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RARITA-SCHWINGER TYPE OPERATORS ON SPHERES AND REAL PROJECTIVE SPACE

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Abstract. In this paper we deal with Rarita-Schwinger type operators on spheres and real projective space. First we define the spherical Rarita-Schwinger type operators and construct their fundamental solutions. Then we establish that the projection operators appearing in the spherical Rarita-Schwinger type operators and the spherical Rarita-Schwinger type equations are conformally invariant under the Cayley transformation. Further, we obtain some basic integral formulas related to the spherical Rarita-Schwinger type operators. Second, we define the Rarita-Schwinger type operators on the real projective space and construct their kernels and Cauchy integral formulas.

1. Introduction

Rarita-Schwinger operators are generalizations of the Dirac operator and arise in representation theory for the Spin and Pin groups. See [3, 4, 6, 14, 15]. We denote a Rarita-Schwinger operator by R_k, where $k = 0, 1, \dots, m, \dots$. When $k = 0$ we have the Dirac operator. The Rarita-Schwinger operators R_k in Euclidean space have been studied in [3, 4, 6, 14, 15]. Here we construct similar Rarita-Schwinger operators together with their fundamental solutions and study their representation theory on the sphere and real projective space.

First J. Ryan [12, 11] in 1997 and P. Van Lancker [13] in 1998 studied the Dirac operators on the sphere. Later, H. Liu and J. Ryan [8] studied the spherical Dirac type operators on the sphere by using Cayley transformations. See also [1]. Using similar methods to define the Rarita-Schwinger operators in $\mathbb{R}^n$, we can define the spherical Rarita-Schwinger type operator on the sphere based on the spherical Dirac operator. We also use similar arguments as in Euclidean space to establish the conformal invariance for the projection operators and the spherical Rarita-Schwinger type equations under the Cayley transformations. See [6]. Further the fundamental solutions to the spherical Rarita-Schwinger type operators are achieved by applying the Cayley transformation. In turn, Stokes' Theorem, Cauchy's Theorem, Borel-Pompeiu Formula, Cauchy Integral Formula and a Cauchy Transform are proved for the sphere. Furthermore, we show that Stokes' theorem is

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conformally invariant under Cayley transformation, and with minor modification, is equivalent to the Rarita-Schwinger version of Stokes' Theorem in Euclidean space appearing in [3, 6] and elsewhere.

By factoring out $\mathbb{S}^n$ by the group $\mathbb{Z}_2 = \{\pm 1\}$ we obtain real projective space, $\mathbb{R}P^n$. On this space, we define the Rarita-Schwinger type operators and construct their kernels over two different bundles over $\mathbb{R}P^n$. Further, we obtain some basic integral formulas from Clifford analysis associated with these operators for the two different bundles. This extends results from [7].

2. Preliminaries

A Clifford algebra, Cl_{n+1}, can be generated from $\mathbb{R}^{n+1}$ by considering the relationship

$$\underline{x}^2 = -\|\underline{x}\|^2$$

for each $\underline{x} \in \mathbb{R}^{n+1}$. We have $\mathbb{R}^{n+1} \subseteq \mathrm{Cl}_{n+1}$. If $e_1, \ldots, e_{n+1}$ is an orthonormal basis for $\mathbb{R}^{n+1}$, then $\underline{x}^2 = -\|\underline{x}\|^2$ tells us that $e_ie_j + e_je_i = -2\delta_{ij}$. Let $A = \{j_1, \ldots, j_r\} \subset \{1, 2, \ldots, n+1\}$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq n+1$. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \ldots e_{j_r}$. Hence for any element $a \in \mathrm{Cl}_{n+1}$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$.

The reversion is given by

$$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A \,,$$

where $|A|$ is the cardinality of A. In particular, $\widetilde{e_{j_1} \ldots e_{j_r}} = e_{j_r} \ldots e_{j_1}$. Also $\widetilde{ab} = \tilde{b}\tilde{a}$ for $a, b \in \mathrm{Cl}_{n+1}$. The Clifford conjugation is defined by

$$\bar{a} = \sum_A (-1)^{|A|(|A|+1)/2} a_A e_A$$

and satisfies $\overline{e_{j_1} \ldots e_{j_r}} = (-1)^r e_{j_r} \ldots e_{j_1}$ and $\overline{ab} = \bar{b}\bar{a}$ for $a, b \in \mathrm{Cl}_{n+1}$.

For each $a = a_0 + a_1e_1 + \cdots + a_{1\ldots n+1}e_1 \ldots e_{n+1} \in \mathrm{Cl}_{n+1}$ the scalar part of $\bar{a}a$ gives the square of the norm of a, namely $a_0^2 + a_1^2 + \cdots + a_{1\ldots n+1}^2$. For more on Clifford algebras and their properties, see [9].

The Pin and Spin groups play an important role in Clifford analysis. The Pin group can be defined as

$$\mathrm{Pin}(n+1) := \{a \in \mathrm{Cl}_{n+1} : a = y_1 \ldots y_p : y_1, \ldots, y_p \in \mathbb{S}^n, p \in \mathbb{N}\}$$

and it is clearly a group under multiplication in Cl_{n+1}.

Now suppose that $y \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Look at $yxy = yx^{\|_y}y + yx^{\perp_y}y = -x^{\|_y} + x^{\perp_y}$ where $x^{\|_y}$ is the projection of x onto y and $x^{\perp_y}$ is perpendicular to y. So yxy gives a reflection of x in the y direction. By the Cartan–Dieudonné Theorem each $O \in O(n+1)$ is the composition of a finite number of reflections. If $a = y_1 \ldots y_p \in \mathrm{Pin}(n+1)$, then $\tilde{a} := y_p \ldots y_1$ and $ax\tilde{a} = O_a(x)$ for some $O_a \in O(n+1)$. Choosing $y_1, \ldots, y_p$ arbitrarily in $\mathbb{S}^n$, we see that the group homomorphism

$$\theta\colon\ \mathrm{Pin}(n+1) \longrightarrow O(n+1)\colon a \longmapsto O_a \,,$$

with $a = y_1 \dots y_p$ and $O_a(x) = ax\tilde{a}$, is surjective. Further $-ax(-\tilde{a}) = ax\tilde{a}$, so 1, $-1 \in \ker(\theta)$. In fact $\ker(\theta) = \{\pm 1\}$. See [9]. The Spin group is defined as

$$\mathrm{Spin}(n+1) := \{a \in \mathrm{Pin}(n+1) : a = y_1 \dots y_p \text{ and } \ p \ \text{ even}\}$$

and it is a subgroup of $\mathrm{Pin}(n+1)$. There is a group homomorphism

$$\theta\colon \mathrm{Spin}(n+1) \longrightarrow SO(n+1)$$

which is surjective with kernel $\{1, -1\}$. See [9] for details.

The Dirac Operator in $\mathbb{R}^n$ is defined to be

$$D := \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}\,.$$

Note $D^2 = -\Delta_n$, where Δ_n is the Laplacian in $\mathbb{R}^n$.

If p_k is a homogeneous polynomial with degree k such that $Dp_k = 0$, we call such a polynomial a left monogenic polynomial homogeneous of degree k.

Let $\mathcal{H}_k$ be the space of Cl_{n+1}-valued harmonic polynomials homogeneous of degree k and $\mathcal{M}_k$ the space of Cl_{n+1}-valued monogenic polynomials homogeneous of degree k. Note if $h_k \in \mathcal{H}_k$, then $Dh_k \in \mathcal{M}_{k-1}$. But $Dup_{k-1}(u) = (-n-2k+2)p_{k-1}(u)$, so

$$\mathcal{H}_k = \mathcal{M}_k \bigoplus u\mathcal{M}_{k-1}\,, \quad h_k = p_k + up_{k-1}\,.$$

This is the so-called Almansi-Fischer decomposition of $\mathcal{H}_k$. See [2, 10].

Note that if $p(u) \in \mathcal{M}_k$ then it trivially extends to $P(v) = p(u + u_{n+1}e_{n+1})$ with $u_{n+1} \in \mathbb{R}$ and $P(v) = p(u)$ for all $u_{n+1} \in \mathbb{R}$. Consequently $D_{n+1}P(v) = 0$ where $D_{n+1} = \sum_{j=1}^{n+1} e_j \frac{\partial}{\partial u_j}$.

If $p(u) \in \mathcal{M}_k$ then for any boundary of a piecewise smooth bounded domain $U \subseteq \mathbb{R}^n$ by Cauchy's Theorem

$$\int_{\partial U} n(u)p(u)d\sigma(u) = 0\,. \tag{1}$$

Suppose now $a \in \mathrm{Pin}(n+1)$ and $u = aw\tilde{a}$ then although $u \in \mathbb{R}^n$ in general w belongs to the hyperplane $a^{-1}\mathbb{R}^n\tilde{a}^{-1}$ in $\mathbb{R}^{n+1}$.

By applying a change of variable, up to a sign the integral (1) becomes

$$\int_{a^{-1}\partial U\tilde{a}^{-1}} an(w)\tilde{a}P(aw\tilde{a})d\sigma(w) = 0. \tag{2}$$

As ∂U is arbitrary then on applying Stokes' Theorem to (2) we see that

$$D_a\tilde{a}P(aw\tilde{a}) = 0\,, quad\text{where} \quad D_a := D_{n+1}\big|_{a^{-1}\mathbb{R}^n\tilde{a}^{-1}}. \tag{3}$$

Suppose U is a domain in $\mathbb{R}^n$. Consider a function of two variables

$$f\colon U \times \mathbb{R}^n \longrightarrow \mathrm{Cl}_{n+1}$$

such that for each $x \in U$, $f(x,u)$ is a left monogenic polynomial homogeneous of degree k in u. Let P_k be the left projection map

$$P_k\colon \mathcal{H}_k \to \mathcal{M}_k,$$

then $R_k f(x,u)$ is defined to be $P_k D_x f(x,u)$. The left Rarita-Schwinger equation is defined to be

$$R_k f(x,u) = 0\,.$$

We also have a right projection $P_{k,r}\colon \mathcal{H}_k \to \overline{\mathcal{M}_k}$, and a right Rarita-Schwinger equation $f(x,u)D_x P_{k,r} = f(x,u)R_k = 0$, where $\overline{\mathcal{M}_k}$ stands for the space of right monogenic polynomials homogeneous of degree k. See [6].

3. Rarita-Schwinger type operators on spheres

Let $\mathbb{R}^n$ be the span of $e_1,\dots,e_n$. Consider the Cayley transformation $C\colon \mathbb{R}^n \to \mathbb{S}^n$, where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$, defined by $C(x) = (e_{n+1}x+1)(x+e_{n+1})^{-1}$, where $x = x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^n$, and e_{n+1} is a unit vector in $\mathbb{R}^{n+1}$ which is orthogonal to $\mathbb{R}^n$. Now $C(\mathbb{R}^n) = \mathbb{S}^n \setminus \{e_{n+1}\}$. Suppose $x_s \in \mathbb{S}^n$ and $x_s = x_{s_1}e_1 + \cdots + x_{s_n}e_n + x_{s_{n+1}}e_{n+1}$, then we have $x = C^{-1}(x_s) = (-e_{n+1}x_s+1)(x_s-e_{n+1})^{-1}$.

The Dirac operator over the n-sphere $\mathbb{S}^n$ has the form $D_s = w(\Lambda + \frac{n}{2})$, where $w \in \mathbb{S}^n$ and $\Lambda = \displaystyle\sum_{i<j,i=1}^{n} e_ie_j(w_i\frac{\partial}{\partial w_j} - w_j\frac{\partial}{\partial w_i})$, see for instance [5, 8, 13].

Let U be a domain in $\mathbb{R}^n$. Consider a function $f_\star\colon U\times\mathbb{R}^n \to \mathrm{Cl}_{n+1}$ such that for each $x\in U$, $f_\star(x,u)$ is a left monogenic polynomial homogeneous of degree k in u. This function reduces to $f(x_s,u)$ on $C(U)\times\mathbb{R}^n$ and $f(x_s,u)$ takes its values in Cl_{n+1} where $x = C^{-1}(x_s)$ and $x_s \in C(U) \subset \mathbb{S}^n$. Further $f(x_s,u)$ is a left monogenic polynomial homogeneous of degree k in u.

Since $\Delta_u D_{s,x_s} = D_{s,x_s}\Delta_u$, then $D_{s,x_s}f(x_s,u)$ is harmonic in u. Hence by the Almansi-Fischer decomposition:

$$D_{s,x_s}f(x_s,u) = f_{1,k}(x_s,u) + uf_{2,k-1}(x_s,u)\,,$$

where $f_{1,k}(x_s,u)$ is a left monogenic polynomial homogeneous of degree k in u and $f_{2,k-1}(x_s,u)$ is a left monogenic polynomial homogeneous of degree $k-1$ in u.

We can also consider a function $g_\star\colon U\times\mathbb{R}^n \to \mathrm{Cl}_{n+1}$ such that for each $x\in U$, $g_\star(x,u)$ is a right monogenic polynomial homogeneous of degree k in u. This function also reduces to a right monogenic polynomial homogeneous $g(x_s,u)$ on $C(U)\times\mathbb{R}^n$.

Let P_k be the left projection map $P_k\colon \mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1} \to \mathcal{M}_k$, then the n-spherical left Rarita-Schwinger type operator R_k^S is defined to be

$$R_k^S f(x_s,u) = P_k D_{s,x_s} f(x_s,u)\,.$$

On the other hand, the n-spherical right Rarita-Schwinger type operator $R_{k,r}^S$ is defined to be

$$g(x_s,u)R_{k,r}^S = g(x_s,u)D_{s,x_s}P_{k,r}\,,$$

where $P_{k,r}$ is the right projection $P_{k,r}\colon \mathcal{H}_k \to \bar{\mathcal{M}}_k$. Consequently, the left and the right n-spherical Rarita-Schwinger type equations are defined to be

$$R_k^S f(x_s,u) = 0 \quad\text{and}\quad g(x_s,u)R_k^S = 0 \quad\text{respectively.}$$

4. Conformal invariance of P_k under the Cayley transformation and its inverse

Consider the Cayley transformation $C(x) = (e_{n+1}x+1)(x+e_{n+1})^{-1} = e_{n+1}(x-e_{n+1})(x+e_{n+1})^{-1} = e_{n+1}(x+e_{n+1}-2e_{n+1})(x+e_{n+1})^{-1} = e_{n+1}+2(x+e_{n+1})^{-1}$. This last term is the Iwasawa decomposition for the Cayley transformation, C. Further, $C^{-1}(x_s) = (-e_{n+1}x_s+1)(x_s-e_{n+1})^{-1} = -e_{n+1}(x_s+e_{n+1})(x-e_{n+1})^{-1} = -e_{n+1}(x_s-e_{n+1}+2e_{n+1})(x_s-e_{n+1})^{-1} = -e_{n+1}+2(x_s-e_{n+1})^{-1}$, and this last term is the Iwasawa decomposition for the inverse, C^{-1}, of the Cayley transformation.

Now let $f(x_s,u) : U_s \times \mathbb{R}^n \to \mathrm{Cl}_{n+1}$ be a monogenic polynomial homogeneous of degree k in u for each $x_s \in U_s$, where U_s is a domain in $\mathbb{S}^n$.

It is shown in [6] that P_k is conformally invariant under a general Möbius transformation over $\mathbb{R}^n$. This trivially extends to Möbius transformations on $\mathbb{R}^{n+1}$. It follows that if we restrict x_s to $\mathbb{S}^n$, then P_k is also conformally invariant under the Cayley transformation C and its inverse C^{-1}, with $x \in \mathbb{R}^n$.

It follows that we have:

Theorem 1.

$$P_{k,w}J(C,x)f\Big(C(x), \frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big) = J(C,x)P_{k,u}f(x_s,u)\,,$$

where $u = \dfrac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}$ *and* $J(C,x) = \dfrac{x+e_{n+1}}{\|x+e_{n+1}\|^n}$ *is the conformal weight for the Cayley transformation.*

Also for U a domain in $\mathbb{R}^n$, and $g(x,u)$ defined on $U \times \mathbb{R}^n$ such that for each $x \in U$, g is monogenic in u and homogeneous of degree k in u, we have:

Theorem 2.

$$P_{k,w}J(C^{-1},x_s)g\Big(C^{-1}(x_s), \frac{(x_s-e_{n+1})w(x_s-e_{n+1})}{\|x_s-e_{n+1}\|^2}\Big) = J(C^{-1},x_s)P_{k,u}g(x,u)\,,$$

where $u = \dfrac{(x_s-e_{n+1})w(x_s-e_{n+1})}{\|x_s-e_{n+1}\|^2}$ *and* $J(C^{-1},x_s) = \dfrac{x_s-e_{n+1}}{\|x_s-e_{n+1}\|^n}$ *is the conformal weight for the inverse Cayley transformation.*

Note that in the previous theorems $a_1(x) := \dfrac{x+e_{n+1}}{\|x+e_{n+1}\|}$ and $a_2(x_s) := \dfrac{x_s-e_{n+1}}{\|x_s-e_{n+1}\|}$ belong to $\mathrm{Pin}(n+1)$. So $w \in \mathbb{R}^{n+1}$ and hence $D_{a_1(x)}f = 0$ and $D_{a_2(x_s)}g = 0$, where for $a \in \mathrm{Pin}(n+1)$ the operator D_a is defined in (3).

5. The intertwining formulas for R_k and R_k^S and the conformal invariance of $R_k^S f = 0$

We can use the intertwining formulas for D_x and D_{s,x_s} given in [8] to establish the intertwining formulas for R_k and R_k^S.

Theorem 3.

$$J_{-1}(C^{-1},x_s)R_{k,u}f(x,u) = R^S_{k,w}J(C^{-1},x_s)f\Big(C^{-1}(x_s),\frac{(x_s-e_{n+1})w(x_s-e_{n+1})}{\|x_s-e_{n+1}\|^2}\Big),$$

where $R_{k,u}$ is the Rarita-Schwinger operator in Euclidean space with respect to $u\in\mathbb{R}^n$, $R^S_{k,w}$ is the spherical Rarita-Schwinger type operator on $\mathbb{S}^n$ with respect to $w\in\mathbb{R}^{n+1}$, $u=\dfrac{(x_s-e_{n+1})w(x_s-e_{n+1})}{\|x_s-e_{n+1}\|^2}$, $J(C^{-1},x_s)=\dfrac{x_s-e_{n+1}}{\|x_s-e_{n+1}\|^n}$ and $J_{-1}(C^{-1},x_s)=\dfrac{x_s-e_{n+1}}{\|x_s-e_{n+1}\|^{n+2}}$.

Proof. In [8] it is shown that $D_x=J_{-1}(C^{-1},x_s)^{-1}D_{s,x_s}J(C^{-1},x_s)$. Consequently,

$$R_{k,u}f(x,u)=P_{k,u}D_xf(x,u)=P_{k,u}J_{-1}(C^{-1},x_s)^{-1}D_{s,x_s}J(C^{-1},x_s)f(C^{-1}(x_s),u).$$

Now applying Theorem 2, the previous equation becomes

$$\begin{aligned}R_{k,u}f(x,u) &= J_{-1}(C^{-1},x_s)^{-1}P_{k,w}D_{s,x_s}J(C^{-1},x_s)\\ &\quad\times f\Big(C^{-1}(x_s),\frac{(x_s-e_{n+1})w(x_s-e_{n+1})}{\|x_s-e_{n+1}\|^2}\Big)\\ &= J_{-1}(C^{-1},x_s)^{-1}R^S_{k,w}J(C^{-1},x_s)\\ &\quad\times f\Big(C^{-1}(x_s),\frac{(x_s-e_{n+1})w(x_s-e_{n+1})}{\|x_s-e_{n+1}\|^2}\Big).\end{aligned}$$

□

We have the similar result for the Rarita-Schwinger operator under the Cayley transformation.

Theorem 4.

$$J_{-1}(C,x)R^S_{k,u}g(x_s,u)=R_{k,w}J(C,x)g\Big(C(x),\frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big),$$

where $R^S_{k,u}$ is the Rarita-Schwinger type operator on the sphere with respect to u and $R_{k,w}$ is the Rarita-Schwinger operator in Euclidean space with respect to w,

$$u=\frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2},\qquad J(C,x)=\frac{x+e_{n+1}}{\|x+e_{n+1}\|^n}\quad\text{and}$$

$$J_{-1}(C,x)=\frac{x+e_{n+1}}{\|x+e_{n+1}\|^{n+2}}.$$

In other words we have the following intertwining relations for R_k and R^S_k:

$$J_{-1}(C^{-1},x_s)R_k=R^S_kJ(C^{-1},x_s) \tag{4}$$

$$J_{-1}(C,x)R^S_k=R_kJ(C,x) \tag{5}$$

As a corollary to Theorems 3 and 4 we have the conformal invariance of equation $R^S_{k,w}f=0$:

Theorem 5. *$R^S_{k,u}g(x_s,u)=0$ if and only if*

$$R_{k,w}J(C,x)g\Big(C(x),\frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big)=0$$

and $R_{k,u}f(x,u)=0$ if and only if

$$R^S_{k,w}J(C^{-1},x_s)f\Big(C^{-1}(x_s),\frac{(x_s-e_{n+1})w(x_s-e_{n+1})}{\|x_s-e_{n+1}\|^2}\Big)=0\,.$$

6. The fundamental solutions of R^S_k and some basic integral formulas

The reproducing kernel of $\mathcal{M}_k$ with respect to integration over $\mathbb{S}^{n-1}$ is given by (see [2], [6])

$$Z_k(u,v):=\sum_\sigma P_\sigma(u)V_\sigma(v)v\,,$$

where

$$P_\sigma(u)=\frac{1}{k!}\Sigma(u_{i_1}-u_1e_1^{-1}e_{i_1})\dots(u_{i_k}-u_1e_1^{-1}e_{i_k}),V_\sigma(v)=\frac{\partial^kG(v)}{\partial v_2^{j_2}\dots\partial v_n^{j_n}}\,,$$

$j_2+\dots+j_n=k$, $i_k\in\{2,\dots,n\}$, $G(v)=\dfrac{-1}{\omega_n}\dfrac{v}{\|v\|^n}$, and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Here summation is taken over all permutations of the monomials without repetition. This function is left monogenic in u and it is a right monogenic polynomial in v. It is homogeneous of degree k in both u and v. See [2] and elsewhere.

Consider the kernel of the Rarita-Schwinger operator in Euclidean n-space

$$E_k(x-y,u,v)=\frac{1}{\omega_nc_k}\frac{x-y}{\|x-y\|^n}Z_k\Big(\frac{(x-y)u(x-y)}{\|x-y\|^2},v\Big)\tag{6}$$

$$=\frac{1}{\omega_nc_k}J(C^{-1},x_s)^{-1}\frac{x_s-y_s}{\|x_s-y_s\|^n}J(C^{-1},y_s)^{-1}Z_k\Big(\frac{(x-y)u(x-y)}{\|x-y\|^2},v\Big)\,,\tag{7}$$

where $c_k=\dfrac{n-2}{n-2+2k}$. See for instance [6]. Note that $(x-y)u(x-y)\in\mathbb{R}^n$ as u, x and $y\in\mathbb{R}^n$.

Now applying the Cayley transformation to the above kernel, we obtain

$$\begin{aligned}E^S_k(x_s,y_s,u,v):&=\frac{1}{\omega_nc_k}J(C^{-1},x_s)J(C^{-1},x_s)^{-1}\frac{x_s-y_s}{\|x_s-y_s\|^n}\\&\quad\times J(C^{-1},y_s)^{-1}Z_k(au\tilde{a},v)\\&=\frac{1}{\omega_nc_k}\frac{x_s-y_s}{\|x_s-y_s\|^n}J(C^{-1},y_s)^{-1}Z_k(au\tilde{a},v)\,,\end{aligned}\tag{8}$$

where $a=a(x_s,y_s)=\dfrac{J(C^{-1},x_s)^{-1}(x_s-y_s)J(C^{-1},y_s)^{-1}}{\|J(C^{-1},x_s)^{-1}\|\,\|(x_s-y_s)\|\,\|J(C^{-1},y_s)^{-1}\|}$.

$E^S_k(x_s,y_s,u,v)$ is the fundamental solution to $R^S_kf(x_s,u)=0$ on $\mathbb{S}^n$. This function is left monogenic in u and it is also right monogenic in v.

In the same way we obtain that

$$\frac{1}{\omega_n c_k} Z_k(u, \tilde{a}va) J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} \tag{9}$$

is a non trivial solution to $g(x_s, v) R^S_{k,r} = 0$. In fact, this function is $E^S_k(x_s, y_s, u, v)$.

Applying the same arguments in [6] to prove the representations (8) and (9) are the same up to a reflection, we have

$$\begin{aligned}
&\frac{1}{\omega_n c_k} Z_k(u, \tilde{a}va) J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} \\
&\quad = -\frac{1}{\omega_n c_k} \tilde{a} Z_k(au\tilde{a}, v) a J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} \\
&\quad = -\frac{1}{\omega_n c_k} J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} \frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|} Z_k(au\tilde{a}, v) \frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|} \\
&\quad = -\frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|} \frac{1}{\omega_n c_k} \frac{x_s - y_s}{\|x_s - y_s\|^n} J(C^{-1}, y_s)^{-1} Z_k(au\tilde{a}, v) \frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|}\,.
\end{aligned}$$

Theorem 6 (Stokes' Theorem for the n-spherical Dirac operator D_s [8])**.** *Suppose U_s is a domain on $\mathbb{S}^n$ and $f, g\colon U_s \times \mathbb{R}^n \to \mathrm{Cl}_{n+1}$ are C^1, then for ∂V_s a sufficiently smooth hypersurface in U_s bounding a subdomain V_s of U_s, we have*

$$\begin{aligned}
&\int_{\partial V_s} g(x_s, u) n(x_s) f(x_s, u) d\Sigma(x_s) \\
&\qquad = \int_{V_s} \big(g(x_s, u) D_{s,x_s}\big) f(x_s, u) + g(x_s, u)\big(D_{s,x_s} f(x_s, u)\big) dS(x_s)\,,
\end{aligned}$$

where $dS(x_s)$ is the n-dimensional area measure on V_s, $d\Sigma(x_s)$ is the $n-1$-dimensional scalar Lebesgue measure on ∂V_s and $n(x_s)$ is the normal vector tangent to the sphere at x_s, orthogonal to ∂V_s and pointing outward.

Definition 1 ([6])**.** *For any Cl_{n+1}-valued polynomials $P(u)$, $Q(u)$, the inner product $(P(u), Q(u))_u$ with respect to $u \in \mathbb{R}^n$ is given by*

$$(P(u), Q(u))_u = \int_{\mathbb{S}^{n-1}} P(u) Q(u) ds(u)\,,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

For any $p_k \in \mathcal{M}_k$, one obtains

$$p_k(u) = (Z_k(u, v), p_k(v))_v = \int_{\mathbb{S}^{n-1}} Z_k(u, v) p_k(v) ds(v)\,.$$

See [2].

Theorem 7 (Stokes' Theorem for the n-spherical Rarita-Schwinger type operator R_k^S)**.** *Let U_s, V_s, ∂V_s be as in Theorem 6. Then for f, $g \in C^1(U_s \times \mathbb{R}^n, \mathcal{M}_k)$, we have*

$$\begin{aligned}\int_{V_s} \Big(\big(g(x_s,u)R_k^S, f(x_s,u)\big)_u + \big(g(x_s,u), R_k^S f(x_s,u)\big)_u\Big) dS(x_s) \\ = \int_{\partial V_s} \big(g(x_s,u), P_k n(x_s) f(x_s,u)\big)_u d\Sigma(x_s) \\ = \int_{\partial V_s} \big(g(x_s,u) n(x_s) P_{k,r}, f(x_s,u)\big)_u d\Sigma(x_s) \\ = \int_{\partial V_s} \big(g(x_s,u) n(x_s) f(x_s,u)\big)_u d\Sigma(x_s)\end{aligned}$$

where $dS(x_s)$ is the n-dimensional area measure on V_s, $n(x_s)$ and $d\Sigma(x_s)$ are as in Theorem 6.

Proof. The proof follows similar lines to the proof of Theorem 6 in [6]. First, by the traditional Clifford version of Stokes' Theorem

$$\begin{aligned}\int_{\partial V_s} (g(x_s,u) n(x_s) f(x_s,u))_u d\Sigma(x_s) \\ = \int_{V_s} \big((g(x_s,u) D_{s,x_s}, f(x_s,u))_u + (g(x_s,u), D_{s,x_s} f(x_s,u))_u\big) dS(x_s)\,.\end{aligned}$$

By applying the Almansi-Fischer decomposition to $g(x_s,u)D_{s,x_s}$ and $D_{s,x_s}f(x_s,u)$ and Definition 1 the right side of the previous equation becomes

$$\int_{V_s} \Big(\big(g(x_s,u)R_k^S, f(x_s,u)\big)_u + \big(g(x_s,u), R_k^S f(x_s,u)\big)_u\Big) dS(x_s)\,.$$

The other identities follow from arguments given in the proof of Theorem 6 in [6]. □

Corollary 1 (Cauchy's Theorem)**.** *If $R_k^S f(x_s,u) = 0$ and $g(x_s,u)R_k^S = 0$ for f, $g \in C^1(U_s \times \mathbb{R}^n, \mathcal{M}_k)$, then we have*

$$\int_{\partial V_s} (g(x_s,u), P_k n(x_s) f(x_s,u))_u d\Sigma(x_s) = 0\,,$$

where ∂V_s is a sufficiently smooth hypersurface in U_s bounding a subdomain V_s of U_s.

Now let us look at Stokes' Theorem for Rarita-Schwinger operators R_k in $\mathbb{R}^n$. Suppose U is a domain on $\mathbb{R}^n$ and $f_\star$, $g_\star\colon U \times \mathbb{R}^n \to \mathrm{Cl}_{n+1}$ are C^1, then for ∂V a sufficiently smooth hypersurface in U bounding a relatively compact subdomain V of U, we have

$$\begin{aligned}\int_V [(g_\star(x,u)R_k, f_\star(x,u))_u + (g_\star(x,u), R_k f_\star(x,u))_u] dx^n \\ = \int_{\partial V} (g_\star(x,u), P_k n(x) f_\star(x,u))_u \, d\sigma(x)\,,\end{aligned}$$

where $d\sigma(x)$ is the scalar Lebesgue measure on ∂V. Now consider the integral on the right hand side

$$\begin{aligned}
&\int_{\partial V} (g_\star(x,u), P_k n(x) f_\star(x,u))_u \, d\sigma(x) \\
&\quad = \int_{\partial V} \int_{\mathbb{S}^{n-1}} g_\star(x,u) P_k n(x) f_\star(x,u) ds(u) d\sigma(x) \\
&\quad = \int_{C(\partial V)} \int_{\mathbb{S}^{n-1}} g_\star(C^{-1}(x_s),u) P_{k,u} J(C^{-1},x_s) n(x_s) \\
&\quad\quad \times J(C^{-1},x_s) f_\star(C^{-1}(x_s),u) ds(u) d\Sigma(x_s)\,,
\end{aligned}$$

where $x_s = C(x)$, $C(\partial V)$ bounds a domain $C(V)$ in $\mathbb{S}^n$, $d\Sigma(x_s)$ is the scalar Lebesgue measure on $C(\partial V)$ and $J(C^{-1},x_s) = \frac{x_s - e_{n+1}}{\|x_s - e_{n+1}\|^n}$. Since $P_{k,u}$ interchanges with $J(C^{-1},x_s)$, the previous integral becomes

$$\begin{aligned}
&\int_{C(\partial V)} \int_{\mathbb{S}^{n-1}} g_\star(C^{-1}(x_s), \frac{(x_s - e_{n+1}) w (x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}) J(C^{-1},x_s) P_{k,w} n(x_s) J(C^{-1},x_s) \\
&\qquad \times f_\star(C^{-1}(x_s), \frac{(x_s - e_{n+1}) w (x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}) ds(w) d\Sigma(x_s)
\end{aligned} \tag{10}$$

where $u = \frac{(x_s - e_{n+1}) w (x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}$.

Consider the integral on the left hand side

$$\begin{aligned}
&\int_V \left[\big(g_\star(x,u) R_k, f_\star(x,u)\big)_u + \big(g_\star(x,u), R_k f_\star(x,u)\big)_u \right] dx^n \\
&\quad = \int_V \int_{\mathbb{S}^{n-1}} [g_\star(x,u) R_{k,r,u} f_\star(x,u) + g_\star(x,u) R_{k,u} f_\star(x,u)] ds(u) dx^n
\end{aligned} \tag{11}$$

Applying Theorem 3, the integral now is equal to

$$\begin{aligned}
&\int_{C(V)} \int_{\mathbb{S}^{n-1}} \Big[g_\star\Big(C^{-1}(x_s), \frac{(x_s - e_{n+1}) w (x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}\Big) J(C^{-1},x_s) R^S_{k,r,w} J(C^{-1},x_s) \\
&\qquad \times f_\star\Big(C^{-1}(x_s), \frac{(x_s - e_{n+1}) w (x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}\Big) \\
&\qquad + g_\star\Big(C^{-1}(x_s), \frac{(x_s - e_{n+1}) w (x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}\Big) J(C^{-1},x_s) R^S_{k,w} J(C^{-1},x_s) \\
&\qquad \times f_\star\Big(C^{-1}(x_s), \frac{(x_s - e_{n+1}) w (x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}\Big)\Big] \, ds(w) \, dS(x_s)
\end{aligned} \tag{12}$$

where $C(V) = V_s$ is a domain in $\mathbb{S}^n$.

Stokes' Theorem for Rarita-Schwinger operators R_k in $\mathbb{R}^n$ tells us that (10) is equal to (12). Therefore Stokes' Theorem for Rarita-Schwinger type operators is conformally invariant under the Cayley transformation.

Now let us consider Stokes' Theorem for R_k^S in $\mathbb{S}^n$.

$$\int_{V_s} \left(\left(g(x_s,u)R_k^S, f(x_s,u)\right)_u + \left(g(x_s,u), R_k^S f(x_s,u)\right)_u\right) dS(x_s)$$
$$= \int_{\partial V_s} \left(g(x_s,u), P_k n(x_s) f(x_s,u)\right)_u d\Sigma(x_s)\,,$$

where V_s, ∂V_s, $dS(x_s)$ and $d\Sigma(x_s)$ are stated as in Theorem 7.

First look at

$$\int_{\partial V_s} \left(g(x_s,u), P_k n(x_s) f(x_s,u)\right)_u d\Sigma(x_s)$$
$$= \int_{\partial V_s} \int_{\mathbb{S}^{n-1}} g(x_s,u), P_k n(x_s) f(x_s,u) ds(u) \Sigma(x_s)$$
$$= \int_{C^{-1}(\partial V_s)} \int_{\mathbb{S}^{n-1}} g\big(C(x),u\big) P_{k,u} J\big(C(x)n(x)\big) J(C,x) f\big(C(x),u\big)\, ds(u)\, d\sigma(x)\,,$$

where $J(C,x) = \dfrac{x+e_{n+1}}{\|x+e_{n+1}\|^n}$, $x = C^{-1}(x_s)$ and $C^{-1}(\partial V_s)$ bounds a domain $C^{-1}(V_s)$ in $\mathbb{R}^n$. Since we can interchange $P_{k,u}$ with $J(C,x)$, the previous integral is equal to

$$\int_{C^{-1}(\partial V_s)} \int_{\mathbb{S}^{n-1}} g\Big(C(x), \frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big) J\big(C(x) P_{k,w} n(x)\big) J(C,x)$$
$$\times f\Big(C(x), \frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big)\, ds(w) d\sigma(x)\,, \tag{13}$$

where $u = \dfrac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}$.

Second we look at

$$\int_{V_s} \left(\left(g(x_s,u)R_k^S, f(x_s,u)\right)_u + \left(g(x_s,u), R_k^S f(x_s,u)\right)_u\right) dS(x_s)$$
$$= \int_{V_s} \int_{\mathbb{S}^{n-1}} \left(g(x_s,u) R_{k,r,u}^S\right) f(x_s,u) + g(x_s,u)\left(R_{k,u}^S f(x_s,u)\right) ds(u)\, dS(x_s)\,.$$

Applying Theorem 4, the integral becomes

$$\int_{C^{-1}(V_s)} \int_{\mathbb{S}^{n-1}} g\Big(C(x), \frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big) J(C,x) R_{k,r,w}^S$$
$$\times J(C,x) f\Big(C(x), \frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big)$$
$$+ g\Big(C(x), \frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big) J(C,x)$$
$$\times R_{k,w}^S J(C,x) f\Big(C(x), \frac{(x+e_{n+1})w(x+e_{n+1})}{\|x+e_{n+1}\|^2}\Big)\, ds(w)\, dS(x_s)\,. \tag{14}$$

Stokes' Theorem for R_k^S on the sphere shows that (13) is equal to (14). Thus Stokes' Theorem for Rarita-Schwinger operators is also conformally invariant under the inverse of the Cayley transformation.

Theorem 8 (Borel-Pompeiu Theorem)**.** *Suppose U_s, V_s and ∂V_s are as in Theorem 6 and $y_s \in V_s$. Then for $f \in C^1(U_s \times \mathbb{R}^n, \mathcal{M}_k)$ we have*

$$\begin{aligned} f(y_s, u') = {} & J(C^{-1}, y_s) \int_{\partial V_s} \left(E_k^S(x_s, y_s, u, v), P_k n(x_s) f(x_s, v)\right)_v d\Sigma(x_s) \\ & - J(C^{-1}, y_s) \int_{V_s} \left(E_k^S(x_s, y_s, u, v), R_k^S f(x_s, v)\right)_v dS(x_s) \end{aligned}$$

where $u' = \dfrac{J(C^{-1}, y_s)^{-1} u J(C^{-1}, y_s)^{-1}}{\|J(C^{-1}, y_s)^{-1}\|^2}$, $dS(x_s)$ is the n-dimensional area measure on $V_s \subset \mathbb{S}^n$, $n(x_s)$ and $d\Sigma(x_s)$ are as in Theorem 6.

Proof. In this proof we will use the representation of $E_k^S(x_s, y_s, u, v)$ given by (9).

Let $B_s(y_s, \epsilon)$ be the ball centered at $y_s \in \mathbb{S}^n$ with radius ϵ.
We denote $C^{-1}(B_s(y_s, \epsilon))$ by $B(y, r)$, and $C^{-1}(\partial B_s(y_s, \epsilon))$ by $\partial B(y, r)$, where $y = C^{-1}(y_s) \in \mathbb{R}^n$ and r is the radius of $B(y, r)$ in $\mathbb{R}^n$. Consider $\overline{B}_s(y_s, \epsilon) \subset V_s$, then we have

$$\begin{aligned} & \int_{V_s} \left(E_k^S(x_s, y_s, u, v), R_k^S f(x_s, v)\right)_v dS(x_s) \\ & \quad = \int_{V_s \setminus B_s(y_s, \epsilon)} \left(E_k^S(x_s, y_s, u, v), R_k^S f(x_s, v)\right)_v dS(x_s) \\ & \quad \quad + \int_{B_s(y_s, \epsilon)} \left(E_k^S(x_s, y_s, u, v), R_k^S f(x_s, v)\right)_v dS(x_s) \,. \end{aligned}$$

Because of the degree of homogeneity of $x_s - y_s$ in E_k^S, the second integral on the right hand goes to zero as ϵ goes to zero. Applying Theorem 7 to the first integral on the right hand we obtain

$$\begin{aligned} & \int_{V_s \setminus B_s(y_s, \epsilon)} \left(E_k^S(x_s, y_s, u, v), R_k^S f(x_s, v)\right)_v dS(x_s) \\ & \quad = \int_{\partial V_s} \left(E_k^S(x_s, y_s, u, v), P_k n(x_s) f(x_s, v)\right)_v d\Sigma(x_s) \\ & \quad \quad - \int_{\partial B_s(y_s, \epsilon)} \left(E_k^S(x_s, y_s, u, v), P_k n(x_s) f(x_s, v)\right)_v d\Sigma(x_s) \,. \end{aligned}$$

Since $f(x_s, v) = (f(x_s, v) - f(y_s, v)) + f(y_s, v)$ and taking into account the degree of homogeneity of $x_s - y_s$ in E_k^S and the continuity of f, we can replace the second integral on the right hand by

$$\int_{\partial B_s(y_s, \epsilon)} \left(E_k^S(x_s, y_s, u, v), P_k n(x_s) f(y_s, v)\right)_v d\Sigma(x_s) \,.$$

Applying Theorem 7, this integral is equal to

$$\begin{aligned}
&\int_{\partial B_s(y_s,\epsilon)} \big(E_k^S(x_s,y_s,u,v),n(x_s)f(y_s,v)\big)_v\, d\Sigma(x_s)\\
&\qquad= \int_{\partial B_s(y_s,\epsilon)}\int_{\mathbb{S}^{n-1}} E_k^S(x_s,y_s,u,v)n(x_s)f(y_s,v)\,d\Sigma(x_s)\,ds(v)\\
&\qquad= \int_{\partial B_s(y_s,\epsilon)}\int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} Z_k(u,\tilde{a}va)\\
&\qquad\quad \times J(C^{-1},y_s)^{-1}\frac{x_s-y_s}{\|x_s-y_s\|^n} n(x_s)f(y_s,v)ds(v)\,d\Sigma(x_s).
\end{aligned}$$

Now applying the inverse of the Cayley transformation to the last integral, we have

$$\begin{aligned}
&\int_{\partial B(y,r)}\int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} Z_k\Big(u,\frac{(x-y)w(x-y)}{\|x-y\|^2}\Big)J(C^{-1},y_s)^{-1}J(C,y)^{-1}\frac{x-y}{\|x-y\|^n}\\
&\qquad \times J(C,x)^{-1}J(C,x)n(x)J(C,x)f\Big(C(y),\frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\,ds(w)d\sigma(x)\,,
\end{aligned}$$

where $d\sigma(x)$ is the $n-1$-dimensional scalar Lebesgue measure on $\partial B(y,r)$ in $\mathbb{R}^n$ and $v=\dfrac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}$, $w\in\mathbb{R}^{n+1}$. In fact, v is a vector in $\mathbb{R}^n$ which is obtained by reflecting w in $\mathbb{R}^{n+1}$ and its last component is a constant.

Place $J(C,x)=(J(C,x)-J(C,y))+J(C,y)$, but $J(C,x)-J(C,y)$ tends to zero as x approaches y. Thus the previous integral can be replaced by

$$\begin{aligned}
&\int_{\partial B(y,r)}\int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} Z_k(u,\frac{(x-y)w(x-y)}{\|x-y\|^2})\frac{x-y}{\|x-y\|^n}n(x)\\
&\qquad\qquad \times J(C,y)f\Big(C(y),\frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\,ds(w)\,d\sigma(x)\,.
\end{aligned}$$

Here $n(x)=\dfrac{y-x}{\|x-y\|}$ is the unit out normal vector. Now the last integral becomes

$$\begin{aligned}
&\int_{\partial B(y,r)}\int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} Z_k\Big(u,\frac{(x-y)w(x-y)}{\|x-y\|^2}\Big)\frac{1}{\|x-y\|^{n-1}}\\
&\qquad\qquad \times J(C,y)f\Big(C(y),\frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\,ds(w)\,d\sigma(x)\,,
\end{aligned}$$

Using Lemma 5 in [6], the integral is now

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} Z_k(u,w)J(C,y)f\Big(C(y),\frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\,ds(w)\\
&\qquad= J(C,y)f\Big(C(y),\frac{J(C,y)uJ(C,y)}{\|J(C,y)\|^2}\Big)\\
&\qquad= J(C^{-1},y_s)^{-1}f\Big(y_s,\frac{J(C^{-1},y_s)^{-1}uJ(C^{-1},y_s)^{-1}}{\|J(C^{-1},y_s)^{-1}\|^2}\Big)\,,
\end{aligned}$$

since $J(C,y)=J(C^{-1},y_s)^{-1}$.

Now by setting $u'=\dfrac{J(C^{-1},y_s)^{-1}uJ(C^{-1},y_s)^{-1}}{\|J(C^{-1},y_s)^{-1}\|^2}$ and multiplying both sides of the above equation by $J(C^{-1},y_s)$, we obtain

$$f(y_s,u')=J(C^{-1},y_s)\int_{\mathbb{S}^{n-1}} Z_k(u,w)J(C,y)f\Big(C(y),\frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\,ds(w)\,.$$

Therefore when ϵ tends to zero we get the desired result. □

Corollary 2. *Let Ψ be a function in $C^\infty(V_s\times\mathbb{R}^n,\mathcal{M}_k)$ and $\operatorname{supp}(\Psi)\subset V_s$. Then*

$$\Psi(y_s,u')=-J(C^{-1},y_s)\int_{V_s}\big(E_k^S(x_s,y_s,u,v),R_k^S\Psi(x_s,v)\big)_v\,dS(x_s)\,,$$

where $u'=\dfrac{J(C^{-1},y_s)^{-1}uJ(C^{-1},y_s)^{-1}}{\|J(C^{-1},y_s)^{-1}\|^2}$.

Corollary 3 (Cauchy Integral Formula for R_k^S)**.** *If $R_k^Sf(x_s,u)=0$, then for $y_s\in V_s$ we have*

$$\begin{aligned}f(y_s,u')&=J(C^{-1},y_s)\int_{\partial V_s}\big(E_k^S(x_s,y_s,u,v),P_kn(x_s)f(x_s,v)\big)_v\,d\Sigma(x_s)\\&=J(C^{-1},y_s)\int_{\partial V_s}\big(E_k^S(x_s,y_s,u,v)n(x_s)P_{k,r},f(x_s,v)\big)_v\,d\Sigma(x_s),\end{aligned}$$

where $u'=\dfrac{J(C^{-1},y_s)^{-1}uJ(C^{-1},y_s)^{-1}}{\|J(C^{-1},y_s)^{-1}\|^2}$.

Definition 2 (Cauchy Transform for R_k^S)**.** *For a domain $V_s\subset\mathbb{S}^n$ and a function $f(x_s,u)\colon V_s\times\mathbb{R}^n\to\mathrm{Cl}_{n+1}$, which is monogenic in u, the T_k-transform of f is defined to be*

$$(T_kf)(y_s,v)=-\int_{V_s}\big(E_k^S(x_s,y_s,u,v),f(x_s,u)\big)_u\,dS(x_s)\,,\quad \textit{for}\quad y_s\in V_s\,.$$

Theorem 9. *For a function ψ in $C^\infty(\mathbb{S}^n\times\mathbb{R}^n,\mathcal{M}_k)$ we have*

$$P_kJ(C^{-1},y_s)D_{s,y_s}\int_{\mathbb{S}^n}-\big(E_k^S(x_s,y_s,u,v),\psi(x_s,u)\big)_u\,dS(x_s)=\psi(y_s,v)\,.$$

Proof. By [8], the integral

$$P_kJ(C^{-1},y_s)D_{s,y_s}\int_{\mathbb{S}^n}-\big(E_k^S(x_s,y_s,u,v),\psi(x_s,u)\big)_u\,dS(x_s)$$

can be replaced by

$$P_kJ(C^{-1},y_s)\int_{\partial B_s(y_s,\epsilon)}-n(x_s)\big(E_k^S(x_s,y_s,u,v),\psi(x_s,u)\big)_u\,dS(x_s)\,,$$

which in turn is equal to

$$P_k J(C^{-1}, y_s) \int_{\partial B_s(y_s,\epsilon)} \int_{\mathbb{S}^{n-1}} -n(x_s) \frac{1}{\omega_n c_k} \frac{x_s - y_s}{\|x_s - y_s\|^n} \times J(C^{-1}, y_s)^{-1} Z_k(au\tilde{a}, v) \psi(x_s, u) ds(u)\, dS(x_s)\,.$$

Since $\psi(x_s, u) = \psi(x_s, u) - \psi(y_s, u) + \psi(y_s, u)$ then using the continuity of ψ, we can replace the previous integral by

$$P_k J(C^{-1}, y_s) \int_{\partial B_s(y_s,\epsilon)} \int_{\mathbb{S}^{n-1}} -n(x_s) \frac{1}{\omega_n c_k} \frac{x_s - y_s}{\|x_s - y_s\|^n} \times J(C^{-1}, y_s)^{-1} Z_k(au\tilde{a}, v) \psi(y_s, u)\, ds(u)\, dS(x_s)\,.$$

Now applying the inverse of the Cayley transformation to the previous integral it becomes

$$\begin{aligned} P_k J(C^{-1}, y_s) &\int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} -J(C,x) n(x) J(C,x) \frac{1}{\omega_n c_k} \\ &\times J(C,x)^{-1} \frac{x-y}{\|x-y\|^n} J(C,y)^{-1} J(C^{-1}, y_s)^{-1} \\ &\times Z_k\Big(\frac{(x-y)u(x-y)}{\|x-y\|^2}, v\Big) \psi\Big(C(y), \frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\, ds(u)\, d\sigma(x) \\ = P_k J(C^{-1}, y_s) &\int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} -J(C,x) n(x) \frac{1}{\omega_n c_k} \frac{x-y}{\|x-y\|^n} \\ &\times Z_k\Big(\frac{(x-y)u(x-y)}{\|x-y\|^2}, v\Big) \psi\Big(C(y), \frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\, ds(u)\, d\sigma(x)\,, \end{aligned}$$

where $u = \dfrac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}$.

Using the fact $J(C,x) = (J(C,x) - J(C,y)) + J(C,y)$, and $J(C,x) - J(C,y)$ tends to zero as x approaches y, the integral can be replaced by

$$\begin{aligned} P_k J(C^{-1}, y_s) &\int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} -\frac{1}{\omega_n c_k} J(C,y) n(x) \frac{x-y}{\|x-y\|^n} \\ &\times Z_k\Big(\frac{(x-y)u(x-y)}{\|x-y\|^2}, v\Big) \psi\Big(C(y), \frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\, ds(u)\, d\sigma(x) \\ = P_k &\int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k r^{n-1}} Z_k\Big(\frac{(x-y)u(x-y)}{\|x-y\|^2}, v\Big) \\ &\times \psi\Big(C(y), \frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\, ds(u)\, d\sigma(x) \end{aligned}$$

Applying Lemma 5 in [6], the integral becomes

$$= P_k \int_{\mathbb{S}^{n-1}} Z_k(u,v)\psi\Big(C(y), \frac{J(C,y)wJ(C,y)}{\|J(C,y)\|^2}\Big)\, ds(u)$$

$$= P_k \int_{\mathbb{S}^{n-1}} Z_k(u,v)\psi(C(y),u)\, ds(u) = P_k\psi\big(C(y),v\big) = \psi(y_s,v)\,.$$

□

7. Rarita-Schwinger type operators on real projective space

We consider $\mathbb{S}^n$ and $\Gamma = \{\pm 1\}$, then $\mathbb{S}^n/\Gamma$ is $\mathbb{R}P^n$, the real projective space. In all that follows $\mathbb{S}^n$ will be a universal covering space of the conformally flat manifold $\mathbb{R}P^n$. So there is a projection map $p\colon \mathbb{S}^n \to \mathbb{R}P^n$. Further for each $x \in \mathbb{S}^n$ we shall denote $p(x)$ by x'. Furthermore if Q is a subset of U then we denote $p(Q)$ by Q'.

Consider the trivial bundle $\mathbb{S}^n \times \mathrm{Cl}_{n+1}$, then we set up a spinor bundle E_1 over $\mathbb{R}P^n$ by making the identification of (x,X) with $(-x,X)$, where $x \in \mathbb{S}^n$ and $X \in \mathrm{Cl}_{n+1}$.

Now we change the spherical Cauchy kernel $G_S(x-y) = \dfrac{-1}{\omega_n}\dfrac{x-y}{\|x-y\|^n}, x,y \in \mathbb{S}^n$, for the spherical Dirac operator into a kernel which is invariant with respect to $\{\pm 1\}$ in the variable $x \in \mathbb{S}^n$. Hence we consider $G_S(x-y) + G_S(-x-y)$. See [7].

Suppose V is a domain lying in the open northern hemisphere. We assume

$$f(x,u)\colon V \times \mathbb{R}^n \to \mathrm{Cl}_{n+1}$$

is a C^1 function in x and monogenic in u. We observe that the projection map $p\colon \mathbb{S}^n \to \mathbb{R}P^n$ induces a well defined function

$$f'(x',u) : V' \times \mathbb{R}^n \to E_1$$

such that $f'(x',u) = f(p^{-1}(x'),u)$, where V' is a well defined domain in $\mathbb{R}P^n$ and $x' = p(x)$.

We define the Rarita-Schwinger type operators on $\mathbb{R}P^n$, which we will call the real projective Rarita-Schwinger type operators, in the following form

$$R_k^{RP^n} f'(x',u) = P_k D_{RP^n,x'} f'(x',u)\,,$$

where $D_{RP^n,x'}$ is the Dirac operator in $\mathbb{R}P^n$ with respect to the variable x'. See [7].

Now we introduce the spherical Rarita-Schwinger kernel which is also invariant with respect to $\{\pm 1\}$ in the variable $x \in \mathbb{S}^n$:

$$\mathcal{E}_k^{S,1}(x,y,u,v) := E_k^S(x,y,u,v) + E_k^S(-x,y,u,v)\,.$$

Through the projection map p (over $x,y \in \mathbb{S}^n$) we obtain a kernel $\mathcal{E}_k^{RP^n,1}(x',y',u,v)$ for $\mathbb{R}P^n$ defined by

$$\mathcal{E}_k^{RP^n,1}(x',y',u,v) = \mathcal{E}_k^{S,1}\big(p^{-1}(x'),p^{-1}(y'),u,v\big)\,.$$

Now suppose that S is a suitably smooth hypersurface lying in the open northern hemisphere of $\mathbb{S}^n$ bounding a subdomain W of V with closure of W in V.

Theorem 10. *If $R_k^S f(x,u)=0$ then for $y\in W$*

$$f(y,w)=J(C^{-1},y)\int_S \left(\mathcal{E}_k^{S,1}(x,y,u,v),P_k n(x)f(x,u)\right)_u d\Sigma(x),$$

where $w=\dfrac{J(C^{-1},y)^{-1}vJ(C^{-1},y)^{-1}}{\|J(C^{-1},y)^{-1}\|^2}$, $n(x)$ is the unit outer normal vector to S at x lying in the tangent space of S^n at x and Σ is the usual Lebesgue measure on S.

Due to the projection map we have also

Theorem 11.

$$f'(y',\hat{v})=J(C^{-1},y')\int_{S'} \left(\mathcal{E}_k^{RP^n,1}(x',y',u,v),P_k dp(n(x))f'(x',u)\right)_u d\Sigma'(x'),$$

where $\hat{v}=\dfrac{J(C^{-1},y')^{-1}vJ(C^{-1},y')^{-1}}{\|J(C^{-1},y')^{-1}\|^2}$, $x'=p(x)$, $y'=p(y)$ and S' is the projection of S. Further Σ' is a induced measure on S' from the measure Σ on S and dp is the derivative of p.

Now we will assume that the domain V is such that $-x\in V$ for each $x\in V$ and the function f is two fold periodic, so that $f(x)=f(-x)$. Now the projection map p gives rise to a well defined domain V' on $\mathbb{R}P^n$ and a well defined function $f'(x',u)\colon V'\to E_1$ such that $f'(x',u)=f(\pm x,u)$ for $p(\pm x)=x'$. Then if $R_k^{RP^n} f'(x',u)=0$, we also have

$$f'(y',\hat{v})=J(C^{-1},y')\int_{S'} \left(\mathcal{E}_k^{RP^n,1}(x',y',u,v),P_k dp(n(x))f'(x',u)\right)_u d\Sigma'(x'),$$

where $\hat{v}$ is stated as in Theorem 11.

If now we suppose that the hypersurface S satisfies $-S=S$ then both y and $-y$ belong to the subdomain V and in this case

$$J(C^{-1},y')\int_{S'} \left(\mathcal{E}_k^{RP^n,1}(x',y',u,v),P_k dp(n(x))f'(x',u)\right)_u d\Sigma'(x')=2f'(y',\hat{v}).$$

We can also construct a second spinor bundle E_2 over $\mathbb{R}P^n$ by making the identification of (x,X) with $(-x,-X)$, where $x\in\mathbb{S}^n$ and $X\in\mathrm{Cl}_{n+1}$, we introduce the kernel:

$$\mathcal{E}_k^{S,2}(x,y,u,v):=E_k^S(x,y,u,v)-E_k^S(-x,y,u,v).$$

This kernel induces through the projection map on the variable $x,y\in\mathbb{S}^n$, the kernel on $\mathbb{R}P^n$

$$\mathcal{E}_k^{RP^n,2}(x',y',u,v)=\mathcal{E}_k^{S,2}\left(p^{-1}(x'),p^{-1}(y'),u,v\right).$$

In this case a solution of Rarita-Schwinger type equation on $\mathbb{R}P^n$

$$f'(x',u)\colon V'\times\mathbb{R}^n\to E_2$$

will lift to a solution of spherical-Rarita-Schwinger type equation: $f(x,u)\colon V\times\mathbb{R}^n\to\mathrm{Cl}_{n+1}$ such that $f(x,u)=-f(-x,u)$.

Suppose that V as before is a domain on $\mathbb{S}^n$ and S is a hypersurface in V bounding a subdomain W of V. Suppose further that $f(x,u)\colon V\times\mathbb{R}^n\to \mathrm{Cl}_{n+1}$ is a solution of the spherical Rarita-Schwinger type equation such that $f(x,u)=-f(-x,u)$. If S lies entirely in one open hemisphere then

$$f(y,w)=J(C^{-1},y)\int_S \big(\mathcal{E}_k^{S,2}(x,y,u,v),P_kn(x)f(x,u)\big)_u\,d\Sigma(x)\,,$$

for each $y\in W$, where $w=\dfrac{J(C^{-1},y)^{-1}vJ(C^{-1},y)^{-1}}{\|J(C^{-1},y)^{-1}\|^2}$.

Via the projection p this integral formula induces the following

$$f'(y',\hat{v})=J(C^{-1},y')\int_{S'}\big(\mathcal{E}_k^{RP^n,2}(x',y',u,v),P_kdp(n(x))f'(x',u)\big)_u\,d\Sigma'(x')\,,$$

where $\hat{v}$ is stated as in Theorem 11.

On the other hand if S is such that $S=-S$ then

$$\int_S\big(\mathcal{E}_k^{S,2}(x,y,u,v),P_kn(x)f(x,u)\big)_u\,d\Sigma(x)=0\,.$$

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