbb{M}$ with $\|\varphi - \psi\| \ge \epsilon$.

(a) Suppose that for some t we have $\epsilon/2 \le |\varphi(t) - \psi(t)|$ so that

$$(\epsilon/2)^2 \le |\varphi(t) - \psi(t)|^2 \le 2(\varphi^2(t) + \psi^2(t))$$

or

$$\varphi^2(t) + \psi^2(t) \ge \epsilon^2/8\,.$$

For all such t we have

$$|(F\varphi)(t) - (F\psi)(t)| \le |\varphi(t) - \psi(t)|\Big[1 - \frac{\epsilon^2}{16}\Big] \le \|\varphi - \psi\|\Big[1 - \frac{\epsilon^2}{16}\Big]\,.$$

(b) Suppose that for some t, we have $|\varphi(t) - \psi(t)| \le \epsilon/2$. Then

$$|(F\varphi)(t) - (F\psi)(t)| \le |\varphi(t) - \psi(t)| \le (1/2)\|\varphi - \psi\|\,.$$

Thus, for all t we have

$$|(F\varphi)(t) - (F\psi)(t)| \le \min\Big[1/2, 1 - \frac{\epsilon^2}{16}\Big]\|\varphi - \psi\|\,.$$

The proof is complete. □

For the rest of the paper we define

$$\mathbb{M} = \{\varphi \in P_T \mid \|\varphi\| \le L\}\,,$$

where $L = \sqrt{3}/3$.

We also need the following condition on the nonlinear term q.

(D5) There exists periodic functions α, $\beta \in L^1[0,T]$, with period T, such that

$$|q(t,x)| \le \alpha(t)|x| + \beta(t)\,,$$

for all $x \in \mathbb{R}$.

Lemma 3.3. *Suppose that* (D5) *hold. Also suppose there exist constants* $\lambda > 0$, $R > 0$, $J \ge 3$ *and* $\gamma > 0$ *such that*

$$|\alpha(t)|L^3 + |\beta(t)| \le \lambda L a(t)\,, \tag{3.4}$$

$$|r(t)| \le R a(t)\,, \tag{3.5}$$

$$\gamma = \max_{t\in[0,T]} |c(t)|\,, \tag{3.6}$$

and

$$J(\gamma + \lambda + R) \le 1\,. \tag{3.7}$$

For A defined by (3.3), *if $\varphi \in \mathbb{M}$, then $|(A\varphi)(t)| \le L/J \le L$ for all t.*

Proof. Let $\varphi \in \mathbb{M}$. Then $\|\varphi\| \le L$. Thus for A defined by (3.3) we have that

$$\begin{aligned}|(A\varphi)(t)| \le {} & \big|c(t)\varphi\big(g(t)\big)\big| \\ & + \rho\int_{t-T}^{t} \big|q\big(u,\varphi^3(g(u))\big)\big| e^{-\int_u^t a(r)dr} du \\ & + \rho\int_{t-T}^{t} \big|r(u)\varphi\big(g(u)\big)\big| e^{-\int_u^t a(r)dr} du\,.\end{aligned}$$

It follows from conditions (D5), (3.4), (3.5), (3.6) and (3.7) that

$$\begin{aligned}|(A\varphi)(t)| &\le \gamma L \\ &\quad + \rho\int_{t-T}^{t} [|\alpha(u)|L^3 + |\beta(u)|] e^{-\int_u^t a(r)dr} du \\ &\quad + \rho R\int_{t-T}^{t} a(u)L e^{-\int_u^t a(r)dr} du \\ &\le \gamma L \\ &\quad + \rho\lambda L\int_{t-T}^{t} a(u) e^{-\int_u^t a(r)dr} du \\ &\quad + \rho R L\int_{t-T}^{t} a(u) e^{-\int_u^t a(r)dr} du \\ &\le (\gamma+\lambda+R)L \le \frac{L}{J} < L\,.\end{aligned}$$

Therefore A maps $\mathbb{M}$ into itself. $\square$

Lemma 3.4. *Suppose* (D1)*,* (D2)*,* (D3)*,* (D4) *and* (D5) *hold. Suppose also that the hypotheses in Lemma 3.3 hold. For* B, A *defined by* (3.2) *and* (3.3)*, if* φ, $\psi \in \mathbb{M}$ *are arbitrary, then*

$$A\varphi + B\psi \colon \mathbb{M} \to \mathbb{M}\,.$$

Moreover, B *is a large contraction on* M *with a unique fixed point in* $\mathbb{M}$*.*

Proof. Let φ, $\psi \in \mathbb{M}$ be arbitrary. Note that $|\psi(t)| \leq \sqrt{3}/3$ implies

$$|\psi(t) - \psi^3(t)| \leq (2\sqrt{3})/9\,.$$

Using the definition of B and the result of Lemma 3.3, we obtain

$$\begin{aligned}
|(A\varphi)(t) + (B\psi)(t)| &\leq \big|c(t)\varphi\big(g(t)\big)\big| + \rho\int_{t-T}^{t}\big|q\big(u,\varphi^3(g(u))\big)\big|e^{-\int_u^t a(r)dr}du \\
&\quad + \rho\int_{t-T}^{t}\big|r(u)\varphi\big(g(u)\big)\big|e^{-\int_u^t a(r)dr}du \\
&\quad + \Big|\rho\int_{t-T}^{t}a(u)|\psi(u) - \psi^3(u)|e^{-\int_u^t a(r)dr}du\Big| \\
&\leq \frac{\sqrt{3}}{3J} + \frac{2\sqrt{3}}{9} \leq L\,.
\end{aligned}$$

Thus $A\varphi + B\psi \in \mathbb{M}$.

We will next show that B is a large contraction with a unique fixed point in $\mathbb{M}$. Proposition 3.2 shows that $\psi - \psi^3$ is a large contraction in the supremum norm. Thus for any ϵ, we found a $\delta < 1$ from the proof of that proposition such that

$$|(B\varphi)(t) - (B\psi)(t)| \leq \rho\int_{t-T}^{t}a(u)\delta||\varphi - \psi||e^{-\int_u^t a(r)dr}du \leq \delta\|\varphi - \psi\|\,.$$

Furthermore, since $0 \in \mathbb{M}$ the above inequality shows that, $B\colon \mathbb{M} \to \mathbb{M}$ when $\psi = 0$. This completes the proof. $\square$

Theorem 3.5. *Let* $(P_T, \|\cdot\|)$ *be the Banach space of continuous* T*-periodic real functions and* $\mathbb{M} = \{\varphi \in P_T \,\|\varphi\| \leq L\}$*, where* $L = \sqrt{3}/3$*. Suppose* (D1)*,* (D2)*,* (D3)*,* (D4)*,* (D5) *and* (3.4)*–*(3.7) *hold. Then equation* (1.1) *possesses a periodic solution* φ *in the subset* $\mathbb{M}$*.*

Proof. By Lemma 2.1, φ is a solution of (1.1) if

$$\varphi = A\varphi + B\varphi\,,$$

where B and A are given by (3.2) and (3.3) respectively. By Lemma 3.1, $A\colon \mathbb{M} \to \mathbb{M}$ is completely continuous. By Lemma 3.4, $A\varphi + B\psi \in \mathbb{M}$ whenever $\varphi, \psi \in \mathbb{M}$. Moreover, $B\colon \mathbb{M} \to \mathbb{M}$ is a large contraction. Thus all the hypotheses of Theorem 2.4 of Krasnoselskii are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}$ such that $\varphi = A\varphi + B\varphi$. Hence (1.1) has a T-periodic solution. This completes the proof. $\square$

4. Existence of positive solutions

In this section we obtain sufficient conditions under which there exists positive solutions of (1.1). We begin by defining some quantities. Let

$$z \equiv \min_{s\in[t-T,t]} e^{-\int_s^t a(r)dr}, \qquad Z \equiv \max_{s\in[t-T,t]} e^{-\int_s^t a(r)dr}.$$

Given constants $0 < L < K$, define the set $\mathbb{M}_p = \{\psi \in P_T : L \le \psi(t) \le K, t \in [0,T]\}$.

In this section we make the following assumptions.

(D6) $c \in C^1(\mathbb{R},\mathbb{R})$ satisfies $c(t+T) = c(t)$ for all t and there exists a $c^* > 0$ such that $c^* < c(t)$ for all $t \in [0,T]$.

(D7) There exits α such that $\|c\| \le \alpha < 1$.

(D8) There exists constants $0 < L < K$ such that

$$\frac{(1-c^*)L}{\rho z T} \le a(u)[\sigma - \sigma^3] + q(u,\sigma^3) - r(u)\sigma \le \frac{(1-\alpha)K}{\rho Z T}$$

for all $\sigma \in \mathbb{M}$ and $u \in [t-T,t]$.

Theorem 4.1. *Suppose that conditions* (D1)*,* (D3)*,* (D4)*,* (D6)*,* (D7) *and* (D8) *hold. Then there exists a positive solution of* (1.1)*.*

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then

$$\begin{aligned} A\varphi(t) + B\psi(t) &= c(t)\varphi(g(t)) + \rho \int_{t-T}^{t} \big[a(u)[\psi(u) - \psi^3(u)] \\ &\quad + q(u,\varphi^3(g(u))) - r(u)\varphi(g(s))\big] e^{-\int_u^t a(r)dr} du \\ &\ge c^* L + \rho z T \frac{(1-c^*)L}{\rho z T} = L\,. \end{aligned}$$

Likewise,

$$A\varphi(t) + B\psi(t) \le \alpha K + \rho Z T \frac{(1-\alpha)K}{\rho Z T} = K\,.$$

Thus condition (i) of Theorem 2.4 is satisfied. From Lemma 3.1 the operator A is completely continuous and from Lemma 3.4 the operator B is a large contraction. Therefore, by Theorem 2.4, the operator H has a fixed point in $\mathbb{M}_p$. This fixed point is a positive solution of (1.1). ☐

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