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ON COMMUTATIVE RINGS WHOSE PRIME IDEALS ARE DIRECT SUMS OF CYCLICS

M. Behboodi and A. Moradzadeh-Dehkordi

Abstract. In this paper we study commutative rings R whose prime ideals are direct sums of cyclic modules. In the case R is a finite direct product of commutative local rings, the structure of such rings is completely described. In particular, it is shown that for a local ring $(R,\mathcal{M})$, the following statements are equivalent: (1) Every prime ideal of R is a direct sum of cyclic R-modules; (2) $\mathcal{M}=\bigoplus_{\lambda\in\Lambda} Rw_\lambda$ where Λ is an index set and $R/\mathrm{Ann}(w_\lambda)$ is a principal ideal ring for each $\lambda\in\Lambda$; (3) Every prime ideal of R is a direct sum of at most $|\Lambda|$ cyclic R-modules where Λ is an index set and $\mathcal{M}=\bigoplus_{\lambda\in\Lambda} Rw_\lambda$; and (4) Every prime ideal of R is a summand of a direct sum of cyclic R-modules. Also, we establish a theorem which state that, to check whether every prime ideal in a Noetherian local ring $(R,\mathcal{M})$ is a direct sum of (at most n) principal ideals, it suffices to test only the maximal ideal $\mathcal{M}$.

1. Introduction

It was shown by Köthe [8] that an Artinian commutative ring R has the property that every module is a direct sum of cyclic modules if and only if R is a principal ideal ring. Later Cohen-Kaplansky [6] obtained the following result: *"a commutative ring R has the property that every module is a direct sum of cyclic modules if and only if R is an Artinian principal ideal ring."* (Recently, a generalization of the Köthe-Cohen-Kaplansky theorem have been given by Behboodi et al., [3] for the noncommutative setting.) Therefore, an interesting natural question of this sort is *"Whether the same is true if one only assumes that every ideal of R is a direct sum of cyclic modules?"* More recently, this question was answered by Behboodi et al. [2] and [4] for the case R is a finite direct product of commutative local rings.

We note that two theorems from commutative algebra due to I. M. Isaacs and I. S. Cohen state that, to check whether every ideal in a ring is cyclic (resp. finitely generated), it suffices to test only the prime ideals (see [7, p. 8, Exercise 10] and [5, Theorem 2]). So this raises the natural question: "If every prime ideal of R is a direct sum of cyclics, can we conclude that all ideals are direct sums of cyclics?"

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This is not true in general. In [2, Example 3.1], for each integer $n \geq 3$, we provide an example of an Artinian local ring R such that the only prime (maximal) ideal of R is a direct sum of n cyclic R-modules, but there exists a two generated ideal of R which is not a direct sum of cyclic R-modules. Therefore, another interesting natural question of this sort is *"What is the class of commutative rings R for which every prime ideal is a direct sum of cyclic modules?"* The goal of this paper is to answer this question in the case R is a finite direct product of commutative local rings. The structure of such rings is completely described.

Throughout this paper, all rings are commutative with identity and all modules are unital. For a ring R we denote by $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ for the set of prime ideals and maximal ideals of R, respectively. We denote the classical Krull dimension of R by $\dim(R)$. Let X be either an element or a subset of R. The annihilator of X is the ideal $\mathrm{Ann}(X) = \{a \in R \mid aX = 0\}$. A ring R is *local* (resp. *semilocal*) in case R has a unique maximal ideal (resp. a finite number of maximal ideals). In this paper $(R, \mathcal{M})$ will be a local ring with maximal ideal $\mathcal{M}$. A non-zero R-module N is called *simple* if it has no submodules except (0) and N.

For a ring R, it is shown that if every prime ideal of R is a direct sum of cyclic R-modules, then $\dim(R) \leq 1$ (Proposition 2.1). Let R be a semilocal ring such that every prime ideal of R is a direct sum of cyclic R-modules. Then: (i) R is a principal ideal ring if and only if every maximal ideal of R is principal (Theorem 2.4); (ii) R is a Noetherian ring if and only if every maximal ideal of R is finitely generated (Theorem 2.5). Also, in Proposition 2.6, it is shown that if for each $\mathcal{M} \in \mathrm{Max}(R)$, $\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ where Λ is an index set and $R/\mathrm{Ann}(w_\lambda)$ is a principal ideal ring for each $\lambda \in \Lambda$, then every prime ideal of R is a direct sum of cyclic modules. However Example 2.7 shows that the converse is not true in general, but it is true when R is a local ring (see Theorem 2.10). In particular, in Theorem 2.10, we show that for a local ring $(R, \mathcal{M})$ the following statements are equivalent:

(1) Every prime ideal of R is a direct sum of cyclic R-modules.

(2) $\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ where Λ is an index set and $R/\mathrm{Ann}(w_\lambda)$ is a principal ideal ring for each $\lambda \in \Lambda$.

(3) Every prime ideal of R is a direct sum of at most $|\Lambda|$ cyclic R-modules where Λ is an index set.

(4) Every prime ideal of R is a summand of a direct sum of cyclic R-modules.

Also, if $(R, \mathcal{M})$ is Noetherian, we show that the above conditions are also equivalent to: (5) $\mathcal{M}$ *is a direct sum of cyclic R-modules* (see Theorem 2.12); which state that, to check whether every prime ideal in a Noetherian local ring $(R, \mathcal{M})$ is a direct sum of (at most n) principal ideals, it suffices to test only the maximal ideal $\mathcal{M}$.

Finally, as a consequence, we obtain: if $R = R_1 \times \cdots \times R_k$, where each R_i $(1 \leq i \leq k)$ is a local ring, then every prime ideal of R is a direct sum of cyclic R-modules if and only if each R_i satisfies the above equivalent conditions (see Corollary 2.14). We note that the corresponding result in the case $R = \prod_{\lambda\in\Lambda} R_\lambda$

where Λ is an infinite index set and each R_λ is a local ring, is not true in general (see Example 2.15).

2. Main results

We begin with the following evident useful proposition (see [4, Proposition 2.5]).

Proposition 2.1. *Let R be a ring. If every prime ideal of R is a direct sum of cyclic R-modules, then for each prime ideal P of R, the ring R/P is a principal ideal domain. Consequently,* $\dim(R) \leq 1$.

Proof. Assume that every prime ideal of R is a direct sum of cyclic R-modules and $P \subseteq Q$ are prime ideals of R. Since Q is a direct sum of cyclics, we conclude that Q/P is a principal ideal of R/P. Thus R/P is a principal ideal domain. On the other hand $\dim(R) = \dim(R/P_0)$ for some minimal prime ideal P_0 of R. Thus $\dim(R) \leq 1$. $\square$

The following two famous theorems from commutative algebra are crucial in our investigation.

Lemma 2.2 (Cohen [5, Theorem 2])**.** *Let R be a commutative ring. Then R is a Noetherian ring if and only if every prime ideal of R is finitely generated.*

Lemma 2.3 (Kaplansky [7, Theorem 12.3])**.** *A commutative Noetherian ring R is a principal ideal ring if and only if every maximal ideal of R is principal.*

The following theorem is an analogue of Kaplansky's theorem.

Theorem 2.4. *Let R be a semilocal ring such that every prime ideal of R is a direct sum of cyclic R-modules. Then R is a principal ideal ring if and only if every maximal ideal of R is principal.*

Proof. ($\Rightarrow$) is clear.
($\Leftarrow$) We can write $R = R_1 \times \cdots \times R_n$ where each R_i is an indecomposable ring (i.e., R_i has no any nontrivial idempotent elements). Clearly every prime ideal of R is a direct sum of cyclic R-modules if and only if every prime ideal of R_i is a direct sum of cyclic R-modules for each $1 \leq i \leq n$. Also, every maximal ideal of R is principal if and only if every maximal ideal of R_i is principal for each $1 \leq i \leq n$. Thus without loss of generality, we can assume that R is an indecomposable ring. Also, by Proposition 2.1, $\dim(R) \leq 1$.

Suppose, contrary to our claim, that R is not a principal ideal ring. Thus by Lemma 2.3, R is not a Noetherian ring. Thus by Lemma 2.2, there exists a prime ideal P of R such that $P = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ where Λ is an infinite index set and $0 \neq w_\lambda \in R$ for each $\lambda \in \Lambda$. Thus P is not a maximal ideal of R and so it is a minimal prime ideal of R.

For each $\lambda \in \Lambda$, there exists a maximal submodule K_λ of Rw_λ and so $\mathrm{Ann}(Rw_\lambda/K_\lambda) = \mathcal{M}$ for some maximal ideal $\mathcal{M}$ of R. Since $\mathrm{Max}(R)$ is finite and $|\Lambda| = \infty$, we can assume that $\{1, 2\} \subseteq \Lambda$ and there exists $\mathcal{M} \in \mathrm{Max}(R)$ such that

$$\mathrm{Ann}(Rw_1/K_1) = \mathcal{M} = \mathrm{Ann}(Rw_2/K_2)\,.$$

Now set $P = Rw_1 \oplus Rw_2 \oplus L$ where L is an ideal of R and $\bar{R} := R/(K_1 \oplus K_2 \oplus L)$. Since $\mathcal{M}(Rw_i/K_i) = (0)$ for $i = 1, 2$ and

$$\bar{P} = P/(K_1 \oplus K_2 \oplus L) \cong (Rw_1 \oplus Rw_2)/(K_1 \oplus K_2) \cong R/\mathcal{M} \oplus R/\mathcal{M}$$

we conclude that $\bar{\mathcal{M}}\bar{P} = (0)$. It follows that $\bar{P}$ is the only non-maximal prime ideal of $\bar{R}$. Thus by Lemma 2.2, $\bar{R}$ is a Noetherian ring (since $\bar{P}$ is finitely generated and every maximal ideal of $\bar{R}$ is cyclic) and so by Lemma 2.3, $\bar{R}$ is a principal ideal ring. But $\bar{P}$ is a direct sum of two isomorphic simple R-modules (so $\bar{P}$ is a 2-dimensional $R/\mathcal{M}$-vector space) and hence it is not a cyclic R-module, a contradiction. □

Also, the following result is an analogue of Cohen's theorem.

Theorem 2.5. *Let R be a semilocal ring such that every prime ideal of R is a direct sum of cyclic R-modules. Then R is a Noetherian ring if and only if every maximal ideal of R is finitely generated.*

Proof. $(\Rightarrow)$ is clear.
$(\Leftarrow)$ We can write $R = R_1 \times \cdots \times R_n$ where each R_i is an indecomposable ring (i.e., R_i has no any nontrivial idempotent elements). Thus without loss of generality, we can assume that R is an indecomposable ring with maximal ideals $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$. Then by Proposition 2.1, $\dim(R) \le 1$. Suppose, contrary to our claim, thus by Lemma 2.2, there exists a prime ideal P of R such that $P = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ where Λ is an infinite index set and $0 \neq w_\lambda \in R$ for each $\lambda \in \Lambda$. Thus P is not a maximal ideal of R and so it is a minimal prime ideal of R. Also, by hypothesis for each $1 \le i \le k$, there exist $x_{i1}, \ldots, x_{in_i} \in R$ such that

$$\mathcal{M}_i = Rx_{i1} \oplus Rx_{i2} \oplus \cdots \oplus Rx_{in_i}\,.$$

Since P is a non-maximal prime ideal, without loss of generality, we can assume that $x_{11}, x_{21}, \ldots, x_{k1} \notin P$. It follows that $Rx_{i2} \oplus \cdots \oplus Rx_{in_i} \subseteq P$ for each $i = 1, \ldots, k$. Set

$$L = (Rx_{12} \oplus \cdots \oplus Rx_{1n_1}) + (Rx_{22} \oplus \cdots \oplus Rx_{2n_2}) + \cdots + (Rx_{k2} \oplus \cdots \oplus Rx_{kn_k})\,.$$

Then $L \subseteq P$ and so $L \subseteq \bigoplus_{\lambda\in\Lambda'} Rw_\lambda$ where Λ' is a finite subset of Λ.

Clearly, for each $\lambda \in \Lambda$, there exists a maximal submodule K_λ of Rw_λ and hence $\mathrm{Ann}(Rw_\lambda/K_\lambda) = \mathcal{M}$ for some maximal ideal $\mathcal{M}$ of R. Since $\mathrm{Max}(R)$ is finite and $|\Lambda| = \infty$, we can assume that $\{1, 2\} \subseteq \Lambda$ and there exists $\mathcal{M} \in \mathrm{Max}(R)$ such that

$$\mathrm{Ann}(Rw_1/K_1) = \mathcal{M} = \mathrm{Ann}(Rw_2/K_2)\,.$$

Now we can assume that $P = Rw_1 \oplus Rw_2 \oplus L_1$ such that $\bigoplus_{\lambda\in\Lambda'} Rw_\lambda \subseteq L_1$. Set

$$\bar{R} = R/(K_1 \oplus K_2 \oplus L_1)\,.$$

Since $\mathcal{M}(Rw_i/K_i) = (0)$ for $i = 1, 2$ and

$$\bar{P} = P/(K_1 \oplus K_2 \oplus L_1) \cong (Rw_1 \oplus Rw_2)/(K_1 \oplus K_2) \cong R/\mathcal{M} \oplus R/\mathcal{M}\,,$$

we conclude that $\bar{\mathcal{M}}\bar{P} = (0)$. It follows that $\bar{P}$ is the only non-maximal prime ideal of $\bar{R}$. On the other hand, for each $1 \le i \le k$, $Rx_{i2} \oplus \cdots \oplus Rx_{in_i} \subseteq \bigoplus_{\lambda\in\Lambda'} Rw_\lambda \subseteq L_1$. Thus we conclude that every maximal ideal of $\bar{R}$ is cyclic. Thus by Theorem 2.4, $\bar{R}$ is a principal ideal ring. But $\bar{P}$ is a direct sum of two isomorphic simple R-modules (so $\bar{P}$ is a 2-dimensional $R/\mathcal{M}$-vector space), and hence it is not a cyclic R-module, a contradiction. $\square$

Proposition 2.6. *Let R be a ring. If for each $\mathcal{M} \in \operatorname{Max}(R)$, $\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ where Λ is an index set and $R/\operatorname{Ann}(w_\lambda)$ is a principal ideal ring for each $\lambda \in \Lambda$, then every prime ideal of R is a direct sum of cyclic modules.*

Proof. Assume that P is a non-maximal prime ideal of R. There exists a maximal ideal $\mathcal{M} \in \operatorname{Max}(R)$ such that $P \subsetneqq \mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$. Thus there exists a $\lambda_0 \in \Lambda$ where Λ is an index set such that $w_{\lambda_0} \notin P$. Thus, $\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda \subseteq P$ and so by modular property, we have

$$P = P \cap \mathcal{M} = (P \cap Rw_{\lambda_0}) \oplus \Big(\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda \Big) .$$

Now since $P \cap Rw_{\lambda_0} \subseteq Rw_{\lambda_0} \cong R/\operatorname{Ann}(Rw_{\lambda_0})$ and $R/\operatorname{Ann}(Rw_{\lambda_0})$ is a principal ideal ring, we conclude that $P \cap Rw_{\lambda_0}$ is cyclic. Therefore, P is a direct sum of cyclic modules. $\square$

However the following example shows that the converse of Proposition 2.6, is not true in general, but we will show in Theorem 2.10, it is true when R is a local ring.

Example 2.7. Let R be the subring of all sequences from the ring $\prod_{i\in\mathbb{N}} \mathbb{Z}_2$ that are eventually constant. Then R is a zero-dimensional Boolean ring with minimal prime ideals $P_i = \{\{a_n\} \in R \mid a_i = 0\}$ and $P_\infty = \{\{a_n\} \in R \mid a_n = 0 \text{ for large } n\}$ (See [1]). Clearly, each P_i is cyclic (in fact $P_i = Rv_i$ where $v_i = (1, 1, \dots, 1, 0, 1, 1, \dots)$) and $P_\infty = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_2 = \bigoplus_{i\in\mathbb{N}} Rw_i$ where $w_i = (0, 0, \dots, 0, 1, 0, 0, \dots)$. Thus every prime ideal of R is a direct sum of cyclic modules. But the factor ring $R/\operatorname{Ann}(v_1) = R/\operatorname{Ann}(0, 1, 1, 1, \dots)$ is not a principal ideal ring (since prime ideal $P_\infty/\operatorname{Ann}(v_1)$ is not a principal ideal of $R/\operatorname{Ann}(v_1)$). Also, if $P_1 = \bigoplus_{\lambda\in\Lambda} Rz_\lambda$ where Λ is an index set with $|\Lambda| > 1$ and $z_\lambda \in P_1$), then there exists $\lambda_0 \in \Lambda$ such that $z_{\lambda_0} = \{a_n\}$ where $a_n = 1$ for large n (since $(0, 1, 1, 1, \cdots) \in P_1$). One can easily see that $\operatorname{Ann}(z_{\lambda_0}) \subseteq P_\infty$ and the prime ideal $P_\infty/\operatorname{Ann}(z_{\lambda_0})$ of the factor ring $R/\operatorname{Ann}(z_{\lambda_0})$ is not principal (so $R/\operatorname{Ann}(z_{\lambda_0})$ is not a principal ideal ring). Thus the converse of Proposition 2.6 is not true in general.

By using Nakayama's lemma, we obtain the following lemma.

Lemma 2.8. *Let R be a ring and M be an R-module such that M is a direct sum of a family of finitely generated R-modules. Then Nakayama's lemma holds for M (i.e., for each $I \subseteq J(R)$, if $IM = M$, then $M = (0)$).*

Lemma 2.9 (See Warfield [9, Proposition 3])**.** *Let R be a local ring and N an R-module. If $N = \oplus_{\lambda\in\Lambda} R/I_\lambda$ where each I_λ is an ideal of R, then every summand of N is also a direct sum of cyclic R-modules, each isomorphic to one of the R/I_λ.*

The following main theorem is an answer to the question "What is the class of local rings R for which every prime ideal is a direct sum of cyclic modules?"

Theorem 2.10. *Let $(R, \mathcal{M})$ be a local ring. Then the following statements are equivalent:*

(1) *Every prime ideal of R is a direct sum of cyclic R-modules.*

(2) $\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ *where Λ is an index set and $R/Ann(w_\lambda)$ is a principal ideal ring for each $\lambda \in \Lambda$.*

(3) *Every prime ideal of R is a direct sum of at most $|\Lambda|$ cyclic R-modules where Λ is an index set.*

(4) *Every prime ideal of R is a summand of a direct sum of cyclic R-modules.*

Proof. (1) $\Rightarrow$ (2) First, we assume that $\mathcal{M}$ is cyclic and so $\mathcal{M} = Rx$ for some $x \in \mathcal{M}$. If $\mathrm{Spec}(R) = \{\mathcal{M}\}$, then by Lemma 2.2, R is a Noetherian ring and by Lemma 2.3, R is a principal ideal ring. Therefore, $R/\mathrm{Ann}(x)$ is a principal ideal ring. If $\mathrm{Spec}(R) \neq \{\mathcal{M}\}$, then for each non-maximal prime ideal P of R, $x \notin P \subsetneqq \mathcal{M}$. Thus $Px = P$ and so by Lemma 2.8, $P = 0$. Thus R is a principal ideal domain and so $R/\mathrm{Ann}(x)$ is a principal ideal ring.

Now assume that $\mathcal{M}$ is not cyclic. Then by hypothesis $\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ such that Λ is an index set with $|\Lambda| \geq 2$ and $0 \neq w_\lambda \in \mathcal{M}$ for each $\lambda \in \Lambda$. If $\mathrm{Spec}(R) = \{\mathcal{M}\}$, then the only maximal ideal of $R/\mathrm{Ann}(w_\lambda)$ is principal for each $\lambda \in \Lambda$. Thus by Lemma 2.2, $R/\mathrm{Ann}(w_\lambda)$ is a Noetherian ring and so by Lemma 2.3, $R/\mathrm{Ann}(w_\lambda)$ is a principal ideal ring for each $\lambda \in \Lambda$. If $\mathrm{Spec}(R) \neq \{\mathcal{M}\}$, then for each non-maximal prime ideal P of R, there exists $\lambda_0 \in \Lambda$ such that $w_{\lambda_0} \notin P$. It follows that $\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda \subseteq P$. Now by modular property we have

$$P = P \cap \mathcal{M} = (P \cap Rw_{\lambda_0}) \oplus \Big(\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda \Big) .$$

It follows that $Pw_{\lambda_0} = (P \cap Rw_{\lambda_0})w_{\lambda_0}$. Also since $w_{\lambda_0} \notin P$, $P \cap Rw_{\lambda_0} = Pw_{\lambda_0}$ and hence $Pw_{\lambda_0} = (Pw_{\lambda_0})Rw_{\lambda_0}$. Now by Lemma 2.8, $Pw_{\lambda_0} = 0$, since Pw_{λ_0} is a direct sum of cyclic R-modules. Therefore, $P = \bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda$. Thus we conclude that $\mathrm{Spec}(R) = \{\mathcal{M}\} \cup \{\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_j\}} Rw_\lambda \mid w_{\lambda_j} \notin \mathrm{Nil}(R)\}$. This shows that for each $\lambda \in \Lambda$, all prime ideals of $R/\mathrm{Ann}(w_\lambda)$ are principal. Thus by Lemmas 2.2 and 2.3, $R/\mathrm{Ann}(w_\lambda)$ is a principal ideal ring for each $\lambda \in \Lambda$.

(2) $\Rightarrow$ (3) Assume that $\mathcal{M}$ is cyclic and so $\mathcal{M} = Rx$ for some $x \in \mathcal{M}$. If $\mathrm{Spec}(R) = \{\mathcal{M}\}$ then the proof is complete. If $\mathrm{Spec}(R) \neq \{\mathcal{M}\}$, then for each non-maximal prime ideal P of R, $x \notin P \subsetneqq \mathcal{M}$. Thus $Px = P$ and so $Px = Px(Rx)$. By hypothesis $R/\mathrm{Ann}(x)$ is a principal ideal ring and so Px is principal. Thus by Lemma 2.8, $Px = 0$ and so $P = 0$.

Now assume that $\mathcal{M}$ is not cyclic and so $\mathcal{M} = \bigoplus_{\lambda\in\Lambda} Rw_\lambda$ such that $|\Lambda| \geq 2$ and $R/\mathrm{Ann}(Rw_\lambda)$ is a principal ideal ring for each $\lambda \in \Lambda$. If $\mathrm{Spec}(R) = \{\mathcal{M}\}$, then the proof is complete. If $\mathrm{Spec}(R) \neq \{\mathcal{M}\}$, then for each non-maximal prime ideal P of R, there exists $\lambda_0 \in \Lambda$ such that $w_{\lambda_0} \notin P$. This implies that $\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda \subseteq P$. Thus by modular property, $P = P \cap \mathcal{M} = (P \cap Rw_{\lambda_0}) \oplus (\bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda)$ and so $Pw_{\lambda_0} = (P \cap Rw_{\lambda_0})w_{\lambda_0}$. Also $P \cap Rw_{\lambda_0} = Pw_{\lambda_0}$, since $w_{\lambda_0} \notin P$ and $Pw_{\lambda_0} = Pw_{\lambda_0}(Rw_{\lambda_0})$. But $Pw_{\lambda_0} = P \cap Rw_{\lambda_0}$ is principal, since $R/\mathrm{Ann}(w_{\lambda_0})$ is a principal ideal ring. Thus by Lemma 2.8, $Pw_{\lambda_0} = 0$ and so $P = \bigoplus_{\lambda\in\Lambda\setminus\{\lambda_0\}} Rw_\lambda$. Thus we conclude that

$$\operatorname{Spec}(R) = \{\mathcal{M}\} \cup \Big\{ \bigoplus_{\lambda \in \Lambda \setminus \{\lambda_j\}} Rw_\lambda \mid w_{\lambda_j} \notin \operatorname{Nil}(R) \Big\},$$

and hence every prime ideal of R is a direct sum of at most $|\Lambda|$ cyclic R-modules.
(3) $\Rightarrow$ (4) is clear.
(4) $\Rightarrow$ (1) is by Lemma 2.9. □

Also, the following result is an answer to the question "What is the class of local rings $(R, \mathcal{M})$ for which $\mathcal{M}$ is finitely generated and every prime ideal is a direct sum of cyclic modules?"

Corollary 2.11. *Let $(R, \mathcal{M})$ be a local ring. Then the following statements are equivalent:*

(1) *R is a Noetherian ring and every prime ideal of R is a direct sum of cyclic R-modules.*
(2) *$\mathcal{M} = \bigoplus_{i=1}^{n} Rw_i$ and $R/\operatorname{Ann}(w_i)$ is a principal ideal ring for each $1 \le i \le n$.*
(3) *Every prime ideal of R is a direct sum of at most n cyclic R-modules.*
(4) *R is a Noetherian ring and every prime ideal of R is a summand of a direct sum of cyclic R-modules.*

Proof. The proof is straightforward by Theorem 2.5 and Theorem 2.10. □

Next, we greatly improve the main theorem above (Theorem 2.10) in the case R is a Noetherian local ring. In fact, we establish the following result which state that, to check whether every prime ideal in a Noetherian local ring $(R, \mathcal{M})$ is a direct sum of (at most n) principal ideals, it suffices to test only the maximal ideal $\mathcal{M}$. We note that this is also a generalization of the Kaplansky Theorem in the case R is a Noetherian local ring.

Theorem 2.12. *Let $(R, \mathcal{M})$ be a Noetherian local ring. Then the following statements are equivalent:*

(1) *Every prime ideal of R is a direct sum of cyclic R-modules.*
(2) *$\mathcal{M} = \bigoplus_{i=1}^{n} Rw_i$ and $R/\operatorname{Ann}(w_i)$ is a principal ideal ring for each $1 \le i \le n$.*
(3) *The maximal ideal $\mathcal{M}$ is a direct sum of n cyclic R-modules.*
(4) *Every prime ideal of R is a direct sum of at most n cyclic R-modules.*
(5) *Every prime ideal of R is a summand of a direct sum of cyclic R-modules.*

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are by Theorem 2.10.
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (4) If $\mathcal{M} = Rx$ is a cyclic R-module, then by Lemma 2.3, R is a principal ideal ring. Assume that $\mathcal{M} = \bigoplus_{i=1}^{n} Rw_i$ where $n \ge 2$. If $\operatorname{Spec}(R) = \{\mathcal{M}\}$, then the proof is complete. Thus we can assume that $\operatorname{Spec}(R) \ne \{\mathcal{M}\}$ and suppose that $P \subsetneqq \mathcal{M}$ is a prime ideal of R. Without loss of generality, we can assume that, $w_1 \notin P$. This implies that $\bigoplus_{i=2}^{n} Rw_i \subseteq P$. Now by modular property we have $P = P \cap \mathcal{M} = (P \cap Rw_1) \oplus (\bigoplus_{i=2}^{n} Rw_i)$, and hence $Pw_1 = (P \cap Rw_1)w_1$. Also, $P \cap Rw_1 = Pw_1$ since $w_1 \notin P$. Thus $Pw_1 = (Pw_1)Rw_1$, and so by Lemma 2.8, $Pw_1 = 0$. Therefore, $P = \bigoplus_{i=2}^{n} Rw_i$. □

Remark 2.13. Let $R = R_1 \times \cdots \times R_k$ where $k \in \mathbb{N}$ and each R_i is a nonzero ring. One can easily see that, each prime ideal P of R is of the form $P = R_1 \times \cdots \times R_{i-1} \times P_i \times R_{i+1} \times \cdots \times R_k$ where P_i is a prime ideal of R_i. Also, if P_i is a direct sum of Λ principal ideals of R_i, then it is easy to see that P is also a direct sum of Λ principal ideals of R. Thus the ring R has the property that whose prime ideals are direct sum of cyclic R-modules if and only if for each i the ring R_i has this property.

We are thus led to the following strengthening of Theorem 2.10.

Corollary 2.14. *Let $R = R_1 \times \cdots \times R_k$ where $k \in \mathbb{N}$ and each R_i is a local ring with maximal ideal $\mathcal{M}_i$ $(1 \leq i \leq k)$. Then the following statements are equivalent:*

(1) *Every prime ideal of R is a direct sum of cyclic R-modules.*
(2) *For each i, $\mathcal{M}_i = \bigoplus_{\lambda_i \in \Lambda_i} Rw_{\lambda_i}$ where each Λ_i is an index set and $R/\operatorname{Ann}(w_{\lambda_i})$ is a principal ideal ring for each $\lambda_i \in \Lambda_i$.*
(3) *Every prime ideal of R is a direct sum of at most $|\Lambda|$ cyclic R-modules, where $|\Lambda| = \max\{|\Lambda_i| \mid i = 1, \ldots, k\}$ and $\mathcal{M}_i = \bigoplus_{\lambda_i \in \Lambda_i} Rw_{\lambda_i}$ for each Λ_i.*
(4) *Every prime ideal of R is a summand of a direct sum of cyclic R-modules.*

Proof. The proof is straightforward by Theorem 2.10 and Remark 2.13. □

We conclude this paper with the following interesting example. In fact, the following example shows that the corresponding of the above result in the case $R = \prod_{\lambda \in \Lambda} R_\lambda$ where Λ is an infinite index set and each R_λ is a local ring (even if for each $\lambda \in \Lambda$, $R_\lambda \cong \mathbb{Z}_2$), is not true in general.

Example 2.15. Let $R = \prod_{\lambda \in \Lambda} F_\lambda$ be a direct product of fields $\{F_\lambda\}_{\lambda \in \Lambda}$ where Λ is an infinite index set. Clearly, $I = \bigoplus_{\lambda \in \Lambda} F_\lambda$ is a non-maximal ideal of R. Thus there exists a maximal ideal P of R such that $I \subsetneqq P$. It was shown by Cohen and Kaplansky [6, Lemma 1] that P is not a direct sum of principal ideals.

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Corresponding author: M. Behboodi,
Department of Mathematical Sciences, Isfahan University of Technology,
P.O.Box 84156-83111, Isfahan, Iran
and
School of Mathematics,
Institute for Research in Fundamental Sciences (IPM)
P.O.Box: 19395-5746, Tehran, Iran
E-mail: mbehbood@cc.iut.ac.ir

Department of Mathematical Sciences, Isfahan University of Technology,
P.O.Box 84156-83111, Isfahan, Iran
E-mail: a.moradzadeh@math.iut.ac.ir