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SOME GENERALIZED COMPARISON RESULTS IN FINSLER GEOMETRY AND THEIR APPLICATIONS

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Abstract. In this paper, we generalize the Hessian comparison theorems and Laplacian comparison theorems described in [16, 18], then give some applications under various curvature conditions.

1. Introduction

Recently, there has been a surge of interest in Finsler geometry, especially in its global and analytic aspects (see [14]). One of the fundamental problems is to study the comparison theorem in Finsler manifold. It has been started in [16, 13, 18], and the following results are obtained by Z. Shen, B. Y. Wu and Y. L. Xin.

Proposition 1.1 (see [18, Theorem 4.1]). *Let (M,F) be a complete Finsler manifold of dimension m, and $r=d_F(p,x)$ is the distance function on M from a fixed point $p\in M$. Suppose that the flag curvature of M satisfies $K(V;W)\le C$ (resp. $K(V;W)\ge C$), then the following inequality holds whenever r is smooth:*

$$\mathrm{Hess}(r)(X,X)\le \ (\textit{resp.}\ \ge)\ \mathrm{ct_C(r)}\big(\mathrm{g_{\nabla r}(X,X)}-\mathrm{g^2_{\nabla r}(\nabla r,X)}\big)\,. \tag{1.1}$$

Proposition 1.2 (see [16, Theorem 8.2], or see [18, Theorem 5.1 and Theorem 5.3]). *Let (M,F) be a complete Finsler manifold of dimension m, and $r=d_F(p,x)$ is the distance function on M from a fixed point $p\in M$.*
(i) *Suppose that the flag curvature of M satisfies $K(V;W)\le C$, then*

$$\triangle r\ge (m-1)\mathrm{ct_C(r)}-\mathrm{S(\nabla r)}\quad \textit{on}\quad (\mathrm{D_p}\backslash \mathrm{p})\cap \mathrm{B_{r_0}(p)}\,; \tag{1.2}$$

(ii) *Suppose that the Ricci curvature of M satisfies* $\mathrm{Ric}\ge (m-1)C$, *then*

$$\triangle r\le (m-1)\mathrm{ct_C(r)}-\mathrm{S(\nabla r)}\quad \textit{on}\quad (\mathrm{D_p}\backslash \mathrm{p})\cap \mathrm{B_{r_0}(p)}\,, \tag{1.3}$$

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where

$$\mathrm{ct_C(r)} = \begin{cases} \sqrt{C}\,\mathrm{cotanh}(\sqrt{C}r) \quad \textit{for} \quad C > 0\,; \\ \dfrac{1}{r} \quad \textit{for} \quad C = 0\,; \\ \sqrt{-C}\,\mathrm{cotanh}(\sqrt{-C}r) \quad \textit{for} \quad C < 0\,. \end{cases} \tag{1.4}$$

In this paper, we generalize the above propositions under a weaker assumptions that the curvature is bounded by a delicate bound given by a radial function, then obtain some applications of them. The article is organized as follows.

In Section 2, we revive some basic facts in Finsler geometry and prepare some tools for the proof of the main theorems.

In Section 3, we establish a Sturm's type comparison theorem, and deduce a comparison result for the solutions of Ricci (in)equalities of the form

$$\rho' + \rho^2 = G \ (\geq G, \leq G)\,, \quad \text{on} \quad (0, T) \tag{1.5}$$

with appropriate asymptotic behavior as $t \to 0^+$.

After these preparations, we obtain the generalized comparison result for the Hessian as follows.

Theorem 1.3. *Let (M, F) be a complete Finsler manifold of dimension m, and $r = d_F(p, x)$ is the distance function on M from a fixed point $p \in M$. Let $D_p = M \backslash cut(p)$ be the domain of the normal geodesic coordinates centered at p. Given a smooth function G on $[0, +\infty)$, let h be the solution of the Cauchy problem*

$$\begin{cases} h'' + Gh = 0\,, \\ h(0) = 0\,, \quad h'(0) = 1\,, \end{cases} \tag{1.6}$$

and $r_0 = \max\{t | h(s) \geq 0, s \in (0, t)\}$. If the radial flag curvature of M satisfies

$$K(\nabla r, \cdot) \geq \ (\textit{resp.} \leq) G(r) \quad \textit{on} \quad B_{r_0}(p)\,, \tag{1.7}$$

then

$$\mathrm{Hess}(r)(X, X) \leq \ (\textit{resp.} \ \geq) \frac{h'}{h}\big(g_{\nabla r}(X, X) - g^2_{\nabla r}(\nabla r, X)\big) \tag{1.8}$$

$$\textit{on} \quad (D_p \backslash p) \cap B_{r_0}(p)\,.$$

Remark 1.4. If $G(r) = C = \mathrm{const}$, it is easy to see

$$\frac{h'}{h} = \mathrm{ct_C(r)}\,, \tag{1.9}$$

then our conclusion turns into Proposition 1.1.

In Section 4, firstly by taking traces in Theorem 1.3, we immediately obtain corresponding estimates for $\triangle r$. In particular, If the radial flag curvature $K(\nabla r, \cdot) \leq$ (resp. $\geq$) $G(r)$ on $B_{r_0}(p)$, it follows that

$$\triangle r \geq \ (\text{resp.} \ \leq) \ (m-1)\frac{h'}{h} - S(\nabla r) \quad \text{on} \quad (D_p \backslash p) \cap B_{r_0}(p)\,. \tag{1.10}$$

Furthermore, the upper estimate of $\triangle r$ holds under the weaker assumption that the radial Ricci curvature is bounded below by $(m-1)G(r)$. Indeed we have the following Laplacian comparison theorem.

Theorem 1.5. *Let (M,F) be a complete Finsler manifold of dimension m, and $r=d_F(p,x)$ is the distance function on M from a fixed point $p\in M$. Let $D_p=M\backslash\operatorname{cut}(p)$ be the domain of the normal geodesic coordinates centered at p. Given a smooth function G on $[0,+\infty)$, let h be the solution of the problem*

$$\begin{cases} h''+Gh\geq 0\,, \\ h(0)=0\,, \quad h'(0)=1\,, \end{cases} \tag{1.11}$$

and $r_0=\max\{t|h(s)\geq 0, s\in(0,t)\}$. Suppose that the radial Ricci curvature of M satisfies $\operatorname{Ric}(\nabla r,\nabla r)\geq(m-1)G(r)$, then

$$\triangle r(x)\leq(m-1)\frac{h'}{h}-S(\nabla r)\quad on\quad D_p\cap(B_{r_0}(p)\backslash p)\,. \tag{1.12}$$

Remark 1.6. If (1.11) be the Cauchy problem (1.6) and $G(r)=C=\text{const}$, then (1.12) yields (1.3).

Next, we derive a more direct and interesting result, which is an extension of the comparison results described in [16, 18] as well.

Theorem 1.7. *Let (M,F) be a complete Finsler manifold of dimension m, and $r=d_F(p,x)$ is the distance function on M from a fixed point $p\in M$, let $D_p=M\backslash\operatorname{cut}(p)$ be the domain of the normal geodesic coordinates centered at p. If $\operatorname{Ric}_M\geq(m-1)G(r)$, where G is a nonincrease smooth function on $[0,+\infty)$and $G\leq -1$. Then*

$$\triangle r(x)\leq(m-1)\sqrt{(shr)^{-2}-G(r)}-S(\nabla r)\quad on\quad D_p\backslash p\,. \tag{1.13}$$

In Section 5, based on above comparison theorems, some applications to area and first eigenvalue estimates are given.

2. Preliminaries

In this section, we briefly revive some basic facts of Finsler manifolds.

Let (M,F) be a m-dimensional complete connected Finsler manifold with Finsler metric $F\colon TM\to[0,+\infty)$. Let $(x,v)=(x^i,v^i)$ be local coordinates on TM, and $\pi\colon TM\setminus 0\to M$ be the natural projection. We denote

$$g_{ij}:=\frac{1}{2}\frac{\partial^2F^2(x,v)}{\partial v^iv^j} \qquad \text{(fundamental tensor)}, \tag{2.1}$$

$$C_{ijk}:=\frac{1}{4}\frac{\partial^3F^2(x,v)}{\partial v^iv^jv^k} \qquad \text{(Cartan tensor)}. \tag{2.2}$$

According to [2], the pulled-back bundle π^*TM admits a unique linear connection, named Chern connection. Its connection forms are characterized by the following structural equations:

$$dx^j \wedge \omega_j^i = 0 \qquad \text{(torsion freeness)}, \tag{2.3}$$

$$dg_{ij} - g_{kj}\omega_i^k - g_{ik}\omega_j^k = 2C_{ijk}\omega^{n+k} \qquad \text{(almost } g\text{-compatibility)}. \tag{2.4}$$

Let $V = v^i \frac{\partial}{\partial x^i}$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric g_V and a linear connection ∇^V on the tangent bundle over $\mathcal{U}$ as follows.

$$g_V(X, Y) = X^i Y^j g_{ij}(x, V), \quad \forall X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^i \frac{\partial}{\partial x^i}, \tag{2.5}$$

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x, V) \frac{\partial}{\partial x^k}. \tag{2.6}$$

By the torsion freeness and g-compatibility of Chern connection, we have (see [2, 18])

$$\nabla^V_X Y - \nabla^V_Y X = [X, Y], \tag{2.7}$$

$$X g_V(Y, Z) = g_V(\nabla^V_X Y, Z) + g_V(Y, \nabla^V_X Z) + 2C_V(\nabla^V_X V, Y, Z), \tag{2.8}$$

where C_V is defined by $C_V(X, Y, Z) = X^i Y^j Z^k C_{ijk}(x, v)$.

The Chern curvature $R^V(X, Y)Z$ for vector fields X, Y, Z on $\mathcal{U}$ is defined by

$$R^V(X, Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z. \tag{2.9}$$

Let V be a geodesic vector and W a tangent vector, which span the 2-plane in $T_x M$, then the flag curvature is defined by

$$K(V; W) = \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V^2(V, W)}, \tag{2.10}$$

and

$$\mathrm{Ric}(V) = \sum_i K(V, E_i) \tag{2.11}$$

is called the Ricci curvature, where $E_1, E_2, \ldots E_m$ is the local g_V-orthonormal frame over $\mathcal{U}$.

Let $\gamma(s)$, $0 \le s \le l$ be a geodesic with unit speed velocity field T. A vector field J along γ is called a Jacobi field if it satisfies the following equation

$$\nabla^T_T \nabla^T_T J + R^T(J, T)T = 0. \tag{2.12}$$

For vector field X and Y along γ, the index form $I_\gamma(X, Y)$ is defined by

$$I_\gamma = \int_0^l \big(g_T(\nabla^T_T X, \nabla^T_T Y) - g_T(R^T(X, T)T, Y) \big)\, dt. \tag{2.13}$$

A frequently used volume form for (M, F) is the so-called Busemann-Hausdorff volume form dV_F which is locally expressed by (see [4])

$$dV_F = \sigma_F(x) dx^1 \wedge \cdots \wedge dx^m, \tag{2.14}$$

where

$$\sigma_F(x) = \frac{\mathrm{vol}(B^m(1))}{\mathrm{vol}((v^i) \in R^m; F(x, v^i \frac{\partial}{\partial x_i}) < 1)}\,. \tag{2.15}$$

For $v \in T_xM\backslash\{0\}$, define

$$\tau(v) = \log \frac{\sqrt{\det(g_{ij}(x,v))}}{\sigma_F}\,, \tag{2.16}$$

and τ is called the distortion of (M, F). To measure the rate of distortion along geodesic, we define

$$S(v) = \frac{d}{ds}\big[\tau\big(\dot\gamma(s)\big)\big]_{s=0}\,, \tag{2.17}$$

where $\gamma(s)$ is the geodesic with $\dot\gamma(0) = v$, S is called the S-curvature (see [15]).

The canonical energy function is defined by

$$E(u) = \frac{\int_M F^*(du)^2 dV_F}{\int_M u^2 dV_F}\,, \quad u \in C^1(M) \quad \text{and} \quad u \neq 0\,, \tag{2.18}$$

where $F^*\colon T^*M \to [0, +\infty)$ is the Finsler metric dual to F. Let $\mathbb{W}^{1,2}(M)$ denote the Sobolev space, and let

$$\mathfrak{V} = \begin{cases} \{u \in \mathbb{W}^{1,2}(M) : \int_M u dV_F = 0\}, & \text{if} \quad \text{M is compact with } \partial M = \emptyset; \\ \{u \in \mathbb{W}^{1,2}(M) : u|_{\partial M} = 0\}, & \text{if} \quad \text{M is compact with } \partial M \neq \emptyset \\ & \quad\quad \text{(the Dirichlet problem)}\,. \end{cases} \tag{2.19}$$

Then E can be extended to be a function on $\mathfrak{V}$. Furthermore, E is differentiable on $\mathfrak{V}$.

Definition 2.1. Critical values λ of E are called the eigenvalues of M and the corresponding critical points u are called the eigenfunctions of M.

It is easy to see that the first eigenvalue

$$\lambda_1 = \inf_{u \in \mathfrak{V}\backslash\{0\}} \frac{\int_M F^*(du)^2 dV_F}{\int_M u^2 dV_F} \tag{2.20}$$

is the smallest eigenvalue of M and $\lambda_1 \geq \frac{1}{4}\mathfrak{C}^2(M)$ (the Cheeger's inquality), where $\mathfrak{C}(M)$is defined as follows

$$\mathfrak{C}(M) = \begin{cases} \inf\{\frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)} | \Omega \subset M, \ \text{if} \ \ \partial M \neq \emptyset\} \\ \inf\{\frac{\mathrm{Vol}(M)}{\min\{\mathrm{Vol}(M_1), \mathrm{Vol}(M_2)\}} | H \text{ be a surface in } M\text{, which} \\ \text{divides } M \text{ into } \{M_i (i = 1, 2)\} \text{ and } \partial M_1 = \partial M_2 = H\}. \end{cases} \tag{2.21}$$

3. The Hessian Comparison Theorems

Let (M,F) be a Finsler manifold, the Legendre transformation $\mathfrak{l}\colon TM \to T^*M$ is defined by

$$\mathfrak{l}(Y) = \begin{cases} g_Y(Y,\cdot)\,, & Y \neq 0\,; \\ 0, & Y = 0. \end{cases} \tag{3.1}$$

Now let $f\colon M \to \mathbb{R}$ be a smooth function on M, and the gradient of f is defined by $\nabla f = \mathfrak{l}^{-1}(df)$, then we have

$$df(X) = g_{\nabla f}(\nabla f, X)\,, \quad X \in TM\,. \tag{3.2}$$

Let $\mathfrak{U} = \{x \in M, \nabla f|_x \neq 0\}$. We define the Hessian Hess(f) of f on $\mathfrak{U}$ as follows (see [18])

$$\mathrm{Hess}(f)(X,Y) = XY(f) - \nabla_X^{\nabla f} Y(f)\,, \quad \forall\ X, Y \in TM|_{\mathfrak{U}}\,. \tag{3.3}$$

By the torsion freeness and g-compatibility of Chern connection, it is clearly that Hess(f) is symmetric, which can be rewritten as

$$\mathrm{Hess}(f)(X,Y) = g_{\nabla f}(\nabla_X^{\nabla f} \nabla f, Y)\,, \quad \forall\ X, Y \in TM|_{\mathfrak{U}}\,. \tag{3.4}$$

Let $\mathrm{hess}(f)(X) = \nabla_X^{\nabla f}\nabla f$, then $\mathrm{Hess}(f)(X,Y) = g_{\nabla f}(\mathrm{hess}(f)(X), Y)$.

Lemma 3.1. *Let $G \in C[0,+\infty)$, and f, $g \in C^1[0,+\infty)$ with f', $g' \in AC(0,+\infty)$ be solutions of the problems*

$$\left\{ \begin{array}{l} f'' + Gf \leq 0\,,\ \text{a.e. on } (0,+\infty), \\ f(0) = 0\,, \quad f'(0) \leq 1 \end{array} \right. \qquad \left\{ \begin{array}{l} g'' + Gg \geq 0\,,\ \text{a.e. on } (0,+\infty), \\ g(0) = 0\,, \quad g'(0) \geq 1\,. \end{array} \right. \tag{3.5}$$

If $f(t) > 0$ for $t \in (0,T)$ and $g'(0) \geq f'(0)$, then $\frac{f'}{f} \leq \frac{g'}{g}$ and $f \leq g$ on $(0,T)$.

Proof. Let $\beta = \sup\{s : g(s) > 0 \text{ on } (0,s)\}$ and $\tau = \min\{\beta, T\}$, then f and g are both positive on $(0,\tau)$. Since the function $g'f - f'g$ is continuous on $[0,+\infty)$, vanishes in $t = 0$, and

$$(g'f - f'g)' = g''f - f''g \geq -Ggf - (-Gfg) = 0\,, \quad \text{on} \quad (0,+\infty)\,, \tag{3.6}$$

we have

$$\frac{g'}{g} \geq \frac{f'}{f}\,, \quad \text{on} \quad (0,\tau)\,. \tag{3.7}$$

Integrating from ε to t $(0 < \varepsilon < t < \tau)$, we have

$$f(t) \leq \frac{f(\varepsilon)}{g(\varepsilon)} g(t)\,, \tag{3.8}$$

and since

$$\lim_{\varepsilon \to 0^+} \frac{f(\varepsilon)}{g(\varepsilon)} = \frac{f'(0)}{g'(0)} \leq 1\,, \tag{3.9}$$

we have

$$f(t) \le g(t) \quad \text{on} \quad [0,\tau)\,. \tag{3.10}$$

Since $f > 0$ on $(0,T)$ by assumption, this in turn forces $\tau = T$. Otherwise, if $\tau = \beta < T$, then $f(\beta) > 0$. While by continuity, $g(\beta) = 0$. This leads to a contradiction. $\square$

Lemma 3.2. *Let $G \in C[0,+\infty)$ and $\rho_i \in AC(0,T_i)$ be solutions of the differential inequalities*

$$\rho_1' + \rho_1^2 + G \le 0 \quad \textit{a.e. on} \quad (0,T_1)\,; \quad \rho_2' + \rho_2^2 + G \ge 0 \quad \textit{a.e. on} \quad (0,T_2)\,, \tag{3.11}$$

satisfying the asymptotic condition

$$\rho_i(t) = \frac{1}{t} + o(1)\,, \quad \textit{as} \quad t \to 0^+\,. \tag{3.12}$$

Then $T_1 \le T_2$ and $\rho_1 \le \rho_2$ on $(0,T_1)$.

Proof. Observe that the function $\rho_i(s) - \frac{1}{s}$ is bounded and integrable in a neighboorhood of $s = 0$, we let

$$\Phi_i(t) = t \cdot \exp\{\int_0^t (\rho_i(s) - \frac{1}{2})\,ds\}\,, \quad \text{a.e. on} \quad [0,T_i)\,, \tag{3.13}$$

then $\Phi_i(0) = 0$, $\Phi_i > 0$ on $(0,T_i)$, $\Phi_i' = \rho_i\Phi_i \in AC(0,T_i)$ and $\Phi_i'(0) = 1$. By straightforward computations, we have

$$\Phi_1'' + G\Phi_1 \le 0 \quad \text{on} \quad (0,T_1)\,; \qquad \Phi_2'' + G\Phi_2 \ge 0 \quad \text{on} \quad (0,T_2)\,. \tag{3.14}$$

An application of Lemma 3.1 shows that $T_1 \le T_2$ and $\rho_1 = \frac{\Phi_1'}{\Phi_1} \le \frac{\Phi_2'}{\Phi_2} \le \rho_2$ on $(0,T_1)$, as required. $\square$

After these preparations, we are going to prove Theorem 1.3.

Proof. Since $\mathrm{Hess}(r)$ is symmetric, there is an orthonormal basic of T_xM consisting of eigenvectors of $\mathrm{Hess}(r)$. Denoting by $\xi_{\max}(x)$ and $\xi_{\min}(x)$, respectively, the greatest and smallest eigenvalues of the $\mathrm{Hess}(r)$ in the orthogonal complement of $\nabla r(x)$, the theorem amounts to showing that on $(D_p \backslash p) \cap B_{r_0}(P)$,

if $K(\nabla r, \cdot) \ge G(r)$, then $\xi_{\max}(x) \le \frac{h'}{h}(r(x))$;

if $K(\nabla r, \cdot) \le G(r)$, then $\xi_{\min}(x) \ge \frac{h'}{h}(r(x))$.

Let $x \in D_p \backslash p$ and let γ be the minimizing geodesic joining p to x, we claim that if $K(\nabla r, \cdot) \ge G(r)$, then the Lipschitz function $\xi_{\max}$ satisfies

$$\begin{cases} \frac{d}{ds}(\xi_{\max} \circ \gamma) + (\xi_{\max} \circ \gamma)^2 + G \le 0 \quad \text{for a.e.} \quad s > 0\,, \\ \xi_{\max} \circ \gamma = \frac{1}{s} + o(1)\,, \quad \text{as } s \to 0^+\,; \end{cases} \tag{3.15}$$

similarly, if $K(\nabla r, \cdot) \le G(r)$, then the Lipschitz function $\xi_{\min}$ satisfies

$$\begin{cases} \frac{d}{ds}(\xi_{\min} \circ \gamma) + (\xi_{\min} \circ \gamma)^2 + G \ge 0 \quad \text{for a.e.} \quad s > 0\,, \\ \xi_{\min} \circ \gamma = \frac{1}{s} + o(1)\,, \quad \text{as} \quad s \to 0^+\,. \end{cases} \tag{3.16}$$

since $\phi = \frac{h'}{h}$ satisfies

$$\phi' + \phi^2 + G = 0 \quad \text{on} \quad (0, r_0)\,, \qquad \phi(s) = \frac{1}{s} + o(s)\,, \quad \text{as} \quad s \to 0^+\,, \tag{3.17}$$

the required conclusion follows immediately from Lemma 3.2. It remains to prove that $\xi_{\max}$ and $\xi_{\min}$ satisfy the required differential inequalities. Now let $\gamma(s)$ be the geodesic parametrized by arc-length issuing from p with $\gamma(s_0) = x$, then γ is an integral curve of ∇r. For every unit vector $Y \in T_xM$ such that $Y \perp \dot\gamma(s_0)$, define a vector field $Y \perp \dot\gamma$, by parallel translation along γ. By the definition of covariant derivative and curvature tensor, we have

$$\begin{aligned}
\nabla^{\nabla r}_{\dot\gamma}\big(\operatorname{hess}(r)(Y)\big) &= \nabla^{\nabla r}_{\dot\gamma}\big(\operatorname{hess}(r)\big)(Y) + \operatorname{hess}(r)(\nabla^{\nabla r}_{\dot\gamma} Y)\\
&= \nabla^{\nabla r}_{\nabla r}\big(\operatorname{hess}(r)\big)(Y)\\
&= \nabla^{\nabla r}_{Y}\big(\operatorname{hess}(r)\big)(\nabla r) + R^{\nabla r}(\nabla r, Y)\nabla r\\
&= \nabla^{\nabla r}_{Y}\big(\big(\operatorname{hess}(r)\big)\nabla r\big) - \operatorname{hess}(r)(\nabla^{\nabla R}_{Y}\nabla r) - R^{\nabla r}(Y, \nabla r)\nabla r\\
&= -\operatorname{hess}(r)\big(\operatorname{hess}(r)(Y)\big) - R^{\nabla r}(Y, \nabla r)\nabla r\,,
\end{aligned} \tag{3.18}$$

that is

$$\nabla^{\nabla r}_{\dot\gamma}\big(\operatorname{hess}(r)(Y)\big) + \operatorname{hess}(r)\big(\operatorname{hess}(r)(Y)\big) = -R^{\nabla r}(Y, \nabla r)\nabla r\,. \tag{3.19}$$

Since Y is parallel,

$$\frac{d}{ds} g_{\nabla r}\big(\operatorname{hess}(r)(Y), Y\big) = g_{\nabla r}\big(\nabla^{\nabla r}_{\dot\gamma}\big((r)(Y)\big), Y\big)\,, \tag{3.20}$$

and we conclude that

$$\frac{d}{ds}\big(\operatorname{Hess}(r)(\gamma)(Y, Y)\big) + g_{\nabla r}\big(\operatorname{hess}(r)(\gamma)(Y), \operatorname{hess}(r)(\gamma)(Y)\big) = -K(\dot\gamma, Y)\,. \tag{3.21}$$

Note that, for any unit vector field $E \perp \nabla r$,

$$\operatorname{Hess}(r)(E, E) \leq \xi_{\max}\,. \tag{3.22}$$

Thus,

$$\operatorname{Hess}(r)(\gamma)(Y, Y)|_{s=s_0} = \xi_{\max} \circ \gamma(s_0). \tag{3.23}$$

then the function $\operatorname{Hess}(r)(\gamma)(Y, Y) - \xi_{\max} \circ \gamma$ attains its maximum at s_0, and its derivative vanishes:

$$\frac{d}{ds}|_{s=s_0} \operatorname{Hess}(r)(\gamma)(Y, Y) - \frac{d}{ds}|_{s=s_0} \xi_{\max} \circ \gamma = 0\,. \tag{3.24}$$

Assume that $K(\nabla r, \cdot) \geq G(r)$, by (3.21) and (3.24), we have, at s_0,

$$\frac{d}{ds}(\xi_{\max} \circ \gamma) + (\xi_{\max} \circ \gamma)^2 + G \leq 0\,, \tag{3.25}$$

which is the desired inequality stated in (3.15). The asymptotic behavior of $\xi_{\max} \circ \gamma$ near $s = 0^+$ follows from the fact that

$$\operatorname{Hess}(r) = \frac{1}{r}\big(g_{\nabla r}(\cdot, \cdot) - g^2_{\nabla r}(\nabla r, \cdot)\big) + o(1)\,, \quad r \to 0^+\,, \tag{3.26}$$

as one can verify by a simple computation in normal coordinates at $p \in M$. The argument in the case where $K(\nabla r, \cdot) \leq G(r)$ is completely similar. $\square$

4. The Laplacian comparison theorems

Let (M, F) be a Finsler manifold, the dual Finsler metric F^* on M is defined by

$$F^*(\varsigma_x) = \sup_{Y \in T_xM \setminus \{0\}} \frac{\varsigma(Y)}{F(Y)}, \quad \forall\, \varsigma \in T^*M, \tag{4.1}$$

and

$$g^{*kl}(\varsigma) = \frac{1}{2} \frac{\partial^2 F^{*2}(\varsigma)}{\partial \varsigma_k \partial \varsigma_l} \tag{4.2}$$

is the corresponding fundamental tensor. Then we have (see [2], [15])

$$F(Y) = F^*\big(\mathfrak{l}(Y)\big), \quad \forall Y \in TM; \quad g^{ij}(Y) = g^{*ij}\big(\mathfrak{l}(Y)\big), \quad \forall Y \in TM. \tag{4.3}$$

The divergence $\operatorname{div} X$ of X is defined as follows.

$$d(X \rfloor dV_F) = \operatorname{div}(X) dV_F. \tag{4.4}$$

It is easy to see that $\operatorname{div} X$ depends only on the volume form dV_F. Then for a vector field $X = X^i \frac{\partial}{\partial X^i}$ on M, we have

$$\operatorname{div} X = \frac{1}{\sigma} \frac{\partial}{\partial x^i}(\sigma X^i) = \frac{\partial X^i}{\partial x^i} + \frac{X^i}{\sigma} \cdot \frac{\partial \sigma}{\partial x^i}. \tag{4.5}$$

The laplacian of f, denoted by $\triangle f$, is defined as

$$\triangle f = \operatorname{div}(\nabla f) = \operatorname{div}\big(\mathfrak{l}^{-1}(df_1)\big). \tag{4.6}$$

By (4.3) and (4.4), we have the following local expression for $\triangle f$,

$$\triangle f = \frac{1}{\sigma(x)} \frac{\partial}{\partial x^i}\Big(\sigma(x) g^{*ij}(df) \frac{\partial f}{\partial x^j}\Big) = \frac{1}{\sigma(x)} \frac{\partial}{\partial X^i}\Big(\sigma(x) g^{ij}(\nabla f) \frac{\partial f}{\partial x^j}\Big), \tag{4.7}$$

By a direct computation, we have (see [16, 18])

$$\triangle f = \sum_{i=1}^{n} \operatorname{Hess}(f)(e_i, e_i) - S(\nabla f). \tag{4.8}$$

As mentioned above, by taking trace in Theorem 1.3, we immediately obtain corresponding estimates for $\triangle r$.

Theorem 4.1. *Let (M, F) be a complete Finsler manifold of dimension m, and $r = d_F(x, p)$ is the distance function on M from a fixed point $P \in M$. Let $D_p = M \setminus \operatorname{cut}(p)$ be the domain of the normal geodesic coordinates centered at p. Given a smooth function G on $[0, +\infty)$, let h be the solution of the Cauchy problem*

$$\begin{cases} h'' + Gh = 0, \\ h(0) = 0, \quad h'(0) = 1, \end{cases} \tag{4.9}$$

and $r_0 = \max\{t | h(s) \geq 0,\ s \in (0, t)\}$. If the radial flag curvature of M satisfies

$$K(\nabla r, \cdot) \geq \ (\textit{resp.} \ \leq)\ G(r) \quad \textit{on} \quad B_{r_0}(p), \tag{4.10}$$

then

$$\triangle r \leq \ (resp.\ \geq)\ (m-1)\frac{h'}{h} - S(\nabla r) \quad on \quad (D_p\backslash p)\cap B_{r_0}(p)\,. \tag{4.11}$$

Next we are going to prove Theorem 1.5.

Proof. Let $D_p = M\backslash \operatorname{cut}(p)$ be the maximal star-shaped domain of the normal coordinates at p. Fix any $x \in D_p \cap (B_{r_0}(p)\backslash\{p\})$ and let $\gamma(s)$ be the minimizing geodesic from p to x parametrized by arc-length. Set $\psi(s) = (\bar{\triangle} r)\circ\gamma(s)$, where $\bar{\triangle} r = \operatorname{tr}_{\nabla r}(\operatorname{Hess}(r)) = \sum\limits_{i=1}^{n}\operatorname{Hess}(r)(e_i, e_i)$, we claim that

$$\begin{cases} \psi' + \frac{1}{m-1}\psi^2 + (m-1)\,G \leq 0\,, & \text{(i)} \\ \psi(s) = \frac{m-1}{s} + 0(1)\,, \quad \text{as} \quad s\to 0^+\,. & \text{(ii)} \end{cases} \tag{4.12}$$

Indeed, note that by tracing in (3.21), we deduce that

$$\frac{d}{ds}(\bar{\triangle}\circ\gamma) + |\operatorname{Hess}(r)|^2(\gamma) = -\operatorname{Ric}(\nabla r, \nabla r)(\gamma)\,. \tag{4.13}$$

By the elementary inequality

$$\frac{(\bar{\triangle} r)^2}{m-1} \leq |\operatorname{Hess}(r)|^2\,, \tag{4.14}$$

which in turn follows easily from the Cauchy-Schwarz inequality, we deduce that

$$\frac{d}{ds}(\bar{\triangle}\circ\gamma) + \frac{(\bar{\triangle}\circ\gamma)^2}{m-1} \leq -\operatorname{Ric}(\nabla r, \nabla r)(\gamma). \tag{4.15}$$

Inequality (4.12)(i) follows from the assumption on Ric.
As for the asymptotic behavior (4.12)(ii) follows from the well-known fact that

$$\operatorname{tr}_{\nabla r}\big(\operatorname{Hess}(r)\big) = \frac{m-1}{r} + o(1)\,, \quad \text{as} \quad r\to 0^+\,. \tag{4.16}$$

Now, by using (4.12) and arguing as in the proof of Theorem 1.3, it is easy to see that (1.12) holds pointwise on $D_p\cap(B_{r_0}(p)\backslash p)$. $\square$

Next we are ready to attest Theorem 1.7, firstly we will need the following lemma.

Lemma 4.2. *Let G be a continuous function on $[0,+\infty)$ and $G\leq -1$. If w be solution of the Cauchy problem*

$$\begin{cases} w'' + Gw = 0\,, \\ w(0) = 0\,, \quad w'(0) = 1\,, \end{cases} \tag{4.17}$$

then $w(t) \geq \operatorname{sh} t$.

Proof. Certainly, there is a unique solution $w(t)$ of (4.17), and $w(t)\geq 0$. Let $w_1(t)$ be the solution of

$$\begin{cases} w_1'' - w_1 = 0\,, \\ w_1(0) = 0\,, \quad w_1'(0) = 1\,. \end{cases} \tag{4.18}$$

Since

$$0 = \int_0^t \{w(w_1'' - w_1) - w_1(w'' + Gw)\}du$$
$$= \int_0^t (ww_1'' - w_1w'')du + \int_0^t (-1 - G)ww_1 du$$
$$\geq (ww_1' - w_1w')|_0^t$$
$$= w(t)w_1'(t) - w_1(t)w'(t), \tag{4.19}$$

we have that $\left(\frac{w(t)}{w_1(t)}\right)' \geq 0$. Then for any $\varepsilon \in (0, t)$, we have

$$\frac{w(t)}{w_1(t)} \geq \frac{w(\varepsilon)}{w_1(\varepsilon)} \,. \tag{4.20}$$

Therefore,

$$\frac{w(t)}{w_1(t)} \geq \lim_{\varepsilon \to 0} \frac{w(\varepsilon)}{w_1(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{w'(\varepsilon)}{w_1'(\epsilon)} = 1 \,, \tag{4.21}$$

that is $w(t) \geq w_1(t) = \operatorname{sh} t$. □

After this preparation,we can prove Theorem 1.7 as follows.

Proof. Let γ: $[0, r(x)] \to M$ be the unit-speed geodesic from p to x, and let $e_1, \ldots, e_{m-1}, e_m = \dot{\gamma}$ be the g_T-orthonormal basic of T_xM.

By parallel translation along γ, we obtain the parallel vector fields $E_1(t), \ldots, E_m(t)$ along γ. For $1 \leq i \leq m-1$, let J_i be the unique Jocobi field along γ such that $J_i(0) = 0$, $J_i(r(x)) = e_i$. Next, let $\varphi(t)$ be an arbitrary piecewise smooth function defined on $[0, r]$ with $\varphi(0) = 0$ and $\varphi(r) = 1$, then $\varphi(t)E_i(t)$ would be piecewise smooth vector fields along γ satisfying $\varphi(0)E_i(0) = 0$ and $\varphi(r)E_i(r) = J_i(r)$. By the basic index lemma (see [2, 18]), we have

$$\operatorname{tr}_{\nabla r}\big(\operatorname{Hess}(r)\big) = \sum_{i=1}^{n} \operatorname{Hess}(r)(e_i, e_i) = \sum_{i=1}^{n-1} I_\gamma(J_i, J_i)$$
$$= \int_0^r \big((m-1)(\varphi')^2 - \operatorname{Ric}\dot{\varphi}^2\big)\, dt$$
$$\leq \int_0^r \big((m-1)(\varphi')^2 - (m-1)G(r)\cdot\varphi^2\big)\, dt$$
$$= (m-1)\int_0^r \big((\varphi')^2 - G\varphi^2\big)\, dt \,. \tag{4.22}$$

The Euler-Lagrange equation of the right-hand side of inequality (4.22) is

$$\varphi'' + G\varphi = 0 \,. \tag{4.23}$$

By Lemma 4.2 and note that $\varphi(t) = \frac{w(t)}{w(r)}$ is the solution of the boundary value problem

$$\begin{cases} \varphi'' + G\varphi = 0 \,, \\ \varphi(0) = 0 \,, \quad \varphi(r) = 1 \,, \end{cases} \tag{4.24}$$

we have

$$\operatorname{tr}_{\nabla r}\big(\operatorname{Hess}(r)\big) \le (m-1)\int_0^r \big((\varphi')^2 + \varphi\varphi''\big)\,dt$$

$$= (m-1)\varphi(r)\varphi'(r) = (m-1)\varphi'(r)\,. \tag{4.25}$$

Since

$$0 < \varphi'(0) = \frac{w'(0)}{w(r)} \le \frac{1}{\operatorname{sh} r}\,, \tag{4.26}$$

then we have

$$\begin{aligned}
\big(\varphi'(r)\big)^2 &= \big(\varphi'(0)\big)^2 + \int_0^r \Big(\frac{d}{dt}\big(\varphi'(t)\big)^2\Big)\,dt \\
&= \big(\varphi'(0)\big)^2 + \int_0^r 2\varphi'\cdot(-G\varphi)\,dt \\
&= \big(\varphi'(0)\big)^2 + \int_0^r G(t)[\varphi^2(t)]'\,dt \\
&\le \Big(\frac{1}{\operatorname{sh} r}\Big)^2 - G(r)\int_0^r [\varphi^2(t)]'\,dt \\
&\le \Big(\frac{1}{\operatorname{sh} r}\Big)^2 - G(r)\,. \qquad (4.27)
\end{aligned}$$

Combining (4.8), (4.25) and (4.27) we obtain the desired result. □

5. Some Applications

In this section, we give some applications of the above estimates. First, we obtain a simple application of Theorem 4.1.

Theorem 5.1. *Let (M,F) be a complete Finsler manifold of dimension m, and $r = d_F(x,p)$ is the distance function on M from a fixed point $p \in M$. Let $D_p = M\backslash \operatorname{cut}(p)$ be the domain of the normal geodesic coordinates centered at p. Given a smooth function G on $[0,+\infty)$. Let h be the solution of the Cauchy problem*

$$\begin{cases} h'' + Gh = 0\,, \\ h(0) = 0\,, \quad h'(0) = 1\,, \end{cases} \tag{5.1}$$

and let $r_0 = \max\{t|h(s) \ge 0,\ s \in (0,t)\}$, and $D_p \subset B_p(r_0)$. If the radial flag curvature of M satisfies

$$K(\nabla r,\cdot) \le G(r)\,, \tag{5.2}$$

then

$$\operatorname{Vol}(\partial D_p) \ge \int_{D_p} ((m-1)\frac{h'}{h} - S(\nabla r))dV_F\,. \tag{5.3}$$

Proof. By $\triangle r \geq \ (m-1)\frac{h'}{h} - S(\nabla r)$, we have

$$\begin{aligned} \operatorname{vol}(\partial D_p) &\geq \int_{\partial D_p} g_Z(Z, \triangledown r) dA = \int_{D_P} \triangle r dV_F \\ &\geq \int_{D_P} \Big((m-1)\frac{h'}{h} - S(\triangledown r) \Big) dV_F \,, \end{aligned} \tag{5.4}$$

where Z is the outer normal along ∂D_p. □

Remark 5.2. If $G(r) = C = \mathrm{const}(C < 0)$, and $S(\nabla r) \leq (m-1)\delta(\delta < \sqrt{-C})$, then

$$\operatorname{vol}(\partial D_p) \geq \int_{D_p} (m-1)\big(\sqrt{-C} - \delta\big) dV_F = (m-1)(\sqrt{-C} - \delta) \operatorname{vol}(D_P) \,. \tag{5.5}$$

Furthermore,

$$\lambda_1 \geq \frac{1}{4}\mathfrak{C}^2(M) = \frac{1}{4}(m-1)^2\big(\sqrt{-C} - \delta\big)^2 \,. \tag{5.6}$$

Next, we study the first eigenvalue under the condition with the lower bound of flag curvature (or Ricci curvature), and we will apply the key idea in [8] to archive this goal. In this section the discussion is based on the estimate on Δr described in Theorem 1.5 and others are similar.

Let (M, F) be a Finsler m-dimension manifold with

$$\mathrm{Ric} \geq (m-1)G(r)\big(G(r) \leq -1\big) \,, \quad \|S\| \geq (m-1)\delta \,. \tag{5.7}$$

Let $\Lambda = \Lambda(m, \delta, R) > 0$ be a number such that there is a function $u \in C^2[0, R]$ with $u' \leq 0$, which satisfies

$$\begin{cases} u''(r) + (m-1)\big(\sqrt{(\operatorname{sh} r)^{-2} - G(r)} - \delta\big)u'(r) + \Lambda u(r) \geq 0 \,, \\ u(R) = 0 \,, \quad u'(0) = 0 \,. \end{cases} \tag{5.8}$$

Then we have

Theorem 5.3. *Let $B_R(p)$ ($R \leq i_p$, where i_p denotes the injectivity radius about p) be an open ball in a complete Finsler m-manifold satisfying* (5.7), *then*

$$\lambda_1\big(B_R(p)\big) \leq \Lambda(m, \delta, R) \,. \tag{5.9}$$

Proof. By (1.13), we have

$$\begin{aligned} \triangle u &= u''(r) + u'(r)\triangle r \\ &\geq u''(r) + (m-1)(\sqrt{(\operatorname{sh} r)^{-2} - G(r)} - \delta)u'(r) \geq -\Lambda u \,, \end{aligned} \tag{5.10}$$

then

$$\begin{aligned} \int_{B_R(p)} F * (du)^2 dV_F &= \int_{B_R(p)} du(\nabla u) dV_F \\ &= -\int_{B_R(p)} u\Delta u dV_F \leq \Lambda \int_{B_R(p)} u^2 dV_F \,. \end{aligned} \tag{5.11}$$

Now the conclusion is obvious. □

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