ARCHIVUM MATHEMATICUM (BRNO)
Tomus 49 (2013), 51–64

ON THE HAMMERSTEIN EQUATION IN THE SPACE OF FUNCTIONS OF BOUNDED φ-VARIATION IN THE PLANE

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Abstract. In this paper we study existence and uniqueness of solutions for the Hammerstein equation

$$u(x) = v(x) + \lambda \int_{I_a^b} K(x,y) f\big(y, u(y)\big)\, dy\,, \quad x \in I_a^b := [a_1, b_1] \times [a_2, b_2]\,,$$

in the space $BV_\varphi^{\mathbb{R}}(I_a^b)$ of function of bounded total $\varphi-$variation in the sense of Riesz, where $\lambda \in \mathbb{R}$, $K\colon I_a^b \times I_a^b \to \mathbb{R}$ and $f\colon I_a^b \times \mathbb{R} \to \mathbb{R}$ are suitable functions.

1. Introduction

One of the most frequently investigated integral equations in nonlinear functional analysis is the Hammerstein integral equation. It serves as a mathematical model for many nonlinear physical phenomena such as electromagnetic fluid dynamics. Furthermore, solutions of some boundary value problems for differential equations are usually equivalent to solutions of Hammerstein integral equations. In particular, in [7, p. 46] was observed that the integral equation in the current paper can be considered as a two independent variable generalization of the Hammerstein equation studied by many researchers. On the other hand, in [3] and [4] it is pointed out that spaces of functions endowed with some type of bounded variation norm appear in a natural way in certain physical phenomena which are described by Hammerstein equations. This research was motivated by the foregoing comments and the works [3], [4], [5] and [6], where their authors study existence and uniqueness of solutions for Hammerstein equation, and another sort of nonlinear integral equations, in diverse spaces of bounded variation functions on an interval. Another source of motivation has been paper [9], where linear Volterra integral equation involving Lebesgue-Stieltjes integral is studied in two independent variables. In the case of several variables, in [8] we can find a study of integral equations in a space of real functions defined on $\mathbb{R}^n$, endowed with a kind bounded variation norm.

2010 *Mathematics Subject Classification*: primary 45G10.
Key words and phrases: existence and uniqueness of solutions of the Hammerstein integral equation in the plane, φ-bounded total variation norm on a rectangle.
This work has been supported by BCV and CDCHT-ULA-C-1796-12-05-AA.
Received October 17, 2012, revised April 2013. Editor O. Došlý.
DOI: 10.5817/AM2013-1-51

The paper is organized as follows: In Section 2, we present some definitions and preliminaries results concerning with functions of bounded $\varphi-$variation in a rectangle I_a^b in the plane, which were considered in [1] and [2]. In Section 3, applying the Banach fixed point theorem, we study the existence and uniqueness of the solutions of the equation

$$u(x) = v(x) + \lambda \int_{I_a^b} K(x,y) f\big(y, u(y)\big)\, dy\,,$$

in the space of bounded total φ-variation functions defined on $I_a^b := [a_1, b_1] \times [a_2, b_2]$, where $x,\, y \in I_a^b$, $K\colon I_a^b \times I_a^b \to \mathbb{R}$, $f\colon I_a^b \times \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then in Section 4, we investigate the local existence and uniqueness of solutions of the equation

$$u(x) = v(x) + \int_{I_a^b} K(x,y) \big(y, u(y)\big)\, dy\,,$$

in the space of functions which has been mentioned above and finally, in Section 5, using a nonlinear alternative of Leray-Schauder type, we prove the global existence of solutions of the above equation in the same space of functions.

2. Preliminaries

This section contains some definitions and properties about the functions of bounded φ-variation on the plane, in the sense of Riesz, which are used in this paper.

Definition 2.1. Let us fix any real numbers $a_1 < b_1$ and $a_2 < b_2$. Let $\{t_i\}_1^m = \{a_1 = t_0 < t_1 < \cdots < t_m = b_1\}$ and $\{s_j\}_1^n = \{a_2 = s_0 < s_1 < \cdots < s_n = b_2\}$ be partitions of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Given a function $u\colon I_a^b := [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$, we define the following quantities:

$$\Delta_{10} u(t_i, a_2) := u(t_i, a_2) - u(t_{i-1}, a_2)\,,$$

$$\Delta_{01} u(a_1, s_j) := u(a_1, s_j) - u(a_1, s_{j-1})\,,$$

$$\Delta_{11} u(t_i, s_j) := u(t_{i-1}, s_{j-1}) + u(t_i, s_j) - u(t_{i-1}, s_j) - u(t_i, s_{j-1})\,.$$

Let us consider the following definition, which is similar to the above Definition 2.1.

Definition 2.2. Let us fix any real numbers $a_1 < b_1$ and $a_2 < b_2$. Let $\{t_i\}_1^m = \{a_1 = t_0 < t_1 < \cdots < t_m = b_1\}$ and $\{s_j\}_1^n = \{a_2 = s_0 < s_1 < \cdots < s_n = b_2\}$ be partitions of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Given a function $K : [a_1, b_1] \times [a_2, b_2] \times I_a^b \to \mathbb{R}$, we define the following quantities:

$$\Delta_{10} K(t_i, a_2, y) := K(t_i, a_2, y) - K(t_{i-1}, a_2, y)\,,$$

$$\Delta_{01} K(a_1, s_j, y) := K(a_1, s_j, y) - K(a_1, s_{j-1}, y)\,,$$

$$\begin{aligned}\Delta_{11} K(t_i, s_j, y) := {} & K(t_{i-1}, s_{j-1}, y) + K(t_i, s_j, y) \\ & - K(t_{i-1}, s_j, y) - K(t_i, s_{j-1}, y)\,.\end{aligned}$$

Definition 2.3. The function $\varphi\colon [0,\infty) \to [0,\infty)$ is called a φ-function, if the following conditions are verified:

1. φ is continuous,
2. $\varphi(t) = 0$ if and only if $t = 0$,
3. $\varphi(t) \to \infty$ as $t \to \infty$ and
4. the function φ is nondecreasing.

Definition 2.4. Given a function $u : I_a^b \to \mathbb{R}$, we define the Riesz φ-variation of the function u in $[a_1, b_1] \times \{a_2\}$ by the formula

$$V^R_{\varphi,[a_1,b_1]}(u) := \sup_{\Pi_1} \sum_{i=1}^{m} \varphi\Big[\frac{|\Delta_{10}u(t_i, a_2)|}{|t_i - t_{i-1}|}\Big] \cdot |t_i - t_{i-1}|\,,$$

where the supremum is taken over the set of all partitions Π_1 of the interval $[a_1, b_1]$.

Definition 2.5. Given a function $u\colon I_a^b \to \mathbb{R}$, we define the Riesz φ-variation of the function u in $\{a_1\} \times [a_2, b_2]$ by putting

$$V^R_{\varphi,[a_2,b_2]}(u) := \sup_{\Pi_2} \sum_{j=1}^{n} \varphi\Big[\frac{|\Delta_{01}u(a_1, s_j)|}{|s_j - s_{j-1}|}\Big] \cdot |s_i - s_{i-1}|\,,$$

where the supremum is taken over the set of all partitions Π_2 of the interval $[a_2, b_2]$.

Definition 2.6. Given a function $u\colon I_a^b \to \mathbb{R}$, we define the Riesz two dimensional φ-variation of the function u in I_a^b by the formula

$$V^R_{\varphi}(u) := \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi\Big[\frac{|\Delta_{11}u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\Big] \cdot |\Delta t_i| \cdot |\Delta s_j|\,,$$

where the supremum is taken over the set of all pairs of partitions (Π_1, Π_2) of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively.

Definition 2.7. Given a function $u\colon I_a^b \to \mathbb{R}$, we define the Riesz total φ-variation of u, which is denoted $TV^R_{\varphi}(u)$, by putting

$$TV^R_{\varphi}(u) := V^R_{\varphi,[a_1,b_1]}(u) + V^R_{\varphi,[a_2,b_2]}(u) + V^R_{\varphi}(u)\,.$$

Definition 2.8. The set of all functions $u\colon I_a^b \to \mathbb{R}$ with finite bounded Riesz total φ-variation is denoted by $V^R_{\varphi}(I_a^b)$; that is,

$$V^R_{\varphi}(I_a^b) := \big\{u : I_a^b \to \mathbb{R} \;:\; TV^R_{\varphi}(u) \;<\; \infty\big\}\,.$$

Remark 2.1. We denote by $BV^R_{\varphi}(I_a^b) := \big\langle V^R_{\varphi}(I_a^b)\big\rangle$ the linear space generated by $V^R_{\varphi}(I_a^b)$.

Theorem 2.1. *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a φ-function. If φ is convex and $\lim\limits_{t\to\infty} \frac{\varphi(t)}{t} = \infty$, then the pair $\big(BV^R_{\varphi}(I_a^b)\;,\;\|\cdot\|^R_{\varphi}\big)$ is a Banach space with the norm*

$$\|u\|^R_{\varphi} := \;\big|u(a_1, a_2)\big| + \inf\big\{\varepsilon > 0 :\; TV^R_{\varphi}\big(\frac{u}{\varepsilon}\big) \le 1\big\}.$$

Remark 2.2 ([2, p. 82])**.** If φ is a convex φ-function such that $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$, then

$$\|u\|_\infty := \sup\{ \ |u(x)| : x \in I_a^b \ \} \leq \|u\|_\varphi^R .$$

Definition 2.9. We denote by $\mathfrak{G}(I_a^b)$ the set of all rectangles $P := [t_1, t_2] \times [x_1, x_2]$ contained in I_a^b, where $|P| := (t_2 - t_1) \cdot (x_2 - x_1)$.

Definition 2.10. A function $F\colon \mathfrak{G}(I_a^b) \to \mathbb{R}$ is absolutely continuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $P_1, \ldots, P_k \in \mathfrak{G}(I_a^b)$ are rectangles of which their interiors are pairwise disjoints and

$$\sum_{j=1}^{k} |P_j| \leq \delta ,$$

then

$$\sum_{j=1}^{k} \big|F\big(P_j\big)\big| < \epsilon .$$

Definition 2.11. Given a function $u\colon I_a^b \to \mathbb{R}$, we define the rectangles function $F_u\colon \mathfrak{G}(I_a^b) \to \mathbb{R}$ by $F_u\big([t_1, t_2] \times [x_1, x_2]\big) = \Delta_{11} u(t_2, x_2)$.

Definition 2.12. A function $u\colon I_a^b \to \mathbb{R}$ is absolutely continuous in I_a^b in the sense of Carathéodory, if the rectangles function F_u is absolutely continuous and the functions $u(a_1, \cdot)\colon [a_2, b_2] \to \mathbb{R}$, $u(\cdot, a_2)\colon [a_1, b_1] \to \mathbb{R}$ are absolutely continuous in the usual sense.

Theorem 2.2 ([2, p. 23])**.** *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a convex φ-function such that $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$ and let us consider a function $u\colon I_a^b \to \mathbb{R}$. Then $TV_\varphi^R(u) < \infty$ if, and only if, u is absolutely continuous in I_a^b in the sense of Carathéodory.*

Theorem 2.3 ([2, p. 22])**.** *A function $u\colon I_a^b \to \mathbb{R}$ is absolutely continuous in I_a^b in the sense of Carathéodory if, and only if, u has the integral representation*

$$u(t,x) = e + \int_{a_1}^{t} f(s)\, ds + \int_{a_2}^{x} g(\eta)\, d\eta + \int\int_{Q(t,x)} h(s,\eta)\, ds d\eta ,$$

where $(t,x) \in I_a^b$, $e \in \mathbb{R}$, f and g are Lebesgue-integrable in $[a_1, b_1]$ and $[a_2, b_2]$, respectively, h is Lebesgue-integrable in I_a^b and $Q(t,x) = [a_1, t] \times [a_2, x]$.

Lemma 2.1. *If φ is a convex φ-function such that $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$ and $u \in BV_\varphi^R(I_a^b)$, then the function u is continuous.*

Proof. This lemma is a consequence of Remark 2.1, Theorems 2.2 and 2.3 and the continuity of the Lebesgue integral with regard to its measure. $\square$

3. Existence and uniqueness of solutions

In this section we study the existence and uniqueness of solutions of the integral equation

$$u(x) = v(x) + \lambda \int_{I_a^b} K(x,y) f\big(y, u(y)\big)\, dy \tag{1}$$

in the Banach space $BV_\varphi^R(I_a^b)$ with the norm $\|u\|_\varphi^R$, where $\lambda \in \mathbb{R}$. From now on, we assume the following hypotheses.

Assumption 3.1. Suppose that $K\colon I_a^b \times I_a^b \to \mathbb{R}$ is a bounded function and $f\colon I_a^b \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, we assume that $K(x,\cdot)\colon I_a^b \to \mathbb{R}$ is measurable for each fixed $x \in I_a^b$, the function f is locally Lipschitz in the second variable and the sign $\int$ stands for the Lebesgue integral.

Assumption 3.2. We denote I_a^b by I. Also, assume that the function $\varphi\colon [0,\infty) \to [0,\infty)$ is a convex φ-function such that $\lim\limits_{t\to\infty} \frac{\varphi(t)}{t} = \infty$.

Assumption 3.3. We assume that there exists a function $w \in BV_\varphi^R(I)$ such that for all pair of real numbers $(t_i, s_j) \in [a_1, b_1] \times [a_2, b_2]$ and each $y \in I$, we have that

1. $\big|\Delta_{10} K(t_i, a_2, y)\big| \le \big|\Delta_{10} w(t_i, a_2)\big|$,
2. $\big|\Delta_{01} K(a_1, s_j, y)\big| \le \big|\Delta_{01} w(a_1, s_j)\big|$,
3. $\big|\Delta_{11} K(t_i, s_j, y)\big| \le \big|\Delta_{11} w(t_i, s_j)\big|$,
4. $\sup\big\{\big|K(a_1, a_2, y)\big| : y \in I\big\} \le \big|\ w(a_1, a_2)\ \big|$.

Now we give an example of the function K which is mentioned in the above Assumption 3.3.

Example 3.1. Let $p\colon I \to \mathbb{R}$ and $q\colon I \to \mathbb{R}$ be two bounded functions such that $0 \le p(a_1, a_2)$, $p \in BV_\varphi^R(I)$ and q is measurable. We define K by

$$K\colon I \times I \to \mathbb{R}; \quad K(x,y) := p(x) q(y)\,.$$

Observe that the function K has the properties which were assumed, where

$$w(x) := \sup\big\{\big|K(a_1, a_2, y)\big| : y \in I\big\} + \sup\big\{|q(y)| : y \in I\big\} \cdot p(x)\,.$$

We shall define a function $F(u)$ from I into $\mathbb{R}$ for each $u \in BV_\varphi^R(I)$.

Definition 3.1. Given $u \in BV_\varphi^R(I)$, we define $F(u)\colon I \to \mathbb{R}$ by

$$F(u)(t,s) := \int_I K(t,s,y) f(y, u(y))\, dy\,.$$

Remark 3.1. Since we assumed that $K\colon I \times I \to \mathbb{R}$ is a bounded function, $f\colon I \times \mathbb{R} \to \mathbb{R}$ is continuous and $K(x,\cdot)\colon I \to \mathbb{R}$ is a measurable function for each fixed $x \in I$, then the above function $F(u)(x) := \int_I K(x,y) f(y, u(y))\, dy$ is well defined.

Now we shall prove some useful lemmas and theorems related to the above function F.

Lemma 3.1. *If $u \in BV_\varphi^R(I)$, then*

$$\Delta_{11}\Big(\frac{F(u)}{\varepsilon}\Big)(t_i,s_j) = \int_I \Delta_{11}\Big(\frac{K}{\varepsilon}\Big)(t_i,s_j,y) f\big(y,u(y)\big)\,dy\,.$$

Proof. We have the following chain of equalities:

$$\begin{aligned}
\Delta_{11}\,\Big(\frac{F(u)}{\varepsilon}\Big)(t_i,s_j) &= \frac{F(u)}{\varepsilon}(t_{i-1},s_{j-1}) + \frac{F(u)}{\varepsilon}(t_i,s_j) \\
&\quad - \frac{F(u)}{\varepsilon}(t_{i-1},s_j) - \frac{F(u)}{\varepsilon}(t_i,s_{j-1}) \\
&= \frac{1}{\varepsilon}\Big[\int_I K(t_{i-1},s_{j-1},y) f\big(y,u(y)\big)\,dy + \int_I K(t_i,s_j,y) f\big(y,u(y)\big)\,dy \\
&\quad - \int_I K(t_{i-1},s_j,y) f\big(y,u(y)\big)\,dy - \int_I K(t_i,s_{j-1},y) f\big(y,u(y)\big)\,dy\Big] \\
&= \int_I \frac{K}{\varepsilon}(t_{i-1},s_{j-1},y) f\big(y,u(y)\big)\,dy + \int_I \frac{K}{\varepsilon}(t_i,s_j,y) f\big(y,u(y)\big)\,dy \\
&\quad - \int_I \frac{K}{\varepsilon}(t_{i-1},s_j,y) f\big(y,u(y)\big)\,dy - \int_I \frac{K}{\varepsilon}(t_i,s_{j-1},y) f\big(y,u(y)\big)\,dy \\
&= \int_I \Delta_{11}\Big(\frac{K}{\varepsilon}\Big)(t_i,s_j,y) f\big(y,u(y)\big)\,dy\,.
\end{aligned}$$

This completes the proof. □

Definition 3.2. We define C and $|I|$ by $C := C(u) := \max_{y\in I}\big|f(y,u(y))\big|$ and $|I| := (b_1 - a_1)\cdot(b_2 - a_2)$, respectively.

Lemma 3.2. *If $u \in BV_\varphi^R(I)$, then*

$$\Big|\Delta_{11}\Big(\frac{F(u)}{\varepsilon}\Big)(t_i,s_j)\Big| \le \Big|\Delta_{11}\Big(\frac{|I|Cw}{\varepsilon}\Big)(t_i,s_j)\Big|\,.$$

Proof. Observe that

$$\begin{aligned}
\Big|\Delta_{11}\Big(\frac{F(u)}{\varepsilon}\Big)(t_i,s_j)\Big| &\le \int_I \Big|\Delta_{11}\Big(\frac{K}{\varepsilon}\Big)(t_i,s_j,y) f\big(y,u(y)\big)\Big|\,dy \\
&\le \int_I \Big|\Delta_{11}\Big(\frac{w}{\varepsilon}\Big)(t_i,s_j)\Big| \cdot \max_{y\in I}\big|f\big(y,u(y)\big)\big|\,dy \\
&= C\Big|\Delta_{11}\Big(\frac{w}{\varepsilon}\Big)(t_i,s_j)\Big| \cdot |I| = \Big|\Delta_{11}\Big(\frac{|I|Cw}{\varepsilon}\Big)(t_i,s_j)\Big|\,.
\end{aligned}$$ □

Lemma 3.3. *If $u \in BV_\varphi^R(I)$, then*

$$V_\varphi^R\Big(\frac{F(u)}{\varepsilon}\Big) \le V_\varphi^R\Big(\frac{|I|Cw}{\varepsilon}\Big)\,.$$

Proof. Since the function φ is nondecreasing, we have that

$$\varphi\Big(\frac{|\Delta_{11}\big(\frac{F(u)}{\varepsilon}\big)(t_i,s_j)|}{|\Delta t_i|\cdot|\Delta s_j|}\Big)\le\varphi\Big(\frac{|\Delta_{11}\big(\ \frac{|I|Cw}{\varepsilon}\ \big)(t_i,s_j)|}{|\Delta t_i|\cdot|\Delta s_j|}\ \Big).$$

After the use of Definition 2.6, we conclude that $V_\varphi^R\big(\frac{F(u)}{\varepsilon}\big)\ \le\ V_\varphi^R\big(\ \frac{|I|Cw}{\varepsilon}\ \big)$. □

In the same way as the above Lemma 3.3, we can deduce the following two lemmas.

Lemma 3.4. *If $u\in BV_\varphi^R(I)$, then*

$$V^R_{\varphi,[a_1,b_1]}\Big(\frac{F(u)}{\varepsilon}\Big)\le V^R_{\varphi,[a_1,b_1]}\Big(\frac{|I|Cw}{\varepsilon}\Big).$$

Lemma 3.5. *If $u\in BV_\varphi^R(I)$, then*

$$V^R_{\varphi,[a_2,b_2]}\Big(\frac{F(u)}{\varepsilon}\Big)\le V^R_{\varphi,[a_2,b_2]}\Big(\frac{|I|Cw}{\varepsilon}\Big).$$

The following theorem is a straightforward consequence of the above lemmas.

Theorem 3.1. *If $u\in BV_\varphi^R(I)$ and $\varepsilon>0$, then*

$$TV^R_{\varphi}\Big(\frac{F(u)}{\varepsilon}\Big)\le TV^R_{\varphi}\Big(\frac{|I|Cw}{\varepsilon}\Big).$$

Lemma 3.6. *If $u\in BV_\varphi^R(I)$ and $\varepsilon>0$, then*

$$\inf\Big\{\varepsilon>0:TV^R_{\varphi}\Big(\frac{F(u)}{\varepsilon}\Big)\le 1\Big\}\le\inf\Big\{\varepsilon>0:TV^R_{\varphi}\Big(\frac{|I|Cw}{\varepsilon}\Big)\le 1\Big\}.$$

The above lemma allows us to prove the following theorem, which plays an important role in this paper.

Theorem 3.2. *If $u\in BV_\varphi^R(I)$, then $F(u)\in BV_\varphi^R(I)$ and*

$$\|F(u)\|_\varphi^R\le|I|\cdot\max_{y\in I}\big|f\big(y,u(y)\big)\big|\cdot\|w\|_\varphi^R.$$

Theorem 3.3. *If u and $\widetilde{u}$ belong to $BV_\varphi^R(I)$, then*

$$\|F(u)-F(\widetilde{u})\|_\varphi^R\le|I|\cdot\max_{y\in I}\big|f\big(y,u(y)\big)-f\big(y,\widetilde{u}(y)\big)\big|\cdot\|w\|_\varphi^R.$$

Proof. This theorem can be deduced in same way as Theorem 3.2, if we take $C:=C(u,v):=\max\limits_{y\in I}\big|f(y,u(y))-f(y,v(y))\big|$ instead of $C:=C(u):=\max\limits_{y\in I}\big|f(y,u(y))\big|$. □

The following lemmas will be useful in order to prove the existence of solutions of integral equation (1).

Lemma 3.7. *Given $r>0$, there exists $C=C(r)>0$ such that $\|\ F(u)\ \|_\varphi^R\le C$ for each $u\in\overline{B}_r:=\big\{u\in BV_\varphi^R(I)\ :\ \|\ u\ \|_\varphi^R\le r\big\}$.*

Definition 3.3. Let v be a given function belonging to $BV_\varphi^R(I)$ and $\lambda \in \mathbb{R}$. Given $u \in BV_\varphi^R(I)$, we define

$$G_\lambda(u) : I \to \mathbb{R};$$

$$\begin{aligned} G_\lambda(u)(t,s) := {} & v(t,s) + \lambda \int_I K(t,s,y) f\big(y, u(y)\big)\, dy \\ & = v(t,s) + \lambda F(u)(t,s)\,. \end{aligned}$$

Lemma 3.8. *If* $u \in BV_\varphi^R(I)$, *then* $G_\lambda(u) \in BV_\varphi^R(I)$.

Lemma 3.9. *Let v be a given function in $BV_\varphi^R(I)$ and $r > 0$. If $\| v \|_\varphi^R < r$, then there exists a real number $D = D(r) > 0$ such that $G_\lambda(u) \in \overline{B}_r$ for each $u \in \overline{B}_r$ and λ with $|\lambda| < D$.*

Lemma 3.10. *Let v be a given function in $BV_\varphi^R(I)$ and consider $r > 0$ such that $\| v \|_\varphi^R < r$. There exists a real number $E = E(r) > 0$ such that if $|\lambda| < E$, then $G_\lambda(u)\colon \overline{B}_r \to \overline{B}_r$ is a contraction.*

Proof. Let D be such as in Lemma 3.9 and let $u, \widetilde{u} \in \overline{B}_r$. Observe that

$$\| G_\lambda(u) - G_\lambda(\widetilde{u}) \|_\varphi^R = |\lambda| \;\; \| F(u) - F(\widetilde{u}) \|_\varphi^R \;.$$

By Theorem 3.3 and since $f\colon I \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in the second variable, there exists a constant $L_r > 0$ such that

$$\|F(u) - F(\widetilde{u})\|_\varphi^R \le |I| L_r \|w\|_\varphi^R \cdot \|u - \widetilde{u}\|_\infty \quad \text{for all} \quad u, \widetilde{u} \in \overline{B}_r\,.$$

Therefore, we have

$$\| G_\lambda(u) - G_\lambda(\widetilde{u}) \|_\varphi^R \le |I||\lambda| L_r \|w\|_\varphi^R \cdot \|u - \widetilde{u}\|_\varphi^R \quad \text{for all} \quad u, \widetilde{u} \in \overline{B}_r\,.$$

Observe that there exists a real number $E := E(r) > 0$ such that if $|\lambda| < E$, then $|I|\; |\lambda| L_r \|w\|_\varphi^R < 1$. Hence we have that if $|\lambda| < \min\{E, D\}$, then $G_\lambda(u)\colon \overline{B}_r \to \overline{B}_r$ is a contraction.

In view of fact concerning the existence of an adequate function which is a contraction, we can use the fixed point theorem of Banach to prove the existence and uniqueness of a solution of integral equation (1). □

Theorem 3.4. *Suppose that $v \in BV_\varphi^R(I)$ and $\| v \|_\varphi^R < r$ for a real number $r > 0$. There is a real number $E = E(r) > 0$ such that if $|\lambda| < E$, then there exists a solution of the integral equation (1) belonging to $BV_\varphi^R(I)$.*

Proof. By a straightforward application of the fixed point theorem of Banach with $G_\lambda\colon \overline{B}_r \to \overline{B}_r$, there exists a unique solution in $\overline{B}_r$ of the above integral equation, which is a solution in $BV_\varphi^R(I)$. □

Theorem 3.5. *Suppose that $v \in BV_\varphi^R(I)$. If $f\colon I \times \mathbb{R} \to \mathbb{R}$ is globally Lipschitz in the second variable with Lipschitz constant $L > 0$, then there is a real number $D > 0$ such that for each real number λ with $|\lambda| < D$, there exists a unique solution of the integral equation (1) belonging to $BV_\varphi^R(I)$.*

Proof. Let us choose a real number $r > 0$ such that $\| v \|_\varphi^R < r$. By Theorem 3.4 there is a real number $E = E(r) > 0$ such that if $|\lambda| < E$, then there exists a solution of the integral equation

$$u(x) = v(x) + \lambda \int_I K(x,y) f\big(y, u(y)\big)\, dy\,,$$

belonging to $BV_\varphi^R(I)$. Let us suppose that a function $\widetilde{u} \in BV_\varphi^R(I)$ is another solution of the above integral equation. Due to Theorem 3.3, we have the estimate

$$\begin{aligned}\|u - \widetilde{u}\|_\varphi^R &\le |\lambda|\ \|F(u) - F(\widetilde{u})\|_\varphi^R \\ &\le |\lambda| \cdot |I| \cdot \max_{y \in I} \big| f(y, u(y)) - f\big(y, \widetilde{u}(y)\big)\big| \cdot \|w\|_\varphi^R \\ &\le |\lambda| \cdot |I| \cdot L \cdot \big\|u - \widetilde{u}\big\|_\infty \cdot \|w\|_\varphi^R \\ &\le |\lambda| \cdot |I| \cdot L \cdot \|w\|_\varphi^R \cdot \big\|u - \widetilde{u}\big\|_\varphi^R .\end{aligned}$$

If we take the real number λ such that

$$|\lambda| < \ D := D(r) := \min \Big\{ E(r),\ \frac{1}{|I| \cdot L \cdot \|w\|_\varphi^R} \Big\},$$

then $u(y) = \widetilde{u}(y)$ for all $y \in I$. □

4. Existence of local solutions

In this section we prove the local existence of solutions of the integral equation

$$u(x) = v(x) + \int_I K(x,y) f\big(y, u(y)\big)\, dy\,. \tag{2}$$

Let us formulate an assumption about function $v\colon I \to \mathbb{R}$ appearing in the above integral equation, which is only assumed in this section.

Assumption 4.1. The function $v\colon I \to \mathbb{R}$ satisfies the following conditions:

1. $V^R_{\varphi,[a_1,b_2]}(v)(s) := \sup\limits_{\Pi_1} \sum\limits_{i=1}^{m} \varphi\Big[\dfrac{|\Delta_{10} v(t_i, s)|}{l|t_i - t_{i-1}|}\Big] \cdot |t_i - t_{i-1}| < \infty$ for all

   $s \in [a_2, b_2]\,,$ where $\Delta_{10} v(t_i, s) := v(t_i, s) - v(t_{i-1}, s)\,,$

2. $V^R_{\varphi,[a_2,b_2]}(v)(t) := \sup\limits_{\Pi_2} \sum\limits_{j=1}^{n} \varphi\Big[\dfrac{|\Delta_{01} v(t, s_j)|}{|s_j - s_{j-1}|}\Big] \cdot |s_j - s_{j-1}| < \infty$ for all

   $t \in [a_1, b_1]\,,$ where $\Delta_{01} v(t, s_j) := v(t, s_j) - v(t, s_{j-1})$ and

3. $V^R_\varphi(v) := \sup\limits_{\Pi_1,\Pi_2} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \varphi\Big[\dfrac{|\Delta_{11} v(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\Big] \cdot |\Delta t_i| \cdot |\Delta s_j| < \infty\,,$ where $\Delta_{11} v(t_i, s_j) := v(t_{i-1}, s_{j-1}) + v(t_i, s_j) - v(t_{i-1}, s_j) - v(t_i, s_{j-1})$, such that the supremum is taken over the set of all pairs of partitions Π_1 and Π_2 of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively.

Remark 4.1. If v satisfies Assumption 4.1, then $v \in V_\varphi^R(J) \subseteq BV_\varphi^R(J)$ for each rectangle J which is contained in I. In particular, if $J = I$, then $v \in V_\varphi^R(I) \subseteq BV_\varphi^R(I)$.

Theorem 4.1. *There is a real number* $\delta > 0$ *such that if* $J := [c_1, d_1] \times [c_2, d_2]$ *is a rectangle contained in* I *and* $|J| := (d_1 - c_1) \cdot (d_2 - c_2) < \delta$, *then there exists a solution of the integral equation*

$$u(x) = v_J(x) + \int_J K(x,y) f\big(y, u(y)\big)\, dy \tag{3}$$

belonging to $BV_\varphi^R(J)$, *where* v_J *is the function* v *restricted to the rectangle* J. *Moreover, if* f *is globally Lipschitz in the second variable, then such a solution is unique.*

Proof. Let u be an element of $BV_\varphi^R(J)$. Define $F_J(u)\colon J \to \mathbb{R}$ by putting

$$F_J(u)(t,s) := \int_J K(t,s,y) f\big(y, u(y)\big)\, dy \quad \text{and}$$

$$G_J(u)\colon J \to \mathbb{R} \quad \text{by}$$

$$\begin{aligned} G_J(u)(t,s) &:= v_J(t,s) + \int_J K(t,s,y) f\big(y, u(y)\big)\, dy \\ &= v_J(t,s) + F_J(u)(t,s)\,. \end{aligned}$$

Let us denote by $\| \, v \, \|_{\varphi,I}^R$ the norm of the function v in the space $BV_\varphi^R(I)$. Fix $r > 0$ such that $\| \, v \, \|_{\varphi,I}^R < r$. We define $\overline{B}_r(J)$ by

$$\overline{B}_r(J) := \{ u \in BV_\varphi^R(J) \ : \| \, u \, \|_\varphi^R \leq r \}\,.$$

By Theorem 3.2, we have that $F_J(u) \in BV_\varphi^R(J)$ for all $u \in BV_\varphi^R(J)$ and there exists a real number $R(r)$ which does not depend on the rectangle J, such that

$$\|F_J(u)\|_\varphi^R \leq |J| \cdot R(r) \cdot \|w\|_\varphi^R\,.$$

Note that there is a real number $\delta_1 := \delta_1(r) > 0$ such that if $|J| < \delta_1$, then

$$\|G_J(u)\|_\varphi^R \leq \| \, v_J \, \|_\varphi^R + \|F_J(u)\|_\varphi^R \leq \| \, v \, \|_{\varphi,I}^R + \|F_J(u)\|_\varphi^R < r\,.$$

By Theorem 3.3 and in view of the assumed fact that $f\colon I \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in the second variable, there exists a constant $L(r) > 0$ which does not depend on the rectangle J, such that

$$\| \, G_J(u) - G_J(\widetilde{u}) \, \|_\varphi^R = \|F_J(u) - F_J(\widetilde{u})\|_\varphi^R \leq |J| \cdot L(r) \cdot \|w\|_\varphi^R \cdot \|u - \widetilde{u}\|_\infty$$

for all pair $u, \widetilde{u} \in \overline{B}_r(J)$. Hence, we get

$$\| \, G_J(u) - G_J(\widetilde{u}) \, \|_\varphi^R \leq |J| \cdot L(r) \cdot \|w\|_\varphi^R \cdot \|u - \widetilde{u}\|_\varphi^R \quad \text{for all pair} \quad u, \ \widetilde{u} \in \overline{B}_r(J)\,.$$

Observe that there exists a real number $\delta_2 := \delta_2(r) > 0$ such that if $|J| < \delta_2$, then $|J| \cdot L(r) \cdot \|w\|_\varphi^R < 1$. Thus, if $|J| < \delta := \min\{\delta_1, \delta_2\}$, then $G_J(u)\colon \overline{B}_r(J) \to \overline{B}_r(J)$

is a contraction. By the theorem of fixed point of Banach, the integral equation

$$u(x) = v_J(x) + \int_J K(x,y) f\big(y, u(y)\big)\, dy$$

has a unique solution $u \in \overline{B}_r(J) \subseteq BV_\varphi^R(J)$. If the function f is globally Lipschitz in the second variable, then by the ideas in the proof of Theorem 3.5 such a solution is unique in the space $BV_\varphi^R(J)$. □

5. Existence of global solutions

Again we consider the integral equation in above Section 4

$$u(x) = v(x) + \int_I K(x,y) f\big(y, u(y)\big)\, dy\,,$$

but now we prove the existence of solutions of in the Banach space $BV_\varphi^R(I)$. Let us recall the following Leray-Schauder alternative, which statement is taken from [5].

Theorem 5.1. *Let U be an open subset of a Banach space $(X, \|\cdot\|)$ with $0 \in U$. Suppose $H\colon \overline{U} \to X$ and assume there exists a continuous nondecreasing function $\phi\colon [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for x, $y \in \overline{U}$ we have $\|H(x) - H(y)\| \le \phi(\|x - y\|)$; here $\overline{U}$ denotes the closure of U in X. In addition assume that $H(\overline{U})$ is bounded and $x \neq \lambda H(x)$ for $x \in \partial U$ and $\lambda \in (0,1]$; here ∂U denotes the boundary of U in X. Then H has a fixed point in U.*

Now we shall prove the main results of this section.

Theorem 5.2. *If $f\colon I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is globally Lipschitzian with respect to the second variable with Lipschitz constant $L > 0$ and $\kappa := \kappa(I, L, w) := |I| \cdot L \cdot \|w\|_\varphi^R < 1$, then there exists a solution of the integral equation (2) belonging to $BV_\varphi^R(I)$ for each fixed function $v \in BV_\varphi^R(I)$.*

Proof. Let r be a fixed positive real number such that $\frac{\|v+F(0)\|_\varphi^R}{1-\kappa} < r$, where F is the function giving in Definition 3.1. Now define the set

$$U := U(r) := \big\{u \in BV_\varphi^R(I) : \|u\|_\varphi^R < r\big\}$$

and the function $H\colon \overline{U} \to BV_\varphi^R(I)$ by the formula

$$H(u)(t,s) := v(t,s) + F(u)(t,s) = v(t,s) + \int_I K(t,s,y) f\big(y, u(y)\big)\, dy\,.$$

By Theorem 3.3, we have that

$$\| H(u) - H(\widetilde{u}) \|_\varphi^R \le |I| \cdot L \cdot \|w\|_\varphi^R \cdot \big\|u - \widetilde{u}\big\|_\varphi^R \quad \text{for all pair} \quad u, \widetilde{u} \in \overline{U}\,.$$

Observe that above inequality implies that $H(\overline{U})$ is bounded. Now we define $\phi\colon [0,\infty) \to [0,\infty)$ by $\phi(z) := \kappa z$, where κ is the constant which was mentioned in the hypothesis of the theorem.

Let us suppose that there is an element $u \in \overline{U}$ such that $u = \lambda H(u)$ for some $\lambda \in (0,1]$. Applying the above inequality, we get

$$\begin{aligned}\| u \|_{\varphi}^{R} &= \| \lambda H(u) \|_{\varphi}^{R} = \lambda \| H(u) \|_{\varphi}^{R} \leq \| H(u) \|_{\varphi}^{R} \\ &\leq \| H(u) - H(0) \|_{\varphi}^{R} + \| H(0) \|_{\varphi}^{R} \\ &\leq \kappa \left\|u\right\|_{\varphi}^{R} + \| H(0) \|_{\varphi}^{R} .\end{aligned}$$

This yields

$$\| u \|_{\varphi}^{R} \leq \frac{\| H(0) \|_{\varphi}^{R}}{1-\kappa} < r .$$

From this inequality and the fact that $u \in \partial U$ implies $\| u \|_{\varphi}^{R} = r$, we deduce that $u \neq \lambda H(u)$ for each $u \in \partial U$ and all $\lambda \in (0,1]$. After the use of Leray-Schauder alternative, we conclude that there exists a solution of the integral equation

$$u(x) = v(x) + \int_I K(x,y) f\big(y, u(y)\big)\, dy .$$

This completes the proof. □

In the prove of the following theorem, we use the same techniques which are used in [3, Theorem 9, p. 275] and [5, Theorem 5, p. 303].

Theorem 5.3. *If*

1. *there exists a continuous nondecreasing function $\psi\colon [0,\infty) \to [0,\infty)$ satisfying $|I| \cdot \|w\|_{\varphi}^{R} \cdot \psi(z) < z$ for each $z > 0$, moreover*
2. $$|f(y,t) - f(y,\widetilde{t})| < \psi(|t - \widetilde{t}|)$$

   for all pair (y,t), $(y,\widetilde{t})$ belonging to $I \times \mathbb{R}$, furthermore
3. *there exists a continuous nondecreasing function $\Psi\colon \mathbb{R} \to \mathbb{R}$ with $\Psi(t) > 0$ for all $t > 0$ and $|f(y,t)| \leq \Psi(|t|)$ for each $(y,t) \in I \times \mathbb{R}$, and*
4. *there exists a real number $r > 0$ such that*

   $$\frac{r}{\|v\|_{\varphi}^{R} + |I| \cdot \|w\|_{\varphi}^{R} \cdot \Psi(r)} > 1 ,$$

   then the integral equation (2) has a solution belonging to $BV_{\varphi}^{R}(I)$ for each fixed function $v \in BV_{\varphi}^{R}(I)$.

Proof. Let r be the real number appearing in the hypotheses. Let us define the set $U := \left\{u \in BV_{\varphi}^{R}(I) : \|u\|_{\varphi}^{R} < r\right\}$ and the function $H\colon \overline{U} \to BV_{\varphi}^{R}(I)$ by putting

$$H(u)(t,s) := v(t,s) + F(u)(t,s) = v(t,s) + \int_I K(t,s,y) f\big(y, u(y)\big)\, dy .$$

By Theorem 3.3, we have

$$\begin{aligned}\| H(u) - H(\widetilde{u}) \|_{\varphi}^{R} &\le |I| \cdot \max_{y\in I} \left|f\big(y, u(y)\big) - f\big(y, \widetilde{u}(y)\big)\right| \cdot \|w\|_{\varphi}^{R} \\ &\le |I| \cdot \|w\|_{\varphi}^{R} \cdot \psi\big(\max_{y\in I}\left|u(y) - \widetilde{u}(y)\right|\big) \le |I| \cdot \|w\|_{\varphi}^{R} \cdot \psi(\| u - \widetilde{u} \|_{\varphi}^{R})\end{aligned}$$

for all pairs u, $\widetilde{u} \in \overline{U}$. If we define $\phi\colon \mathbb{R} \to \mathbb{R}$ by $\phi(z) := |I| \cdot \|w\|_{\varphi}^{R} \cdot \psi(z)$, then we have

$$\| H(u) - H(\widetilde{u}) \|_{\varphi}^{R} \le \phi\Big(\| u - \widetilde{u} \|_{\varphi}^{R}\Big)$$

for all pairs $u, \widetilde{u} \in \overline{U}$. From this inequality we can deduce that $H(\overline{U})$ is bounded.

Now, let us suppose that $\|u\|_{\varphi}^{R} = r$ and $u = \lambda H(u)$ for some real number $\lambda \in (0, 1]$. Observe that we can write

$$u(x) = \lambda\Big(v(x) + \int_I K(x, y) f\big(y, u(y)\big)\, dy\Big).$$

By Theorem 3.2, we obtain

$$\begin{aligned}r = \|u\|_{\varphi}^{R} &\le \|v\|_{\varphi}^{R} + \|F(u)\|_{\varphi}^{R} \le \|v\|_{\varphi}^{R} + |I| \cdot \|w\|_{\varphi}^{R} \cdot \max_{y\in I} |f(y, u(y))| \\ &\le \|v\|_{\varphi}^{R} + |I| \cdot \|w\|_{\varphi}^{R} \cdot \Psi(\|u\|_{\varphi}^{R}) = \|v\|_{\varphi}^{R} + |I| \cdot \|w\|_{\varphi}^{R} \cdot \Psi(r).\end{aligned}$$

Therefore

$$\frac{r}{\|v\|_{\varphi}^{R} + |I| \cdot \|w\|_{\varphi}^{R} \cdot \Psi(r)} \le 1,$$

which contradicts to a part of the hypothesis. The Leray-Schauder alternative implies that there exists a solution of the integral equation

$$u(x) = v(x) + \int_I K(x, y) f\big(y, u(y)\big)\, dy.$$

The proof is complete. □

Acknowledgement. We would like to thank the anonymous referee for the valuable comments and suggestions.

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