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SOME SURJECTIVITY THEOREMS WITH APPLICATIONS

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Abstract. In this paper a new class of mappings, known as locally λ-strongly ϕ-accretive mappings, where λ and ϕ have special meanings, is introduced. This class of mappings constitutes a generalization of the well-known monotone mappings, accretive mappings and strongly ϕ-accretive mappings. Subsequently, the above notion is used to extend the results of Park and Park, Browder and Ray to locally λ-strongly ϕ-accretive mappings by using Caristi-Kirk fixed point theorem. In the sequel, we introduce the notion of generalized directional contractor and prove a surjectivity theorem which is used to solve certain functional equations in Banach spaces.

1. Introduction and preliminaries

In the beginning of the last quarter of the 20th century many problems related to nonlinear operators were studied in the framework of Banach spaces by several researchers. In this context, the remarkable work of Altman [1]–[3], Ray and Walker [14] and others is worth mentioning. Altman [1]–[3] obtained some surjectivity theorems for nonlinear mapings which had a directional contractor. A transfinite induction argument was applied in his work to prove surjectivity theorems for nonlinear mappings by using the well-known Caristi-Kirk [8] fixed point theorem as a tool. On the other hand, Browder [7] initiated the study of ϕ-accretive mappings in Banach spaces under appropriate geometric conditions. This class of mappings has been further studied by Browder [4]–[6], Kirk [11], Ray [13] and many others.

In this paper a new class of mappings, known as locally λ-strongly ϕ-accretive mappings, where λ and ϕ have special meanings, is introduced.

This class of mappings constitutes a generalization of the well-known monotone mappings, accretive mappings and strongly ϕ-accretive mappings. Subsequently, in Section 2, the above notion is used to extend the results of Park and Park [12], Browder [6] and Ray [13] to locally λ-strongly ϕ-accretive mappings by using Caristi-Kirk fixed point theorem (cf. [8, 10, 11]). In the sequel, we introduce the notion of generalized directional contractor in Section 3 and prove a surjectivity theorem which is used to solve certain functional equations in Banach spaces.

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Definition 1.1. Let X and Y be Banach spaces with Y^* the dual of Y, and let $\phi\colon X \to Y^*$ be a mapping satisfying:

(i)**:** $\phi(X)$ is dense in Y^*,

(ii)**:** for each $x \in X$ and each $\alpha \geq 0$,

$$\|\phi(x)\| \leq \|x\|, \quad \|\phi(\alpha x)\| = \alpha\|\phi(x)\| .$$

Then:

A mapping $P\colon X \to Y$ is said to be

(a): ϕ-accretive if for all $u, v \in X$,

$$\langle Pu - Pv, \phi(u - v)\rangle \geq 0 . \tag{1.1}$$

(b): strongly ϕ-accretive if there exists a constant $c > 0$ such that, for all $u, v \in X$,

$$\langle Pu - Pv, \phi(u - v)\rangle \geq c\|u - v\|^2 . \tag{1.2}$$

(c): locally strongly ϕ-accretive if for each $y \in Y$ and $r > 0$, there exists a constant $c > 0$ such that: if $\|Px - y\| \leq r$, then, for all $u \in X$ sufficiently near to x, we have

$$\langle Pu - Px, \phi(u - x)\rangle \geq c\|u - x\|^2 . \tag{1.3}$$

Note that (c) presents a localized version of (b). Historically, ϕ-accretive mappings were introduced in an effort to unify the theories for monotone mappings (when $Y = X^*$) and for accretive mappings (when $Y = X$). These mappings have been studied by Browder [4]–[7], Kirk [11] and Ray [13] among others.

The following result of Browder [7, Theorem 4] is of fundamental importance.

Theorem 1.2. *Let X and Y be Banach spaces and $P\colon X \to Y$ a strongly ϕ-accretive mapping. If Y^* is uniformly convex and P is locally Lipschitzian, then $P(X) = Y$.*

For a Banach space X, the duality mapping J from X into 2^{X^*} is given by

$$J(x) = \{j \in X^* : \langle x,\ j\rangle = \|x\|^2 = \|j\|^2\} \qquad (x \in X)$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. It is well known that J is single-valued in case X is strictly convex, and it is uniformly continuous on bounded subsets of X whenever X^* is uniformly convex.

We now introduce the following definition.

Definition 1.3. A Lipschitzian mapping $P\colon X \to Y$ with Lipschitzian constant M is said to be locally λ-strongly ϕ-accretive if for each $y \in Y$ and $r > 0$, there exist constants λ, c with $c/2M > \lambda \geq 0$ such that: if $\|Px - y\| \leq r$ and $j \in J$, the duality mapping on Y, then, for all $u \in X$ sufficiently near to x,

$$\langle Pu - Px, \phi(u - x) + \lambda M^{-1} j(Pu - Px)\rangle \geq c\|u - x\|^2 . \tag{1.4}$$

It may be remarked that a 0-strongly ϕ-accretive mapping $P\colon X \to Y$ is strongly ϕ-accretive as defined in [7].

Ray [13] extended Browder's theorem [7] by applying a theorem of Ekeland [9] and showed that a localized class of strongly ϕ-accretive mappings must be surjective under appropriate geometric assumptions on Y and continuity assumptions on P. Indeed, he proved the following.

Theorem 1.4. *Let X and Y be Banach spaces and $P\colon X \to Y$ a locally Lipschitzian and locally strongly ϕ-accretive mapping. If Y^* is strictly convex and J is continuous, and if $P(X)$ is closed in Y, then $P(X) = Y$.*

Park and Park [12] proved the following surjectivity theorem.

Theorem 1.5 ([12, Theorem 2])**.** *Let X and Y be Banach spaces and $P\colon X \to Y$ a locally Lipschitzian and locally strongly ϕ-accretive mapping. If the duality mapping J of Y is strongly upper semicontinuous and $P(X)$ is closed, then $P(X) = Y$.*

Note that if P is strongly ϕ-accretive, then $P(X)$ is closed in Y. Therefore, as a consequence of Theorem 1.5, we have the following:

Corollary 1.6 ([12, Theorem 1])**.** *Let X and Y be Banach spaces and $P\colon X \to Y$ a locally Lipschitzian and strongly ϕ-accretive mapping. If the duality mapping J of Y is strongly upper semicontinuous and $P(X)$ is closed, then $P(X) = Y$.*

Throughout, $B(x, r) = \{w \in E : \|w - x\| \le r\}$ will denote a closed ball in a Banach space E, where $E = X$ or $E = Y$ in this case.

2. A surjectivity result for λ-strongly ϕ-accretive mappings

The following is our main result for the above class of mappings.

Theorem 2.1. *Let X and Y be Banach spaces and $P\colon X \to Y$ a locally Lipschitzian and locally λ-strongly ϕ-accretive mapping. If the duality mapping J of Y is strongly upper semicontinuous and $P(X)$ is closed, then $P(X) = Y$.*

The proof of our main result is prefaced by the following lemma of Park and Park [12].

Lemma 2.2. *For any $y \in Y, y^* \in J(y)$, and $\epsilon > 0$, there exists an $h \in X$ such that $\|h\| \ge 1$ and $\|\phi(h) - y^*\|y\|^{-1}\| < \epsilon$.*

Notice that for any $y^* \in J(y) \in 2^{Y^*}$ we have

$$\|y\|^2 \le \|z\|^2 - 2\langle z - y, y^*\rangle \quad \text{for any} \quad z \in Y\,. \tag{2.1}$$

Proof of Theorem 2.1. As $P(X)$ is closed, to prove the theorem it is just sufficient to show that $P(X)$ is open. It is well known that $J(y) \neq \emptyset$ for each $y \in Y$, so we can choose $y^* \in J(y)$. For a given $x_0 \in X$, choose $\epsilon_1 > 0$ so small that P is Lipschitzian with constant M on $B(x_0, 2\epsilon_1)$. Choose $\lambda > 0$ and $\epsilon_2 > 0$ so that (1.4) holds on $B(Px_0, 2M\epsilon_1)$ whenever $\|u - x_0\| \le 2\epsilon_2$; set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and set $r = \min\{c\epsilon/(1 + \sqrt{1 + 4c\lambda M^{-1}}), M\epsilon\}$. Now it suffices to show that $B(Px_0, r) \subset P(X)$. To this end, suppose $y \in B(Px_0, r)$ and $y \notin P(X)$. It follows that $\mathrm{dist}(y, P(X)) > 0$. Let $d = \mathrm{dist}(y, P(X))$ and $D = \{x \in B(x_0, \epsilon) : \|y - Px\| \le r\}$. Clearly, $x_0 \in D$ so that D is nonempty.

Moreover, D is closed. Therefore, D is complete. For any $x \in D$, by Lemma 2.2, there exists $h \in X$ such that $\|h\| \geq 1$ and

$$\langle \phi(h), (y - Px)^* \|y - Px\|^{-1} \rangle \leq (c/2M - \lambda)\,. \tag{2.2}$$

Set $x_t = x + th, t > o$. By (1.4), for t sufficiently small we have

$$\langle Px_t - Px, \phi(x_t - x) + \lambda M^{-1} j(Px_t - Px) \rangle \geq c\|x_t - x\|^2\,.$$

Thus

$$\begin{aligned}\langle Px_t - Px, \phi(x_t - x) \rangle &\geq c\|x_t - x\|^2 - \lambda M^{-1} \langle Px_t - Px, j(Px_t - Px) \rangle \\ &\geq c\|x_t - x\|^2 - \lambda M^{-1} \|Px_t - Px\| \|j(Px_t - Px)\|\end{aligned}$$

or

$$\begin{aligned}\langle Px_t - Px, \phi(h) \rangle &\geq c\,t\|h\|^2 - \lambda M^{-1} t^{-1} \|Px_t - Px\| \|j(Px_t - Px)\| \\ &\geq c\,t\|h\|^2 - \lambda \|h\| \|Px_t - Px\| \\ &\geq c\,t\|h\| - \lambda \|Px_t - Px\| \\ &\geq (c/M - \lambda) \|Px_t - Px\|\,.\end{aligned}$$

As P is locally Lipschitzian we have for $x,\ x_t \in B(x_0, 2\epsilon_1)$

$$\|Px_t - Px\| \leq M\|x_t - x\|\,.$$

By applying (2.2),

$$\begin{aligned}\langle Px_t - Px, (y - Px)^* \rangle &= \langle Px_t - Px, \|y - Px\|\, \phi(h) - \|y - Px\|\, \phi(h), (y - Px)^* \rangle \\ &\geq (c/M - \lambda) \|Px_t - Px\|\, \|y - Px\| \\ &\quad - (c/2M - \lambda) \|Px_t - Px\|\, \|y - Px\| \\ &\geq (c/2M) \|Px_t - Px\|\, \|y - Px\|\ . \qquad (2.3)\end{aligned}$$

From (2.1) and (2.3) we have

$$\begin{aligned}\|y - Px_t\|^2 &\leq \|y - Px\|^2 - 2\langle Px_t - Px, (y - Px_t)^* \rangle \\ &= \|y - Px\|^2 - 2\langle Px_t - Px, (y - Px)^* - (y - Px)^* + (y - Px_t)^* \rangle \\ &= \|y - Px\|^2 - 2\langle Px_t - Px, (y - Px)^* \rangle \\ &\quad + 2\langle Px_t - Px, (y - Px)^* - (y - Px_t)^* \rangle \\ &\leq \|y - Px\|^2 - 2\langle Px_t - Px, (y - Px)^* \rangle \\ &\quad + 2\|Px_t - Px\| \|(y - Px)^* - (y - Px_t)^*\| \\ &\leq \|y - Px\|^2 - (cd/M) \|Px_t - Px\| \\ &\quad + 2\|Px_t - Px\| \|(y - Px)^* - (y - Px_t)^*\|\,.\end{aligned}$$

Since $y - Px_t \to y - Px$ as $t \to 0$ and J is strongly upper semicontinuous we may select $t > 0$ so small that $\|(y - Px)^* - (y - Px_t)^*\| \leq (cd/2M)$. Then it follows that

$$\|y - Px_t\|^2 \leq \|y - Px\|^2 - (cd/2M)\|Px_t - Px\|\,.$$

Recall that for sufficiently small t, we have

$$\langle Px_t - Px, \phi(x_t - x)\rangle \geq c\|x_t - x\|^2 - \lambda M^{-1}\|Px_t - Px\|\|j(Px_t - Px)\|\,.$$

This yields

$$\|x_t - x\| \leq \frac{1 + \sqrt{1 + 4c\lambda M^{-1}}}{2c}\|Px_t - Px\|\,.$$

So

$$[c^2 d/M(1 + \sqrt{1 + 4c\lambda M^{-1}})]\|x_t - x\| \leq \|y - Px\|^2 - \|y - Px_t\|^2\,.$$

Thus we find that $\|y - Px\|^2 - \|y - Px_t\|^2 \geq 0$. Hence $\|y - Px_t\| \leq \|y - Px\| \leq r$ and $x_t \in B(x_0, 2\epsilon)$.

Notice that $x_t \in B(x_0, 2\epsilon)$ and $\|y - Px_t\| \leq r$ imply

$$\begin{aligned}\|x_t - x\| &\leq \frac{1 + \sqrt{1 + 4c\lambda M^{-1}}}{2c}\|Px_t - Px\| \\ &\leq \frac{1 + \sqrt{1 + 4c\lambda M^{-1}}}{2c}(\|Px_t - y\| + \|y - Px_0\|) \\ &\leq r\frac{1 + \sqrt{1 + 4c\lambda M^{-1}}}{c} \leq \epsilon\,.\end{aligned}$$

Let $\psi(x) = [M(1 + \sqrt{1 + 4c\lambda M^{-1}})/c^2 d]\|y - Px\|^2$ and define $g\colon D \to D$ such that $gx = x_t$. Then

$$\|x - gx\| \leq \psi(x) - \psi(gx)\,.$$

Observe that D, being a closed subset of X, is complete. Since ψ is the continuous map from the complete metric space D into nonnegative reals, by the Caristi-Kirk fixed point theorem (cf. [8, 10, 11]) g has a fixed point in D. Note that $\|x_t - x\| = t\|h\| \neq 0$, a contradiction. This completes the proof. □

Remark 2.3. We remark that Theorem 2.1 generalizes results of Park and Park [12] and hence those of Browder [7] and Ray [13]. Further, geometrical structures of Y^* in Theorem 2.1 are not required as opposed to [7] and [13].

Example 2.4. Let $X = Y = \mathbb{R}$. Then $Y^* = \mathbb{R}^* = \mathbb{R}$. Define $\phi\colon X \to Y^*$ implicitly which satisfy conditions (i) and (ii) and $P\colon X \to Y$ explicitly by

$$Px = \frac{2c}{1 + \sqrt{1 + 4c\lambda M^{-1}}}x + \beta \quad \text{for all} \quad x \in X\,, \quad \beta \in \mathbb{R}\,.$$

Notice that the condition

$$\langle Pu - Px, \phi(u - x) + \lambda M^{-1}j(Pu - Px)\rangle \geq c\|u - x\|^2$$

for all $u \in X$ sufficiently near to x yields

$$\|u-x\| \leq \frac{1+\sqrt{1+4c\lambda M^{-1}}}{2c}\|Pu-Px\|\,. \tag{2.4}$$

Indeed, for all $u \in X$ sufficiently near to x,

$$\langle Pu-Px, \phi(u-x)+\lambda M^{-1}j(Pu-Px)\rangle \geq c\|u-x\|^2$$

implies that

$$\langle Pu-Px, \phi(u-x)\rangle + \langle Pu-Px, \lambda M^{-1}j(Pu-Px)\rangle \geq c\|u-x\|^2\,,$$

i.e.,

$$c\|u-x\|^2 \leq \|Pu-Px\|\|\phi(u-x)\| + \lambda M^{-1}\|Pu-Px\|\|j(Pu-Px)\|\,,$$

i.e.,

$$c\|u-x\|^2 \leq \|Pu-Px\|\|u-x\| + \lambda M^{-1}\|Pu-Px\|^2\,.$$

By solving the above quadratic in $d = \frac{\|u-x\|}{\|Pu-Px\|}$ we can easily find (2.4). Clearly, P satisfies the above condition for all $x \in X$ and all $u \in X$ sufficiently near to x. By definition of P, it is evident that $P(X) = Y$.

3. Generalized directional contractor and its application

In this section, we establish a surjectivity theorem for some nonlinear operators by using the notion of generalized directional contractor. In the sequel we apply our result to obtain a solution of certain functional equations.

Altman's fundamental paper [1] contains the following useful notion of directional contractor:

Let X and Y be two Banach spaces. Let $P\colon D(P) \subset X \to Y$ be a nonlinear operator from a linear subspace $D(P)$ of X to Y, $\Gamma(x)\colon Y \to D(P)$ a bounded linear operator associated with $x \in D(P)$. Suppose there exists a positive number $q = q(P) < 1$ such that for any $x \in D(P)$ and $y \in Y$, there exist $\epsilon = \epsilon(x,y) \in (0,1]$ satisfying

$$\|P(x+\epsilon\Gamma(x)y) - Px - \epsilon y\| \leq q\,\epsilon\,\|y\|\,.$$

Then $\Gamma(x)$ is called a directional contractor for P at $x \in D(P)$ and $\Gamma\colon D(P) \subset X \to L(Y,X)$ is called a directional contractor for P, where $L(Y,X)$ denotes the set of all linear continuous maps of Y into X. If there exists a constant $B(>0)$ such that $\|\Gamma(x)\| \leq B$ for all $x \in D(P)$, then Γ is called a bounded directional contractor for P.

We now introduce the concept of generalized contractor as follows:

Definition 3.1. Let X and Y be two Banach spaces. Let $P\colon D(P) \subset X \to Y$ be a nonlinear operator from a linear subspace $D(P)$ of X to Y, $\Gamma(x)\colon Y \to D(P)$ a bounded linear operator associated with $x \in D(P)$. Suppose there exists a positive number $q = q(P) < 1$ such that for any $x \in D(P)$ and $y \in Y$, there exist $\epsilon = \epsilon(x, y) \in (0, 1]$ and a nonincreasing function $c\colon [0, \infty) \to (0, q^{-1/2})$ satisfying

$$\|P(x + \epsilon\Gamma(x)y) - Px - \epsilon y\| \leq q\,\epsilon\, c(\|x\|)\,\|y\|\,.$$

Then $\Gamma(x)$ is called a generalized directional contractor for P at $x \in D(P)$ and $\Gamma\colon D(P) \subset X \to L(Y, X)$ is called a generalized directional contractor for P, where $L(Y, X)$ denotes the set of all linear continuous maps of Y into X. If there exists a constant $B(> 0)$ such that $\|\Gamma(x)\| \leq B$ for all $x \in D(P)$, then Γ is called a bounded generalized directional contractor for P. It follows from the above definition that $\Gamma(x)y = 0$ implies $y = 0$ i.e., $\Gamma(x)$ is injective.

Notice that every generalized directional contractor is a directional contractor and an inverse Gâuteaux derivative is a directional contractor. Recall that $P\colon D(P) \subset X \to Y$ is said to have closed graph if $x_n \to x$, $x_n \in D(P)$ and $Px_n \to y$ imply $x \in D(P)$ and $y = Px$.

By applying the ideas of Ray and Walker [14], we are now ready to prove a surjectivity theorem for generalized directional contractor.

Theorem 3.2. *Let X and Y be two Banach spaces. A nonlinear map $P : D(P) \subset X \to Y$ which has closed graph and a bounded generalized directional contractor Γ is surjective.*

Proof. Define a metric ρ on $D(P)$ by

$$\rho(x, y) = \max\{\|x - y\|, (1 + q^{1/2})^{-1}\|Px - Py\|\}\,.$$

As $D(P)$ has closed graph, $(D(P), \rho)$ is a complete metric space. Suppose $w \notin R(P)$ (the range of P). For any $x \in D(P)$ we set $y = w - Px$. Since P has a bounded generalized directional contractor Γ we have, for some $0 < \epsilon(x, y) \leq 1$,

$$\|P(x + \epsilon\Gamma(x)y) - Px - \epsilon y\| \leq q\,\epsilon\, c(\|x\|)\|y\|\,. \tag{3.1}$$

Set $\epsilon\Gamma(x)y = h$. Then we have

$$\|h\| = \|\epsilon\Gamma(x)y\| \leq \epsilon B\|y\| = \epsilon B\|w - Px\|\,. \tag{3.2}$$

From (3.1) we have

$$\|P(x + h) - w + (1 - \epsilon)(w - Px)\| \leq q\,\epsilon\, c(\|x\|)\|w - Px\|$$

which yields

$$\|P(x + h) - w\| - (1 - \epsilon)\|w - Px\| \leq q\,\epsilon\, c(\|x\|)\|w - Px\|\,.$$

Therefore we have

$$\epsilon\|w - Px\| - q\,\epsilon\, c(\|x\|)\|w - Px\| \leq \|w - Px\| - \|w - P(x + h)\|\,,$$

i.e.,

$$\epsilon(1 - q\, c(\|x\|))\|w - Px\| \leq \|w - Px\| - \|w - P(x + h)\|\,. \tag{3.3}$$

Again from (3.1) we have

$$\|P(x + h) - Px\| - \epsilon\|w - Px\| \leq q\,\epsilon\, c(\|x\|)\|w - Px\|$$

which yields

$$\|P(x+h)-Px\| \le \epsilon(1+q\,c(\|x\|))\|w-Px\|\,. \tag{3.4}$$

From (3.3) and (3.4) we have

$$\begin{aligned}\|P(x+h)-Px\| &\le \big(1+q\,c(\|x\|)\big)\big(1-q\,c(\|x\|)\big)^{-1}(\|w-Px\|-\|w-P(x+h)\|)\\ &\le \big(1+q^{1/2}\big)\big(1-q^{1/2}\big)^{-1}(\|w-Px\|-\|w-P(x+h)\|)\,. \qquad (3.5)\end{aligned}$$

Using (3.2) again we have

$$\begin{aligned}\|h\| &\le \epsilon B\|w-Px\|\\ &\le B\big(1-q\,c(\|x\|)\big)^{-1}(\|w-Px\|-\|w-P(x+h)\|)\\ &\le B\big(1-q^{1/2}\big)^{-1}(\|w-Px\|-\|w-P(x+h)\|)\,. \qquad (3.6)\end{aligned}$$

Let $a=\max(B,1)$ and $\varphi(x)=a\Big(1-q^{1/2}\Big)^{-1}\|w-Px\|$. Then φ is continuous with respect to metric ρ. Therefore if we set $fx=x+h$, then $fx\neq x$. Indeed if $h=0$ then from (3.2) we have

$$\epsilon\|y\| \le q\epsilon\, c(\|x\|)\|y\| \le q^{1/2}\epsilon\,\|y\| < \epsilon\,\|y\|\,.$$

But since $w\notin R(P)$, $y=Px-w\neq 0$. Therefore $fx\neq x$ and $\rho(x,fx)\le \varphi(x)-\varphi(fx)$. This is a contradiction to Caristi-Kirk fixed point theorem ([4], see also [6]). Hence we conclude that $w\in R(P)$. □

Let X and Y be two Banach spaces. Let $P\colon D(P)\subset X\to Y$, and let $x\in X$. We now consider a special class of generalized directional contractors. Let $\Gamma(x)(P)$ be a set of generalized directional contractors for P at $x\in D(P)$ called class (S) if there exist a positive number $q=q(P)<1$, a constant $B>0$ and a nonincreasing function $c\colon[0,\infty)\to(0,\,q^{-1/2})$ with the following property:
For each $y\in\Gamma(x)(P)$, there exist a positive number $\epsilon=\epsilon(x,y)\le 1$ and an element $\bar{x}\in D(P)$ such that:

(S_1): $\|P\bar{x}-Px-\epsilon y\| \le q\epsilon c(\|x\|)\|y\|$ and

(S_2): $\|\bar{x}-x\| \le \epsilon B\|y\|$.

Now we apply the above results to obtain a solution of certain functional equations.

Theorem 3.3. *Let X and Y be two Banach spaces. Let $P\colon D(P)\subset X\to Y$ has closed graph. For $x\in D(P)$, let $\Gamma(x)(P)$ denote the class (S). Suppose that y_0 is such that for each $x\in D(P)$, the element y_0-Px belongs to the closure of the set $\Gamma(x)(P)$ defined by (S_1) and (S_2). Then the equation $Px-y_0=0, x\in D(P)$ has a solution.*

Proof. Suppose, if possible, $Px-y_0=0$, $x\in D(P)$ has no solution. Set $y=y_0-Px\neq 0$, then by hypothesis $y\in\overline{\Gamma(x)(P)}$. So we can choose y' in $\Gamma(x)(P)$ and a $\alpha>0$ such that $\|y-y'\|\le\alpha\|y\|$. Note that $\alpha<1$ and does not depend on x.

Since $y' \in \Gamma(x)(P)$, there exists $\bar{x}$ such that

$$\|P\bar{x} - Px - \epsilon y'\| \leq q\epsilon\, c(\|x\|)\|y'\|\,. \tag{3.7}$$

From the above inequality we have

$$\|P\bar{x} - y_0 + y_0 - Px - \epsilon y'\| \leq q\epsilon\, c(\|x\|)\|y'\|\,.$$

As $y = y_0 - Px$, we obtain

$$\|P\bar{x} - y_0 + y - \epsilon y'\| \leq q\epsilon\, c(\|x\|)\|y'\|\,. \tag{3.8}$$

Choose $q' > 0$ such that $q < q' < 1$. After having chosen q' we may choose $\alpha > 0$ sufficiently small such that $(\alpha + 1) \leq qq'$. Since $\|y - y'\| \leq \alpha\|y\|$, we have $\|y'\| \leq (1 + \alpha)\|y\|$. From this and (3.8) we have

$$\|P\bar{x} - y_0 + y - \epsilon y'\| \leq q'\epsilon\, c(\|x\|)\|y\|\,.$$

On the other hand, we have

$$(\|P\bar{x} - y_0 + (1-\epsilon)y\| - \|P\bar{x} - y_0 + y - \epsilon y'\|) \leq \epsilon\,\|y - y'\| \leq \epsilon\,\alpha\,\|y\|\,.$$

Hence from (3.8) and the above inequalities, we have

$$\|P\bar{x} - y_0 + (1-\epsilon)y\| - \epsilon\,\alpha\,\|y\| \leq q'\epsilon\, c(\|x\|)\|y\|\,.$$

From this we have

$$\|P\bar{x} - y_0\| - (1-\epsilon)\|y\| - \epsilon\,\alpha\,\|y\| \leq q'\epsilon\, c(\|x\|)\|y\|\,.$$

Therefore we obtain

$$\epsilon(1 - q'c(\|x\|) - \alpha)\|y\| \leq \|y\| - \|P\bar{x} - y_0\|$$

and which implies that

$$\epsilon(1 - q'^{-1/2} - \alpha)\|y\| \leq \|y\| - \|P\bar{x} - y_0\|\,.$$

If we choose $\alpha > 0$ so that $\beta = 1 - q'^{-1/2} - \alpha > 0$, then we obtain

$$\epsilon\,\beta\|y_0 - Px\| \leq \|Px - y_0\| - \|P\bar{x} - y_0\|\,. \tag{3.9}$$

But from (3.7) we have

$$\begin{aligned}\|P\bar{x} - Px\| &\leq \epsilon(qc(\|x\|) + 1)\|y'\| \\ &\leq \epsilon(qc(\|x\|) + 1)(\alpha + 1)\|y\|\end{aligned}$$

or

$$\|P\bar{x} - Px\| \leq \epsilon(q^{1/2} + 1)(\alpha + 1)\|y_0 - Px\|.$$

Using (3.9), the above inequality yields

$$\|P\bar{x} - Px\| \leq (q^{1/2} + 1)(\alpha + 1)\beta^{-1}(\|Px - y_0\| - \|P\bar{x} - y_0\|)\,.$$

Since $\|\bar{x} - x\| \leq \epsilon B\|y'\| \leq \epsilon B(\alpha + 1)\|y\| = \epsilon B(\alpha + 1)\|y_0 - Px\|$ we have

$$\|\bar{x} - x\| \leq B(\alpha + 1)\beta^{-1}(\|Px - y_0\| - \|P\bar{x} - y_0\|)\,.$$

We now define a metric ρ on $D(P)$ by

$$\rho(x, y) = \max\{\|x - y\|, (1 + q^{1/2})^{-1}\|Px - Py\|\}\,.$$

Set $fx = \bar{x}$. Since $y' \neq 0 (\alpha < 1)$, we have $x \neq \bar{x}$. Take $a = \max\{B, 1\}$ and set $\varphi(x) = a(\alpha + 1)\beta^{-1}\|Px - y_0\|$. Then

$$\rho(x, fx) \leq \varphi(x) - \varphi(fx)\,.$$

This is a contradiction to Caristi-Kirk fixed point theorem (cf. [8, 10, 11]). Hence we conclude that $Px - y_0 = 0, x \in D(P)$ has a solution. $\square$

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