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PARTIAL DCPO'S AND SOME APPLICATIONS

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Abstract. We introduce partial dcpo's and show their some applications. A partial dcpo is a poset associated with a designated collection of directed subsets. We prove that (i) the dcpo-completion of every partial dcpo exists; (ii) for certain spaces X, the corresponding partial dcpo's of continuous real valued functions on X are continuous partial dcpos; (iii) if a space X is Hausdorff compact, the lattice of all S-lower semicontinuous functions on X is the dcpo-completion of that of continuous real valued functions on the space; (iv) a topological space has an injective hull iff it is homeomorphic to the pre-Scott space of a continuous partial dcpo whose way-below relation satisfies the interpolation property.

1. Introduction

Domain theory is a branch of mathematics about a special class of partially ordered sets. It has essential applications in theoretical computer science as well as other areas of mathematics. In general topology, one type of application is to represent some topological spaces in terms of intrinsic order topologies on posets. For instance, the injective T_0 spaces are represented as the continuous lattices with their Scott topologies [23], sober spaces that have an injective hull are represented as the continuous dcpo's with their Scott topologies (see [17] and [11]), and complete metric spaces are represented as the maximal point spaces of certain continuous dcpo's [3] (see also [18]). More generally, every T_1 space is homeomorphic to the maximal point space of some algebraic poset (see [27] and [7]). For other works on maximal point spaces, see [15], [20], [22] and [19].

In the present paper we examine two other links of domain theory with topology. Given a topological space X, let $C(X, \mathbb{R})$ be the set of all continuous real valued functions (with respect to the usual topology on $\mathbb{R}$) on X. Then $C(X, \mathbb{R})$ is a lattice under the pointwise order of functions. However, $C(X,\mathbb{R})$ is not a directed complete poset (a directed subset need not have a supremum). In [28] (also in [14] and [16]), a directed complete poset $E(P)$, called the dcpo-completion of P, is constructed for every poset P, which in certain sense, is the smallest extension of P

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to a directed complete poset. It is thus natural to wonder what the dcpo-completion of $C(X,\mathbb{R})$ could be. It seems hard to represent the dcpo-completion of $C(X,\mathbb{R})$ as a class of naturally defined real valued functions on X. In addition, the poset $C(X,\mathbb{R})$ need not be a continuous poset (in the sense of [10]) even for the Euclidean space $X=[0,1]$. In the following, we introduce partially directed complete posets (partial dcpos, for short), of which all posets are special cases. We show that if X is a compact Hausdorff space, then $C(X,\mathbb{R})$ is a continuous partial dcpo. Furthermore, the dcpo-completion of every partial dcpo exists and for certain spaces X, the dcpo-completion of $C(X,\mathbb{R})$, when it is taken as a special partial dcpo, is the poset $SLSC(X,\mathbb{R}^T)$ of certain lower semicontinuous functions from X to $\mathbb{R}^T=\mathbb{R}\cup\{+\infty\}$ with the usual topology.

A topology (pre-Scott topology), generalizing the Scott topology on a poset, is defined for each partial dcpo and it is proved that the lattice of all closed subsets of a partial dcpo P is isomorphic to the lattice of Scott closed sets of its dcpo-completion. We also prove that a topological spaces has an injective hull iff it is homeomorphic to a continuous partial dcpo with the pre-Scott topology and the way-below relation satisfies the interpolation property. Spaces having an injective hull were first systematically studied by Banaschewski [1], and later by Hoffmann [11], Lawson, Erné and others. Recently, Erné showed that if X is a weak monotone convergence space and has an injective hull, then X is homeomorphic to the Scott space of a continuous poset [6].

2. Partially directed-complete posets

If A is a subset of a poset P, we denote $\downarrow A=\{x\in P:\exists y\in A,\ x\le y\}$ and $\uparrow A=\{x\in P:\exists y\in A,\ y\le x\}$. For any $x\in P$, we take $\downarrow x=\downarrow\{x\}$ and $\uparrow x=\uparrow\{x\}$. A subset A of P is an upper (lower) set if $A=\uparrow A$ (resp. $A=\downarrow A$). A non-empty subset D of a poset is called a directed set if any two elements in D have an upper bound in D.

A subset U of a poset P is Scott open if it is an upper set and for any directed subset $D\subseteq P$, $\sup D\in U$ implies $D\cap U\neq\emptyset$ whenever $\sup D$ exists. The Scott open sets form a topology, called the Scott topology on P and denoted by $\sigma(P)$. The symbol ΣP is used for the space $(P,\sigma(P))$. Every ΣP is a T_0 space and it is T_1 only if P has the discrete order.

The Scott topology on the poset of real numbers (with the usual order of numbers) coincides with the upper topology generated by $\{(r,\infty):r\in\mathbb{R}\}$.

A mapping $f\colon P\to Q$ between two posets is Scott continuous if for each directed set D of P, $f(\sup D)=\sup f(D)$ whenever $\sup D$ exists. It's well known that a mapping $f\colon P\to Q$ is Scott continuous iff it is continuous with respect to the Scott topology (see Lemma 1 below for a more general result).

A poset P is a directed complete poset (or dcpo for short) if the supremum $\sup D$ exists for every directed subset $D\subseteq P$.

For more about dcpo's, continuous posets and Scott topology see the excellent monograph [10].

Definition 1. A *partial dcpo* is a pair (P, Ψ), where P is a poset and Ψ is a collection of directed subsets of P, such that

(1) $\sup D$ exists for each $D \in \Psi$;

(2) for any directed subset E of P, $E \in \Psi$ if and only if $\downarrow E \in \Psi$;

(3) $\{x\} \in \Psi$ for any $x \in P$.

Example 1. (1) For every poset P, let $\mathcal{D}_P$ be the set of all directed subsets D of P such that $\sup D$ exists in P. Then $(P, \mathcal{D}_P)$ is a partial dcpo. We shall call $\mathcal{D}_P$ the standard family of directed sets. Obviously, for any partial dcpo (P, Ψ), $\Psi \subseteq \mathcal{D}_P$.

In the following, if a poset P is taken as a partial dcpo without specifying Ψ we shall mean that $\Psi = \mathcal{D}_P$. In particular, if P is a dcpo, then the family Ψ denotes the set of all directed subsets of P.

(2) Let Q be a subposet of a poset P and Ψ be the set of all directed subsets D of Q such that $\sup_P D$ exists and $\sup_P D \in Q$. Then (Q, Ψ) is a partial dcpo.

(3) Let X be a topological space and $C(X, \mathbb{R})$ be the set of all continuous real valued functions defined on X. With the pointwise order, $C(X, \mathbb{R})$ is a poset. Let Ψ be the set of all directed subset D of $C(X, \mathbb{R})$ such that the function f defined by $f(x) = \sup\{g(x) : g \in D\}(x \in X)$ exists and is continuous. Then $(C(X, \mathbb{R}), \Psi)$ is a partial dcpo.

(4) Let X be a topological space and $F(X)$ be the set of all non-empty closed sets of X. Let $\Psi = \{\mathcal{F} \subseteq F(X) : \mathcal{F}$ is a filter base and $\bigcap \mathcal{F} \neq \emptyset\}$. Then with the inverse inclusion order, $(F(X), \Psi)$ is a partial dcpo.

(5) Let P be a poset. Define $\mathcal{M}(P) = \{A \subseteq P : A$ has a largest element$\}$. Then for any $x \in P, \{x\} \in \mathcal{M}(P)$ and $\downarrow x \in \mathcal{M}(P)$. In general, $A \in \mathcal{M}(P)$ if and only if there is $x \in P$ such that $A = \{x\} \cup E$ with E a subset of $\downarrow x$. Then $(P, \mathcal{M}(P))$ is a partial dcpo. Obviously, if (P, Ψ) is a partial dcpo, then $\mathcal{M}(P) \subseteq \Psi$.

(6) An ω-chain C in a poset P is a countable subset of P which is a chain. For any poset P, let $\mathcal{D}_\omega(P) = \{E \subseteq P : \downarrow E = \downarrow C$ for some $\omega -$ chain $C\}$. Then $(P, \mathcal{D}_\omega(P))$ is a partial dcpo.

Example 2. Given a T_0 space (X, τ), the specialization order $\leq_\tau$ (or just denoted by $\leq$) is the partial order on X defined by $x \leq_\tau y$ if and only if $x \in \mathrm{cl}(\{y\})$ (the closure of $\{y\}$) for $x, y \in X$. Every open set $U \in \tau$ is an upper set with respect to the specialization order. A directed subset D of $(X, \leq_\tau)$ is called τ-defined if (i) sup$D$ exists and (ii) for any $U \in \tau$, $D \cap U \neq \emptyset$ whenever sup$D \in U$. Let Φ_τ be the collection of all τ-defined directed subsets of X. Then conditions (1) and (3) of Definition 1 are clearly satisfied by Φ_τ. As every $U \in \tau$ is an upper subset of $(X, \leq_\tau)$, it is also straightforward to show that condition (2) is satisfied by Ψ_τ. Thus we have a partial dcpo (X, Ψ_τ) for any T_0 space X, which will be called the induced partial dcpo of space X. One can easily verify that a directed subset of X (with respect to the specialization order) is τ-defined iff sup$D$ exists and the net $S = \{s(d) : d \in D\}$ converges to sup$D$, where $s(d) = d$ for each $d \in D$.

We will use $(X, \leq_\tau)'s$ to construct a right adjoint functor from the category of T_0 space to the category of partial dcpo's in the last section.

The lemma below can be verified easily.

Lemma 1. *Given a T_0 space (X, τ), a directed subset $D \subseteq X$ is τ-defined iff* $\mathrm{cl}(D) = c(\{x\})$ *for some $x \in X$.*

Note that for a given poset P, there could be two different collections Ψ_1 and Ψ_2 such that both (P, Ψ_1) and (P, Ψ_2) are partial dcpos.

A homomorphism $f\colon (P, \Psi) \to (Q, \Phi)$ between two partial dcpo's is a map $f\colon P \to Q$ such that for each $D \in \Psi$, $f(D) \in \Phi$ and $f(\sup D) = \sup f(D)$.

If $f\colon (P, \Psi) \to (Q, \Phi)$ is a homomorphism and x, $y \in P$ with $x \leq y$, then $\downarrow y \in \Psi$, so $f(y) = f(\sup \downarrow y) = \sup f(\downarrow y) \geq f(x)$. Thus every homomorphism between partial dcpo's is monotone.

Let $PDCPO$ be the category of partial dcpo's and homomorphisms between them. The category $DCPO$ of all dcpo's and Scott continuous functions is a full subcategory of $PDCPO$.

Definition 2. A *dcpo-completion* of a partial dcpo (P, Ψ) is a pair (M, η), where M is a dcpo (with the standard family of directed subsets) and $\eta\colon (P, \Psi) \to M$ is a homomorphism in $PDCPO$ (called the universal homomorphism) such that for any homomorphism $f\colon (P, \Psi) \to Q$ into a dcpo Q, there is a unique Scott continuous map $\hat{f}\colon M \to Q$ such that $f = \hat{f} \circ \eta$.

Our main concern in this section is: does the dcpo-completion exist for every partial dcpo? Equivalently, is the category $DCPO$ reflective in $PDCPO$? In [28] (also in [16] and [14]), it is proved that every poset has a dcpo-completion (in terms of the current definition, it means that for each poset P, the partial dcpo $(P, \mathcal{D}_P)$ has a dcpo-completion).

In order to answer the above question, we first introduce a topology on each partial dcpo, which is similar to the Scott topology on posets.

A subset U of a partial dcpo (P, Ψ) is called pre-Scott open if it is an upper set and for any $D \in \Psi$, $\sup D \in U$ implies $D \cap U \neq \emptyset$. The complements of pre-Scott open sets are called pre-Scott closed sets.

The pre-Scott open sets of a partial dcpo P form a topology on P, denoted by $\sigma_\Psi(P)$ and called the pre-Scott topology on P. We will denote the space $(P, \sigma_\Psi(P))$ by $\Sigma(P, \Psi)$. The set of all pre-Scott closed sets is denoted by $\Gamma_\Psi(P)$. As for any topological space, $\sigma_\Psi(P)$ and $\Gamma_\Psi(P)$ are complete lattices under the inclusion order. A subset A of a partial dcpo (P, Ψ) is pre-Scott closed iff it is a lower set and for any $D \in \Psi$, $D \subseteq A$ implies $\sup D \in A$.

Let (P, Ψ) be a partial dcpo. A subset A of P is called D-closed if $\sup E \in A$ whenever $E \in \Psi$ and $E \subseteq A$. Obviously, the intersection of any collection of D-closed sets is D-closed. Now assume that A and B are D-closed sets of P, and $E \in \Psi$ such that $E \subseteq A \cup B$. Then there must be a directed set $F \subseteq E$ such that $\downarrow F \supseteq E$ and either $F \subseteq A$ or $F \subseteq B$. Then as $\downarrow F = \downarrow E$, so $F \in \Psi$ and $\sup F \in A$ or $\sup F \in B$. But $\sup F = \sup E$, thus we have $\sup E \in A \cup B$. So $A \cup B$ is D-closed.

Therefore, the set of all D-closed sets of P forms the set of closed sets of a topology on P, called the D-topology. Complements of D-closed sets are called D-open sets.

Remark 1. (1) Every Scott open set is D-open.

(2) Every lower set is D-open.
(3) A subset is pre-Scott open iff it is a D-open upper set.

The D-closure (the closure with respect to the D-topology) of $A \subseteq P$ will be denoted by $\mathrm{cl}_d^{\Psi}(A)$ (or just $\mathrm{cl}_d(A)$ where no confusion may occur).

The following lemma can be proved by a routine checking. For the sick of convenience to readers, we give a brief proof.

Lemma 2. *Let $f\colon (P, \Psi) \to (Q, \Phi)$ be a mapping between two partial dcpo's such that $f(D) \in \Phi$ for each $D \in \Psi$. Then the following statements are equivalent:*
(1) *f is a homomorphism.*
(2) *f is continuous with respect to the pre-Scott topologies.*
(3) *f is monotone and is continuous with respect to the D-topologies.*

Proof. (1) implies (2). Let f be a homomorphism. Then f can be shown to be monotone by Definition 1(3). For each pre-Scott open set U of Q, $f^{-1}(U)$ is an upper set because f is monotone. If $D \in \Psi$ and $\sup D \in f^{-1}(U)$, then $\sup f(D) \in U$, implying $f(D) \cap U \neq \emptyset$. Thus $D \cap f^{-1}(U) \neq \emptyset$. Hence $f^{-1}(U)$ is pre-Scott open in P, and thus f is continuous.

(2) implies (1). Assume that f is continuous. Let $D \in \Psi$ and $f(\sup D) \not\leq \sup f(D)$. Put $a = \sup f(D)$. The set $Q - \downarrow a$ is D-open and $\sup D \in f^{-1}(Q - \downarrow a)$. Thus $D \cap f^{-1}(Q - \downarrow a) \neq \emptyset$, this then implies $f(D) \cap (Q - \downarrow a) \neq \emptyset$, which is impossible because $f(D) \subseteq \downarrow a = \downarrow \sup f(D)$. So for any $D \in \Psi$, $f(\sup D) \leq \sup f(D)$ and thus $f(\sup D) = \sup f(D)$, showing that f is a homomorphism.

(1) implies (3). This is trivial.

(3) implies (2). Let f be monotone and continuous with respect to the D-topology. For any pre-Scott open set U of Q, $f^{-1}(U)$ is an upper set as well as a D-open set, and hence it is pre-Scott open. □

The corollary below follows from a general fact on continuous mappings and closure operators.

Corollary 1. *If $f\colon P \to Q$ is a homomorphism between partial dcpo's, then for any $A \subseteq P, f(\mathrm{cl}_d(A)) \subseteq \mathrm{cl}_d(f(A))$.*

Lemma 3. *If f and g are Scott continuous mappings from dcpo P to dcpo Q, and $A \subseteq P$ with $f|_A = g|_A$, then $f(x) = g(x)$ for every $x \in \mathrm{cl}_d(A)$.*

Proof. Let $B = \{x \in P : f(x) = g(x)\}$. It is clear that B is a D-closed set and contains A, thus $\mathrm{cl}_d(A) \subseteq B$, implying $f(x) = g(x)$ for all $x \in \mathrm{cl}_d(A)$. □

Note that if L is a complete lattice (regarded as the partial dcpo $(L, \mathcal{D}_L)$), then for any subset A of L, the D-closure $\mathrm{cl}_d(A)$ of A is a sub dcpo of L (it is actually the smallest sub dcpo containing A), that is, it is closed under taking supremum of directed sets. In particular, $\mathrm{cl}_d(A)$ is a dcpo under the inherited order from L.

Theorem 1. *For any partial dcpo (P, Ψ), $(E(P), \eta_P)$ is a dcpo-completion of (P, Ψ), where $E(P)$ is the D-closure of $\kappa(P) = \{\downarrow x : x \in P\}$ in the complete lattice $\Gamma_\Psi(P)$ and the universal mapping $\eta_P\colon P \to E(P)$ sends each $x \in P$ to $\downarrow x$.*

Proof. First, by the above remarks, $E(P)$ is a dcpo with respect to the inclusion order, where for any directed $\mathcal{B} \subseteq E(P)$, $\sup \mathcal{B}$ in $E(P)$ equals its supremum in $\Gamma_\Psi(P)$ which equals $\mathrm{cl}_d(\bigcup \mathcal{B})$.

For any $B \in \Psi$, $a = \sup B$ exists in P, and $a \in \mathrm{cl}_d(\bigcup\{\downarrow x : x \in B\}) = \sup\{\eta_P(x) : x \in B\} = \sup \eta_P(B)$ (here the supremum is taken in $E(P)$). As $\mathrm{cl}_d(\bigcup\{\downarrow x : x \in B\})$ is a lower set of P, $\downarrow a \subseteq \sup \eta_P(B)$. But it is trivial that $\downarrow a \supseteq \sup \eta_P(B)$, hence $\eta_P(\sup B) = \sup \eta_P(B)$ holds for every $B \in \Psi$, therefore η_P is a homomorphism in $PDCPO$.

Now let $f\colon P \to Q$ be any homomorphism in $PDCPO$ from (P, Ψ) to a dcpo Q. Then, by Lemma 1, f is continuous with respect to the pre-Scott topology on P and the Scott topology on Q. Thus the mapping $f^{-1}\colon \Gamma(Q) \to \Gamma_\Psi(P)$, sending each $A \in \Gamma(Q)$ to its inverse image under f, is a mapping preserving arbitrary meets and finite unions. Let $f^*\colon \Gamma_\Psi(P) \to \Gamma(B)$ be the mapping that sends $X \in \Gamma_\Psi(P)$ to $\mathrm{cl}(f(X))$ (the closure of $f(X)$ in $(Q, \sigma(Q))$). Then f^* is the lower adjoint of f^{-1}. Thus f^* preserves arbitrary joins, and so it is Scott continuous between the two complete lattices $\Gamma_\Psi(P)$ and $\Gamma(Q)$.

By Corollary 1, $f^*(\mathrm{cl}_d(\kappa(P))) \subseteq \mathrm{cl}_d(f^*(\kappa(P)))$. For each $x \in P, f^*(\downarrow x) = \mathrm{cl}(f(\downarrow x))$. Since f is monotone, $f(\downarrow x) \subseteq \downarrow f(x) \subseteq \downarrow f(\downarrow x)$, implying $\mathrm{cl}(f(\downarrow x)) = \downarrow f(x)$. Now $f^*(\downarrow x) = \mathrm{cl}(f(\downarrow x)) = \downarrow f(x) \in \kappa(Q) = \{\downarrow y : y \in Q\}$. This shows that $f^*(\kappa(P)) \subseteq \kappa(Q)$. Since Q is a dcpo, one easily verify that the D-closure of $\kappa(Q)$ in $\Gamma(Q)$ equals itself, that is $\mathrm{cl}_d(\kappa(Q)) = \kappa(Q)$. Now we deduced that $f^*(\mathrm{cl}_d(\kappa(P))) \subseteq \mathrm{cl}_d(f^*(\kappa(P))) \subseteq \mathrm{cl}_d(\kappa(Q)) = \kappa(Q)$.

Let $f^o\colon\ \mathrm{cl}_d(\kappa(P)) \to \kappa(Q)$ be the restriction of f^* to the domain $\mathrm{cl}_d(\kappa(P))$ and the codomain $\kappa(Q)$. The mapping $\xi\colon \kappa(Q) \to Q$ defined by $\xi(\downarrow y) = y$ is obviously an isomorphism.

Now let $\hat{f} = \xi \circ f^o$. Then $\hat{f}\colon\ \mathrm{cl}_d(\kappa(P)) \to Q$ is a Scott continuous mapping satisfying $f = \hat{f} \circ \eta_P$.

If $h\colon\ \mathrm{cl}_d(\kappa(P)) \to Q$ is any Scott continuous mapping satisfying $f = h \circ \eta_P$, then for each $x \in P$, $h(\downarrow x) = h \circ \eta_P(x) = f(x) = \hat{f} \circ \eta_P(x)) = \hat{f}(\downarrow x)$, that is, $h(C) = \hat{f}(C)$ holds for all $C \in \kappa(P)$. By Lemma 3, $h = \hat{f}$. The proof is completed. □

In [16], Keimel and Lawson took another approach to obtain the dcpo-completion of a poset P. First consider the standard soberification $\epsilon_P\colon P \to P^s$ of the Scott space $\Sigma P = (P, \sigma)$, where P^s consists of all the irreducible Scott closed sets. Note that the specialization order on P^s equals the inclusion order. Let P^d be the D-closure of the image of P in P^s (here the D-topology on a T_0 space X is the D-topology on X with the specialization order and the standard family of directed sets). Since the union of a directed family of irreducible closed sets is an irreducible set, so P^s is a D-closed set of $\Gamma(P)$. Also $\kappa(P) = \{\downarrow x : x \in P\}$ is a subset of P^s, so the D-closure of $\kappa(P)$ in $\Gamma(P)$ equals its D-closure in P^s. It thus follow that P^d equipped with the relative specialization order of P^s is the dcpo-completion of the poset P.

Now given a partial dcpo (P, Ψ), the soberification P^s_Ψ of $(P, \sigma_\Psi(P))$ consists of all the irreducible pre-Scott closed sets. Again, P^s_Ψ is a D-closed subset of $\Gamma_\Psi(P)$, so the D-closure of $\kappa(P)$ in P^s_Ψ equals its D-closure in $\Gamma_\Psi(P)$.

By the proof of Theorem 1 and the above remarks, we have the following result:

Proposition 1. *For a partial dcpo (P, Ψ), let $\eta\colon P \to X$ be a soberification of $(P, \sigma_\Psi(P)$. Then the D-closure of $\eta(P)$ in X equipped with the relative specialization order on X is the dcpo-completion of (P, Ψ).*

Definition 3. A subset B of a dcpo P is called a *basis* of P if there is a function $\phi\colon P \to \mathcal{P}(B)$ from P to the power set of B such that the following conditions are satisfied:

(B1) for each $a \in P$, $\phi(a)$ is a directed set and $\sup \phi(a) = a$;

(B2) for any directed set $D \subseteq P, \phi(\sup D) = \bigcup\{\phi(a) : a \in D\}$.

Theorem 2. *Let B be a basis of a dcpo P. Then (P, η_B) is a dcpo-completion of the partial dcpo (B, Ψ), where $\Psi = \{C \subseteq B : C$ is directed and $sup_P C \in B\}$ and $\eta_B\colon B \to P$ is the embedding map.*

Proof. By the definition of Ψ, η_B is a homomorphism from (B, Ψ) to P. Now assume that $f\colon (B, \Psi) \to Q$ is a partial dcpo homomorphism from B to a dcpo Q. Define $\hat{f}\colon P \to Q$ by $\hat{f}(a) = \sup f(\phi(a))$ for each $a \in P$. Then $\hat{f} \circ \eta_B = f$.

For any directed set $D \subseteq P$, $\hat{f}(\sup D) = \sup f(\phi(\sup D)) = \sup f(\bigcup\{\phi(x) : x \in D\}) = \sup \bigcup\{f(\phi(x)) : x \in D\} = \sup\{\sup f(\phi(x)) : x \in D\} = \sup\{\hat{f}(x) : x \in D\}$. Thus $\hat{f}$ is a Scott continuous mapping between P and Q. Let $h\colon P \to Q$ be any Scott continuous mapping satisfying $h \circ \eta_B = f$. Then for any $a \in P$, $a = \sup \phi(a)$, so $h(a) = \sup h(\phi(a)) = \sup f(\phi(a))$, the last equation holds because $\phi(a) \subseteq B$ and for any $x \in B, h(x) = h \circ \eta_B(x) = f(x)$. Hence $h(a) = \hat{f}(a)$ for all $a \in P$. Therefore (P, η_B) is a dcpo-completion of (B, Ψ). □

Example 3. (1) Let $\mathbb{Q}$ be the poset of all rational numbers with the usual order of numbers. Then $\mathbb{Q}$ is a subset of $\mathbb{R}^T = \mathbb{R} \cup \{\infty\}$. Define $\phi\colon \mathbb{R}^T \to \mathbb{Q}$ by $\phi(x) = \{y \in Q : y < x\}$. Then ϕ satisfies the conditions of Definition 3, so $\mathbb{Q}$ is a basis of $\mathbb{R}^T$. By Theorem 2, $\mathbb{R}^T$ is a dcpo-completion of $(\mathbb{Q}, \Psi)$, where Ψ is the set of all non-empty $D \subseteq \mathbb{Q}$ with $\sup D \in \mathbb{Q}$.

Clearly, if B is a basis of P and $B \subseteq A \subseteq P$, then A is also a basis of P, thus P is also a dcpo-completion of (A, Ψ). It follows that the dcpo-completion of $\mathbb{R}$, the poset of all real numbers, is also $\mathbb{R}^T$.

(2) Let X be a non-empty set and $(\mathcal{P}(X), \subseteq)$ be the complete lattice of all subsets of X. The set $\mathbf{B}$ of all finite subsets of X is a basis of $\mathcal{P}(X)$ (define $\phi(A) = \{D \in \mathbf{B} : D \subseteq A\}$). Hence $\mathcal{P}(X)$ is a dcpo-completion of $(\mathbf{B}, \Psi)$.

The major application of Theorem 2 is in the proof of Theorem 5.

The concept of bases of a domain (i.e. continuous dcpo) L was defined in [10]: $B \subseteq L$ is a basis of L if for each $x \in L$, (i)$\{y \in L : y \ll x\} \cap B$ is a directed set and (ii) $x = \sup(\{y \in L : y \ll x\} \cap B)$ ([10, Definition III-4.1]). A basis B of a domain L in this sense is clearly a basis as defined in Definition 3, here we just let $\phi(x) = \{y \in L : y \ll x\} \cap B$ for each $x \in L$.

The reader may wonder whether a dcpo that has a basis in the sense of Definition 3 must be a domain. The following example gives a negative answer.

Example 4. Let $L = \{0, 1, a\} \cup \{b_n : n = 1, 2, 3, ...\}$ and define $0 < a < 1$, $0 < b_n < b_{n+1} < 1$ for all n. It is well-known that $(L, \leq)$ is not a meet-continuous lattice (let alone a domain). Let $B = \{0, a\} \cup \{b_n : n = 1, 2, 3, \ldots\}$ and assign $\phi(x) = \{x\}$ if $x \neq 1$, and $\phi(1) = \{b_n : n = 1, 2, 3, \ldots\}$. Then B is a basis of L. It is easily verified that this B is the smallest basis (i.e. if C is another basis of L, then $B \subseteq C$).

Definition 4. Let P be a dcpo. For any x, $y \in P$, define $x \dashv y$ if for any directed subset D of P, $\sup D = y$ implies $x \leq d$ for some $d \in D$.

A dcpo P is called *below-continuous* if for each $a \in P$, $\mathrm{T}_a = \{x \in P : x \dashv a\}$ is a directed set and $a = \sup T_a$.

Remark 2. (1) If $x \dashv y$ then $x \leq y$ (consider the directed set $D = \{y\}$), so T_a is a lower set for each $a \in P$.

(2) $x \leq y \dashv z$ implies $x \dashv z$. Unlike the way-below relation $\ll$, in general $x \dashv y \leq z$ do not imply $x \dashv z$.

(3) If P is below-continuous, then the relation $\dashv$ satisfies the interpolation property: if $x \dashv y$ then there is $z \in P$ such that $x \dashv z \dashv y$. The proof is similar to that for the way-below relation of continuous posets (see [10, Theorem I-1.9]).

(4) Let L be a complete lattice which is meet-continuous (i.e. $x \wedge \sup D = \sup\{x \wedge d : d \in D\}$ holds for any element x and directed subset D). Then $x \dashv y$ holds in L iff $x \ll y$. Thus L is below-continuous iff it is continuous. For example, for any topological space X, the lattice $\mathcal{O}(X)$ of all open sets of X is below-continuous iff it is continuous.

Theorem 3. *A dcpo is below-continuous iff it has a basis.*

Proof. Assume that dcpo P is below-continuous. Let $B = P$ and define $\phi(x) = \{y \in P : y \dashv x\}$. Then B is a basis of P. To see this, we only need to verify condition (B2) of Definition 3, but this follows directly from Remark 2.

Now let P have a basis B determined by the function $\phi\colon P \to \mathcal{P}(B)$. Then for each $x \in P$, one can easily verify that $\downarrow\phi(x) = \{y \in P : y \dashv x\}$, which is a directed set because $\phi(x)$ is directed. Also $\sup \downarrow\phi(x) = \sup \phi(x) = x$. Thus P is below-continuous. □

Example 4 gives a complete lattice that is below-continuous but not continuous.

For any poset P, let $\mathrm{Idl}(P)$ be the poset of all ideals of P (directed and lower subsets of P). Recall that a dcpo P is continuous iff the mapping $\sup\colon \mathrm{Idl}(P) \to P$ has a lower adjoint (see [10, Theorem I-1.10]).

The more differences and similarities between continuous dcpo's and below-continuous dcpo's can be seen in the following proposition.

Proposition 2. *Let P be a dcpo. Then the following statements are equivalent:*

(1) *P is below-continuous;*

(2) *the mapping sup: $Idl(P) \to P$ has a Scott continuous right inverse, i.e there is a Scott continuous mapping $g\colon P \to Idl(P)$ such that $\sup \circ g = id_P$;*

(3) *there is a non-empty subset B of P such that the mapping sup: $Idl(B) \to P$ has a Scott continuous right inverse.*

Example 5. Let $M = \{a_n : n \in N\} \cup \{b_n : n \in N\} \cup \{0, 1\}$ with the order given by $0 < a_n < a_{n+1} < 1$, $0 < b_n < b_{n+1} < 1$ for all n. Then M is a complete lattice. There is no non-zero element x satisfying $x \dashv 1$.

Remark 3. If P and Q are below-continuous, then their Cartesian product is below-continuous. If P is below-continuous and D is a retraction of P (i.e. there are Scott continuous functions f, g such that $g \circ f = \mathrm{id}_Q$), then Q is below-continuous.

It is well-known that if P is a continuous dcpo, then the Scott space ΣP is sober. The reader may also wonder whether the Scott space of each below-continuous dcpo is sober. The answer is negative: the dcpo $\mathbb{N} \times (\mathbb{N} \cup \{\propto\})$ constructed by Johnstone [12] is below-continuous and its Scott space is not sober.

3. The lattice of Scott closed sets of dcpo-completion

Recall that a subset U of a partial dcpo (P, Ψ) is pre-Scott open if it is an upper set and $\sup D \in U$ implies $D \cap U \neq \emptyset$ for any $D \in \Psi$. The set of all pre-Scott open (closed) sets of P is denoted by $\sigma_\Psi(P)$ ($\Gamma_\Psi(P)$, respectively). In this section, we show that for any partial dcpo (P, Ψ), the complete lattice $\Gamma_\Psi(P)$ is isomorphic to the lattice $\Gamma(E(P))$ of Scott closed sets of the dcpo-completion $E(P)$ of P.

Note that for a partial dcpo (P, Ψ) and each $A \subseteq P$, $\mathrm{cl}(A)$ denotes the closure of A with respect to the Scott topology and $\mathrm{cl}_d(A)$ denotes the D-closure of A. For any $A \subseteq P$, $\mathrm{cl}_d(A) \subseteq \mathrm{cl}(A)$ because $\mathrm{cl}(A)$ is D-closed.

Recall that for each poset P, $\mathcal{D}_P$ denotes the family of all directed subsets of P whose supremum exist.

Lemma 4. *Suppose that A is a subset of a poset P such that $P = \mathrm{cl}_d(A)$, where the family of directed subsets on P is taken to be $\mathcal{D}_P$. Then for any $F \in \Gamma(P)$, $F = \mathrm{cl}(F \cap A)$.*

Proof. Since F is a lower set, by Remark 1, it is D-open. As A is dense in P with respect to the D-topology, A is dense in every D-open set, thus $\mathrm{cl}_d(F \cap A) \supseteq F$. But $\mathrm{cl}_d(F \cap A) \subseteq \mathrm{cl}(F \cap A) \subseteq \mathrm{cl}(F) = F$, thus $F = \mathrm{cl}(F \cap A)$. □

A pair (g, f) of monotone mappings $f\colon P \to Q$, $g\colon Q \to P$ between posets is a Galois connection or an adjunction between P and Q if for any $x \in P$, $y \in Q$, $x \leq g(y)$ iff $f(x) \leq y$. In the adjunction (g, f), g is called the upper adjoint and f the lower adjoint. If $h\colon L \to M$ is a mapping between complete lattices and preserves the sups (infs) of arbitrary subsets, then h has an upper (lower, resp.) adjoint. A left adjoint f is surjective (injective, resp.) iff it's right adjoint is injective (surjective, resp.). See Proposition O-3.7 of [10].

Theorem 4. *If $E(P)$ is a dcpo-completion of a partial dcpo (P, Ψ), then $\Gamma_\Psi(P)$ is isomorphic to $\Gamma(E(P))$.*

Proof. Let $\eta\colon P \to E(P)$ be the universal homomorphism of P into the dcpo-completion. Then η is a continuous mapping with respect to the pre-Scott topology on P and the Scott topology on $E(P)$, so the mapping $\eta^*\colon \Gamma_\Psi(P) \to \Gamma(E(P))$, which sends X to $\mathrm{cl}(\eta(X))$, is a left adjoint to the mapping $\eta^{-1}\colon \Gamma(E(P)) \to \Gamma_\Psi(P)$. We show that η^* is an isomorphism. By the proof of Theorem 1, $\mathrm{cl}_d(\eta(P)) = E(P)$.

Thus by Lemma 4, for each $F \in \Gamma(E(P)), F = \mathrm{cl}(F \cap \eta(P))$. Let $G = \eta^{-1}(F)$. Then, as η is continuous, $G \in \Gamma_\Psi(P)$ and $\eta^*(G) = \mathrm{cl}(\eta(G))$. Note that $\eta(G) = \eta(\eta^{-1}(F)) = F \cap \eta(P)$, so $\mathrm{cl}(\eta(G)) = \mathrm{cl}(F \cap \eta(P)) = F$. Thus η^* is surjective. For each $A \in \Gamma_\Psi(P)$, the function $j\colon P \to \mathbf{2}$ with $j^{-1}(\{0\}) = A$, is a homomorphism in $PDCPO$. As $E(P)$ is a dcpo-completion of P, there is a Scott continuous function $h\colon E(P) \to \mathbf{2}$ such that $h \circ \eta = j$. Then $A = j^{-1}(\{0\}) = \eta^{-1}(h^{-1}(\{0\}))$, where $h^{-1}(\{0\}) \in \Gamma(E(P))$. Thus $\eta^{-1}\colon \Gamma(E(P)) \to \Gamma_\Psi(P)$ is surjective. Therefore, by Proposition O-3.7 of [10], the left adjoint η^* of η^{-1} is injective. It follows that the mapping η^* is an isomorphism between $\Gamma(P)$ and $\Gamma(E(P))$. The proof is completed. $\square$

Remark 4. (1) From Theorem 4, it follows that for any partial dcpo (P, Ψ), the lattice $\sigma_\Psi(P)$ of all pre-Scott open sets of P is isomorphic to the Scott open set lattice $\sigma(E(P))$ of the dcpo $E(P)$.

(2) Given a poset P and any collection $\mathcal{F}$ of directed subsets of P whose supremum exist, then $\mathcal{F}$ defines a topology $\mathcal{O}_\mathcal{F}(P)$ on P: $U \in \mathcal{O}_\mathcal{F}(P)$ if U is an upper set and for any $D \in \mathcal{F}$, $\sup D \in U$ implies $D \cap U \neq \emptyset$. One may wonder what type the complete lattice $\mathcal{O}_\mathcal{F}(P)$ would be. Let $\Psi = \{E \subseteq P : E$ is directed and $\downarrow E = \downarrow D$ for some $D \in \mathcal{F}$ or $\downarrow E = \downarrow x$ for some $x \in P\}$. Then (P, Ψ) is a partial dcpo and $U \in \mathcal{O}_\mathcal{F}(P)$ if and only if U is pre-Scott open. Hence $\mathcal{O}_\mathcal{F}(P) = \sigma_\Psi(P)$. Then by Theorem 4, the lattice $\mathcal{O}_\mathcal{F}(P)$ is isomorphic to the Scott open set $\sigma(Q)$ of some dcpo Q.

For example, an ω-open set U of a poset P is an upper set such that for each increasing sequence $D = \{a_n : n \in N\}$, $supD \in U$ implies $D \cap U \neq \emptyset$. The lattice of all ω-open sets of P is now isomorphic to the lattice of Scott open sets of a dcpo.

There is, however, still no a characterization of Scott open set (closed set) lattices of dcpo's.

Given a partial dcpo (P, Ψ), we define the binary relation $\ll_\Psi$ on P by $x \ll_\Psi y$ iff for any $D \in \Psi, \sup D \geq y$ implies $x \leq d$ for some $d \in D$. For each $a \in P$, let $W_\Psi(a) = \{x \in P : x \ll_\Psi a\}$. Clearly, $x \leq y$ implies $W_\Psi(x) \subseteq W_\Psi(y)$.

A partial dcpo (P, Ψ) is called continuous if for each $x \in P$, $W_\Psi(x) \in \Psi$ and $\sup W_\Psi(x) = x$.

We write $\ll$ for $\ll_\Psi$ if $\Psi = \mathcal{D}_P$, the set of all directed subsets of P which have a supremum. If P is a poset such that $(P, \mathcal{D}_P)$ is continuous, then P is just called a continuous poset (see [10, Definition I-1.6]).

Lemma 5. *Let (P, Ψ) be a partial dcpo and Ψ be closed under taking directed unions. If (P, Ψ) is continuous then the relation $\ll_\Psi$ satisfies the interpolation property, that is, for any $x \ll_\Psi y$, there exits $z \in P$ such that $x \ll_\Psi z \ll_\Psi y$.*

The proof is similar to that for continuous posets (see [10, Theorem I-1.9]).

If (P, Ψ) is a continuous partial dcpo and the relation $\ll_\Psi$ satisfies the interpolation property, then for each $x \in P, \{y \in P : x \ll_\Psi y\}$ is pre-Scott open.

The following result can be proved in a similar way as for dcpo's (see [10, Theorem II-1.14]).

Lemma 6. *Let (P, Ψ) be a partial dcpo such that Ψ is closed under taking directed unions. Then (P, Ψ) is continuous iff $(\Gamma_\Psi(P), \subseteq)$ is a completely distributive lattice.*

4. The partial dcpo of continuous functions

For any topological space X, let $C(X, \mathbb{R})$ denote the set of continuous real valued functions $f\colon X \to \mathbb{R}$. With respect to the pointwise order, $C(X,\mathbb{R})$ is a lattice.

The following example shows that under the pointwise order, $C([0,1],\mathbb{R})$ is not a continuous poset.

Example 6. Let $f(x) \equiv 1$ ($x \in [0,1]$) be the function with the constant value 1. Given any element g in $C(X,\mathbb{R})$, let $a = g(0)$. For each natural number n, define $f_n \in C([0,1],\mathbb{R})$ by

$$f_n(x) = \begin{cases} 1 & \frac{1}{n} \leq x \leq 1\,, \\ n(2-a)x + (a-1) & 0 \leq x \leq \frac{1}{n}\,. \end{cases}$$

Then $\{f_n\}$ is a directed subset of $C(X,\mathbb{R})$, and the supremum of $\{f_n\}$ in $C(X,\mathbb{R})$ equals f. However, $g \not\leq f_n$ for any n. Thus $g \not\ll f$. Hence there is no $g \in C(X,\mathbb{R})$ satisfying $g \ll f$, so $C(X,\mathbb{R})$ is not a continuous poset.

Let X be a topological space and $(C(X,\mathbb{R}),\Psi)$ be the partial dcpo defined in Example 1(3). The next questions we are to consider are: (1) for what X, is $(C(X,\mathbb{R}),\Psi)$ a continuous partial dcpo? (2) what is the dcpo-completion of $(C(X,\mathbb{R}),\Psi)$?

In the following, for simplicity, we use $\ll$ to denote the relation $\ll_\Psi$ in $(C(X,\mathbb{R}),\Psi)$.

The following lemma is folklore of analysis. The key part of it's proof is similar to the proof of Dini's Theorem.

Lemma 7. *Let X be a compact space. Then for any f, $g \in C(X, \mathbb{R})$, $f \ll g$ iff $f(x) < g(x)$ holds for every $x \in X$.*

A space X is called F-compact if for any f, $g \in C(X,\mathbb{R})$ with $f(x) < g(x)$ for all $x \in X$, $f \ll g$ holds in $(C(X,\mathbb{R}),\Psi)$.

Proposition 3. *A completely regular space is F-compact iff it is compact.*

Proof. By Lemma 7 we only need to verify the necessity. Let X be F-compact. Assume, by contrapositive, that X is not compact. Then there is an open cover $\mathcal{U}$ of X that has no finite subcover. This is equivalent to that there is an open neighbourhood assignment $x \mapsto U(x)$ such that for any finite set $G \subseteq X$, $\bigcup\{U(x)\colon x \in G\} \neq X$. For each $x \in X$, there is a continuous function $f_x\colon X \to [0,1]$ such that $f_x(x) = 1$, and $f_x(X - U(x)) = \{0\}$. Consider the family $\mathcal{F} = \{\sup_{x\in G} f_x : G \subseteq X \text{ is non-empty and finite}\}$. Then $\mathcal{F}$ is a directed subset of $C(X,\mathbb{R})$ and the pointwise supremum of $\mathcal{F}$ has the constant value 1. Thus $\sup\mathcal{F} = C_1$. The constant function $C_{\frac{1}{2}}$ satisfies $C_{\frac{1}{2}} < C_1$, but there is no finite $G \subseteq X$ such that $C_{\frac{1}{2}} \leq \sup_{x\in G} f_x$ because the value of $\sup_{x\in G} f_x$ at each $y \in X - \bigcup\{U(x) : x \in G\}$ equals 0. This contradicts the assumption. Hence X must be compact. □

Now if X is F-compact, then for any $f \in C(X, \mathbb{R})$, the set $\{g \in C(X, \mathbb{R}) : g \ll f\} = \{g \in C(X, \mathbb{R}) : g(x) < f(x)$ for all $x \in X\}$ is a directed set and its supremum is f. Thus we have

Corollary 2. *For any F-compact space X, $(C(X, \mathbb{R}), \Psi)$ is a continuous partial dcpo.*

Let $\mathbb{R}^T = \mathbb{R} \cup \{+\infty\}$. A function $f\colon X \to \mathbb{R}^T$ is called lower semicontinuous if for any $r \in \mathbb{R}^T$, $\{x \in X : f(x) > r\}$ is open in X. The set of all lower semicontinuous functions on X is denoted by $LSC(X, \mathbb{R}^T)$. Under the pointwise order, $LSC(X, \mathbb{R}^T)$ is a lattice and $C(X, \mathbb{R})$ is a sublattice of $LSC(X, \mathbb{R}^T)$.

It is well known that the pointwise supremum of any non-empty collection of lower semicontinuous functions is also lower semicontinuous. Thus every non-empty subset in $LSC(X, \mathbb{R}^T)$ has a supremum.

A function $f\colon X \to \mathbb{R}^T$ is called an S-lower semicontinuous function if it is the (pointwise) superemum of some continuous functions in $C(X, \mathbb{R})$(note, we did not define continuous function from X to $\mathbb{R}^T$). Let $SLSC(X, \mathbb{R}^T)$ be the set of all S-lower semicontinuous functions on X. Clearly every non-empty subset of $SLSC(X, \mathbb{R}^T)$ has a supremum in $SLSC(X, \mathbb{R}^T)$.

In [24], it was proved that a space X is perfectly normal iff every lower semicontinuous function is the supremum of a sequence of continuous functions. Thus for each perfectly normal space X, $LSC(X, \mathbb{R}^T) = SLSC(X, \mathbb{R}^T)$.

Lemma 8. *Let X be a compact space. If $\{f_i : i \in I\}$ is a directed subset of $LSC(X, \mathbb{R}^T)$, f is a continuous function and $\sup\{f_i : i \in I\} > f$ (i.e. $\sup\{f_i(x) : i \in I\} > f(x)$ holds for each $x \in X$), then $f_i > f$ for some $i \in I$.*

Proof. For each $x \in X$, there is $i(x)$ such that $f_{i(x)}(x) > f(x)$. Let $a = \frac{f(x)+f_{i(x)}(x)}{2}$. Since $f_{i(x)}$ is lower semicontinuous and f is continuous, one can find an open set U_x containing x such that $f(y) < a$, $f_{i(x)}(y) > a$ for all $y \in U_x$. Now $\{U_x : x \in X\}$ forms an open cover of X, so it has a finite subcover, say $\{U_{x_1}, U_{x_2}, \ldots, U_{x_m}\}$. Choose an f_i with $f_i \geq f_{i(x_k)}$ for each $k = 1, 2, \ldots, m$. Then clearly $f_i(x) > f(x)$ holds for each $x \in X$. □

Theorem 5. *Let X be a compact space. Then $(SLSC(X, \mathbb{R}^T), \eta)$ is a dcpo-completion of $(C(X, \mathbb{R}), \Psi)$, where η is the embedding map.*

Proof. By Theorem 2, it is enough to show that $C(X, \mathbb{R})$ is a basis of $SLSC(X, \mathbb{R}^T)$. For each $f \in SLSC(X, \mathbb{R}^T)$, let $\phi(f) = \{h \in C(X, \mathbb{R}) : h < f\}$. Then $\phi(f)$ is a directed subset of $C(X, \mathbb{R})$ and $\sup \phi(f) = f$. From Lemma 8 one deduces that for any directed subset $\{f_i : i \in I\}$ of $SLSC(X, \mathbb{R}^T)$, $\phi(\sup\{f_i : i \in I\}) = \bigcup\{\phi(f_i) : i \in I\}$ holds. Thus $C(X, \mathbb{R})$ is a basis of $SLSC(X, \mathbb{R}^T)$ and the proof is completed. □

Let X be a compact space and $f \in SLSC(X, \mathbb{R}^T)$. Then $f = \sup\{g \in C(X, \mathbb{R}) : g < f\}$ and $\{g \in C(X, \mathbb{R}) : g < f\}$ is directed. Also by Lemma 7, if $g \in C(X, \mathbb{R}^T)$ with $g < f$ then $g \ll f$. So we have the following result.

Corollary 3. *For any compact space X, $SLSC(X, \mathbb{R}^T)$ is a continuous dcpo.*

Note that for each space X, the poset $SLSC(X, \mathbb{R}^T)$ is more than being a dcpo. In fact, every non-empty subset of $SLSC(X, \mathbb{R}^T)$ has a supremum in it. One can also check that if $f, g \in SLSC(X, \mathbb{R}^T)$ then $f \wedge g \in SLSC(X, \mathbb{R}^T)$, where $(f \wedge g)(x) = \min\{f(x), g(x)\}$. Thus $SLSC(X, \mathbb{R}^T)$ is a lattice.

5. Continuous partial dcpo's and spaces with an injective hull in the category of T_0 spaces

A T_0 space X is injective if for any topological embedding $i\colon Z \hookrightarrow Y$ and continuous mapping $f\colon Z \to X$ there is a continuous $\hat{f}\colon Y \to X$ such that $f = \hat{f} \circ i$. One of the most important results in domain theory, proved by Scott, is that a space X is injective if and only if it is homeomorphic to the space ΣL of some continuous lattice L (i.e. it is a complete lattice which is continuous as a poset). Thus one has the following corresponding classes of posets and T_0 spaces:

$$\{\text{ continuous lattices }\} \quad \Longleftrightarrow \quad \{\text{ injective } T_0 \text{ spaces }\} \tag{5.1}$$

A topological embedding $i\colon A \hookrightarrow B$ is called an essential extension if, whenever $h \circ i\colon A \to C$ is an embedding for some continuous $h\colon B \to C$, then h is an embedding. Banaschewski [1] has shown that every T_0 space X has a unique(up to equivalence) maximal essential extension $X \hookrightarrow \lambda X$. If λX is injective, X is said to have an injective hull. If X is an injective space, then one can easily show that X is equivalent to λX, thus X has an injective hull.

In [11] and [17], it was proved that a T_0 space is sober and has an injective hull iff it is homeomorphic to the Scott space of a continuous dcpo. Noting that the Scott space ΣP is sober for any continuous dcpo P, so the correspondence (5.1) was extended as follows:

$$\begin{array}{ccc}
\{\text{ continuous lattices }\} & \Longleftrightarrow & \{\text{ injective } T_0 \text{ spaces }\} \\
\downarrow & & \downarrow \\
\{\text{ continuous dcpo's }\} & \Longleftrightarrow & \{\text{ sober spaces with an injective hull }\}
\end{array}$$

For a general T_0 space X, the following characterizations have been established in [11].

Theorem 6. *Let X be a T_0 space. Then the following conditions are equivalent:*
(1) *X has an injective hull.*
(2) *the lattice $\mathcal{O}(X)$ of open sets of X is completely distributive.*
(3) *the lattice $\Gamma(X)$ of closed sets of X is completely distributive.*

In [5], Erné characterized above T_0 spaces (called T_0 C-spaces) by means of C-ordered sets. C-ordered sets are the abstraction of continuous posets equipped with their way-below relations. In [6], Erné also characterized the weak monotone convergence C-space as the Scott spaces of continuous posets (a T_0 space is weak monotone convergence if for any directed set $D \subseteq X$, the corresponding net converges to $\sup D$ whenever $\sup D$ exisits).

In this section, we will use partial dcpo's to establish a Scott type of representation for the general spaces that have an injective hull. The main result is the following:

Theorem 7. *A topological space X has an injective hull iff there is a continuous partial dcpo (P,Ψ) such that $\ll_\Psi$ satisfies the interpolation property and X is homeomorphic to $(P,\sigma_\Psi(P))$.*

We first establish an adjunction between the category PDCPOS and the category Top_0 of T_0 topological spaces.

First, for any T_0 space (X,τ) let $G(X)=(X,\Psi_\tau)$ which is defined in Example 2. If $f\colon (X,\tau)\to(Y,\sigma)$ is a continuous mapping between T_0 spaces, then $f\colon X\to Y$ is monotone with respect to the specialization orders on X and Y. For any $D\in\Psi_\tau$, as mentioned in Example 2, the net $S_D=\{s(d): d\in D\}$, where $s(d)=d$, converges to $\sup D$ in X. So $f(D)$ is a directed set of Y and the net $\{f(s(d)) : d\in D\}$ converges to $f(\sup D)$. Obviously $f(\sup D)$ is an upper bound of $f(D)$. Assume that a is an upper bound of $f(D)$ and $f(\mathrm{sup} D)\not\leq a$, then $f(\mathrm{sup}D)$ belongs to $Y-\downarrow a = Y-\mathrm{cl}(\{a\})$ (note that $\downarrow b=\mathrm{cl}(\{b\})$ holds for any point $b\in Y$, where $\downarrow b=\{y\in Y: y\leq b\}$). So $\sup D$ lies in the open set $f^{-1}(Y-\downarrow a)$, therefore $D\cap f^{-1}(Y-\downarrow a)\neq\emptyset$ because D is from Ψ_τ. Choose an element c in $D\cap f^{-1}(Y-\downarrow a)$, then $f(c)\not\leq a$, which contradicts the assumption on a. It follows that $f(\sup D)=\sup f(D)$. Now if $V\subseteq Y$ is an open set and $\sup f(D)\in V$, then $f(\sup D)\in V$ and so $\sup D\in f^{-1}(V)$. Again, as $D\in\Psi_\tau$ we have $D\cap f^{-1}(V)\neq\emptyset$, which then implies $f(D)\cap V\neq\emptyset$. Therefore $f(D)\in\Psi_\sigma$. Hence f is a homomorphism between the two partial dcpo's (X,Ψ_τ) and (Y,Ψ_σ).

Hence we have a functor $G\colon \mathrm{Top}_0\to PDCPO$, where for any T_0 space (X,τ), $G(X)=(X,\Psi_\tau)$, and $G(f)=f$ for each continuous mapping f from X to Y.

Conversely, given any partial dcpo (P,Ψ), let $F(P)=\Sigma(P,\Psi)=(P,\sigma_\Psi(P))$ and $F(f)=f$ for any partial dcpo homomorphism $f\colon (P,\Psi)\to(Q,\Phi)$ (note that f is indeed a continuous map from (P,σ_Ψ) to (Q,σ_Φ)).

Consider an arbitrary T_0 space (X,τ). By the definition of Ψ_τ we see that the identity mapping $\mathrm{id}_X\colon X\to X$ is a continuous mapping from $FG(X)$ to X. If $f\colon F(P)\to(X,\tau)$ is a continuous mapping for some partial dcpo (P,Ψ), then $f\colon (P,\Psi)\to(X,\Psi_\tau)$ is a homomorphism in PDCPO, and it is the unique homomorphism satisfying $f=\mathrm{id}_X\circ F(f)$.

Proposition 4. *The functors F and G between Top_0 and $PDCPO$ defined above form an adjunction, more specifically, F is left adjoint to G.*

Recall that a T_0 space is a monotone convergence space (or d-space) if X is a dcpo under the specialization order and all open sets of X are Scott open (with respect to the specialization order). Now from the above discussion we see that for any T_0 space X, the topology on (X,τ) is coarser than the pre-Scott topology on the partial dcpo (X,Ψ_τ). It makes sense to consider the spaces X whose topology coincides with the pre-Scott topology on (X,Ψ_τ). In the next part, we shall see a class of such spaces.

Theorem 8 ([1]). *A T_0 space X has an injective hull iff for every open subset V of X and every $u \in V$ there is some open set S of X with $u \in S$ subject to the requirement that there is some $t \in V$ such that $t \leq x$ (i.e. $t \in \mathrm{cl}\{x\}$) for all $x \in S$.*

By [5], a topological space X is called a C-space if for any open set U and $x \in U$, there is $y \in U$ such that $x \in int(\uparrow y)$. Here $int(-)$ is the interior operator. The C-spaces are also called α-spaces by Ershov [8]. By Theorem 8, it is trivial that a T_0 space X has a an injective hull iff it is a C-space.

Lemma 9. *If (P, Ψ) is a continuous partial dcpo such that $\ll_\Psi$ satisfies the interpolation property, then $(P, \sigma_\Psi(P))$ is a T_0 C-space. In this case, $\int(\uparrow x) = \{y : x \ll_\Psi y\}$.*

Proof. First, as $\ll_\Psi$ satisfies the interpolation property, each $\{y : x \ll_\Psi y\}$ is a pre-Scott open set contained in $int(\uparrow x)$. Now if $y \in int(\uparrow x)$, then $\sup\{z \in P : z \ll_\Psi y\} = y$ and y is in the pre-Scott open set $int(\uparrow x)$, and $\{z \in P : z \ll_\Psi y\} \in \Psi$, so there is $z \ll_\Psi y$ such that $z \in \int(\uparrow x)$, which then implies $x \leq z \ll_\Psi y$, so $x \ll_\Psi y$. Hence $\int(\uparrow x) = \{y : x \ll_\Psi y\}$. Now if U is pre-Scott open and $x \in U$, then again, as P is continuous, there is $z \ll_\Psi x$ with $z \in U$. Then $x \in \{y : z \ll_\Psi y\} = int(\uparrow z) \subseteq U$. Hence the space $(P, \sigma_\Psi(P))$ is a T_0 C-space. The proof is completed. $\square$

Now we show that every T_0 C-space is homeomorphic to $(P, \sigma_\Psi(P))$ for some continuous partial dcpo P.

Let X be a topological space. Define the binary relation $\preceq$ on X by: $x \preceq y$ iff $y \in int(\uparrow x)$ [11].

In [11], it was claimed that the following result holds for any space X that has an injective hull. As it is already known that every T_0 C-space has an injective hull, so the following lemma holds. For the convenience of readers, we give a direct brief proof.

Lemma 10. *Let X be a C-space and $\leq$ be the specialization order on X.*

(a) *$z \leq x \prec y \leq w$ implies $z \prec w$.*

(b) *For any $b \in X$, the set $\hat{b} = \{x : x \preceq b\}$ is a directed set and* $\sup \hat{b} = b$.

(c) *The relation $\preceq$ satisfies the interpolation property: if $x \preceq y$ then there exists z such that $x \preceq z \preceq y$.*

Proof. (a) follows from the definition of $\prec$ straightforwardly.

(b) If $x \preceq b$ and $y \preceq b$, then $b \in \int(\uparrow x) \cap \int(\uparrow y)$. Since X is a C-space and $\int(\uparrow x) \cap \int(\uparrow y)$ is an open set, there is $z \in \int(\uparrow x) \cap \int(\uparrow y)$ satisfying $b \in \int(\uparrow z) \subseteq \int(\uparrow x) \cap \int(\uparrow y)$. It follows then that $z \preceq b$ and $x \leq z$, $y \leq z$. So $\hat{b} = \{x : x \preceq b\}$ is a directed set. Obviously b is an upper bound of $\hat{b} = \{x : x \preceq b\}$. If u is an upper bound of $\hat{b}$ and $b \not\leq u$, then $b \in X - \downarrow u = X - \mathrm{cl}(\{u\})$. But then there would be $e \in X - \mathrm{cl}(\{u\})$ with $b \in \int(\uparrow e)$, i.e. $e \preceq b$ and $e \not\leq u$. This contradiction shows that b is the least upper bound of $\hat{b}$.

(c) If $x \preceq y$, $y \in \int(\uparrow x)$. Since X is a C-space, there exists $z \in \int(\uparrow x)$ such that $y \in \int(\uparrow z)$. Now it's clear that $x \preceq z \preceq y$. $\square$

For a T_0 space (X, τ), let $\ll_\tau$ denote the relation such that $x \ll_\tau y$ iff for any $D \in \Psi_\tau$, $\sup D \geq y$ implies $x \leq d$ for some $d \in D$.

Lemma 11. *Let (X, τ) be a T_0 C-space. Then*

(a) *(X, Ψ_τ) is a continuous partial dcpo with $\ll_\tau$ satisfying the interpolation property; and*

(b) *$X = (X, \sigma_\Psi(X))$.*

Proof. (a) By Lemma 10, $\hat{y} = \{z \in X : z \preceq y\}$ is a directed set and $\sup \hat{y} = y$. For any open set U containing y, there is $x \preceq y$ and $x \in U$. Thus $y \in cl(\hat{y})$, implying $\mathrm{cl}(\hat{y}) = \mathrm{cl}(\{y\})$. Hence, by Lemma 1, $\hat{y} \in \Psi_\tau$. Now if $x \ll_\tau y$, then as $\hat{y} \in \Psi_\tau$, $x \leq z$ for some $z \in \hat{y}$, thus $x \in \hat{y}$. Conversely, assume that $x \in \hat{y}$ and $D \in \Psi_\tau$ with $\sup D \geq y$. Then, as $y \in \int(\uparrow x)$ so $\sup D \in \int(\uparrow x)$, implying $D \cap \int(\uparrow x) \neq \emptyset$. So there is $d \in D$ such that $d \geq x$. Therefore $x \ll_\tau y$. Now for any $y \in X$, $\{x \in X : x \ll_\tau y\} = \hat{y} \in \Psi_\tau$ and $\sup\{x \in X : x \ll_\tau y\} = \sup \hat{y} = y$. It thus shows that (X, Ψ_τ) is a continuous partial dcpo. The interpolation property of $\ll_\tau$ follows from Lemma 10.

(b) We need to verify that τ equals the set of all pre-Scott open sets of (X, Ψ_τ). First, by the proof of (a) we see that $\ll_\tau = \preceq$. Let $U \in \tau$, then for any $x \in U$ there is $y \in U$ with $x \in \int(\uparrow y)$, this means, by (a), that $x \in \{z : y \ll_\tau z\} \subseteq U$. But each $\{z : y \ll_\tau z\}$ is pre-Scott open, so U is pre-Scott open. Conversely, assume that $U \subseteq X$ is pre-Scott open. For any $x \in U$, since $x = \sup \hat{x} \in U$ and $\hat{x} \in \Psi_\tau$, so $\hat{x} \cap U \neq \emptyset$. Hence there is $y \ll_\tau x$ such that $y \in U$. Then $x \in \{z : y \ll_\tau z\} \subseteq U$. But $\{z : y \ll_\tau z\} = \{z : y \preceq z\} = \int(\uparrow y) \in \tau$. Hence $U \in \tau$. □

Now from Lemma 9 and Lemma 11 we obtain the Theorem 7.

Remark 5. (1) A T_0 space X is called monotone convergence if $(X, \leq)$ is a dcpo and every directed subset converges to its supremum. Thus if (X, τ) is a monotone convergence space and C-space, then (X, Ψ_τ) is a continuous dcpo, hence, as X is homeomorphic to $\Sigma(X, \Psi_\tau)$, so X is a sober space (the Scott space of each continuous dcpo is sober). Hence a monotone convergence C-space is sober (the converse conclusion is true for any T_0 space).

(2) A T_0 space X is called a weak monotone convergence space if for any directed subset $D \subseteq X$ with $\sup D$ exists then D converges (as a net) to $\sup D$. Thus a T_0 space (X, τ) is a weak monotone convergence space iff Ψ_τ equals the standard family of directed sets on $(X, \leq)$ (Example 1(1)). It is easily seen that for each poset P, the Scott space $\Sigma P = (P, \sigma(P))$ is a weak monotone convergence space. From this it follows that a C-space X is a weak monotone convergence space iff there is a continuous poset P such that X is homeomorphic to ΣP (see also [6]).

We summarize the results on spaces that have an injective hull by the following theorem.

Theorem 9. *Let X be a T_0 space.*

(1) *X is injective iff X is homeomorphic to ΣL for a continuous lattice L.*

(2) *X is sober and has an injective hull iff X is homeomorphic to ΣP for a continuous dcpo P.*

(3) *X is a weak monotone convergence space and has an injective hull iff X is homeomorphic to ΣP for a continuous poset P.*

(4) *X has an injective hull iff X is homeomorphic to $\Sigma(P, \Psi)$ for a continuous partial dcpo P with $\ll_\Psi$ satisfying the interpolation property.*

Remark 6. (1) If (X, τ) is homeomorphic to ΣP for a partial dcpo (P, Ψ), then we can assume $P = X$ and the order on P is the same as the specialization order on X. It then follows that $\Psi \subseteq \Psi_\tau$, thus each $U \in \sigma_\Psi(P)$ is pre-Scott open with respect to Ψ_τ. Therefore $(X, \tau) = \Sigma(X, \Psi_\tau)$. So we have the following conclusion: (X, τ) is homeomorphic to $\Sigma(P, \Psi)$ for some partial dcpo (P, Ψ) iff (X, τ) is homeomorphic to (X, Ψ_τ).

(2) For any poset P, let Ψ be the set of all directed subsets D such that $\downarrow D = \downarrow x$ for some $x \in X$ (equivalently, if D has a largest element), then $\sigma_\Psi(P)$ is the Alexandrov topology on P – the set of all upper sets of P. For this Ψ, $\ll_\Psi = \leq$. So (P, Ψ) is continuous and $\ll_\Psi$ satisfies the interpolation property.

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