ARCHIVUM MATHEMATICUM (BRNO)
Tomus 49 (2013), 35–41

WIRTINGER INEQUALITY AND NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract. Picone identity for a class of nonlinear differential equations is established and various qualitative results (such as Wirtinger-type inequality and the existence of zeros of first components of solutions) are obtained with the help of this new formula.

1. Introduction

Let $\alpha > 0$ and define $\varphi_\alpha(s) = |s|^{\alpha-1}s$ if $s \neq 0$, and $\varphi_\alpha(0) = 0$. The nonlinear differential system to be considered in this paper is of the form

$$
\begin{aligned}
u' &= A(t)u + B(t)\varphi_{1/\alpha}(v) \\
v' &= -C(t)\varphi_\alpha(u) - D(t)v\,,
\end{aligned}
\tag{1.1}
$$

where A, B, C and D are continuous real-valued functions on a given interval I and $B(t) > 0$ in I.

If $A(t) \equiv D(t)$ in I, then (1.1) is the (nonlinear) Hamiltonian system

$$
u' = \partial H/\partial v, \quad v' = -\partial H/\partial u\,,
\tag{1.2}
$$

where

$$
H(t;u,v) = \frac{1}{\alpha+1}C(t)|u|^{\alpha+1} + A(t)uv + \frac{\alpha}{\alpha+1}B(t)|v|^{1+\frac{1}{\alpha}}\,.
\tag{1.3}
$$

In the special case $A(t) \equiv 0$ in I, the system (1.1) is equivalent with the scalar second-order half-linear differential equation

$$
\big(P(t)\varphi_\alpha(u')\big)' + R(t)\varphi_\alpha(u') + Q(t)\varphi_\alpha(u) = 0\,,
\tag{1.4}
$$

where the coefficient functions are

$$
P(t) = B(t)^{-\alpha}\,, \quad R(t) = D(t)B(t)^{-\alpha}, \quad Q(t) = C(t)\,.
$$

If, moreover, also $D(t) \equiv 0$ in I, then (1.4) reduces to the half-linear Sturm-Liouville equation

$$
\big(P(t)\varphi_\alpha(u')\big)' + Q(t)\varphi_\alpha(u) = 0\,.
\tag{1.5}
$$

2010 *Mathematics Subject Classification*: primary 34C10.
Key words and phrases: nonlinear differential system, Picone identity, Wirtinger inequality.
Received December 20, 2012. Editor O. Došlý.
DOI: 10.5817/AM2013-1-35

While qualitative theory for scalar equations (1.4) and (1.5) is well-developed, only little is known about the general system (1.1), particularly in the case where $A(t) \neq 0$ or $A(t) \neq D(t)$ in I (for some results concerning the case $\alpha = 1$ see [4] and [6]). We choose to mention the comparison result of the Sturm type due to Elbert [1] who compared (1.1) with another system of the same type

$$
\begin{aligned}
x' &= a(t)x + b(t)\varphi_{1/\alpha}(y) \\
y' &= -c(t)\varphi_\alpha(x) - d(t)y
\end{aligned}
\tag{1.6}
$$

and proved that if $b(t) > 0$ on I and $(x(t), y(t))$ is a solution of (1.6) such that the function $x(t)$ has consecutive zeros at $t_1, t_2 \in I$ and (1.1) is a Sturmian majorant for (1.6) in the sense that

$$\bigl[B(t) - b(t)\bigr]|\xi|^{\alpha+1} + \Bigl[A(t) - a(t) - \frac{d(t) - D(t)}{\alpha}\Bigr]\xi\varphi_\alpha(\eta) + \frac{C(t) - c(t)}{\alpha}|\eta|^{\alpha+1} \geq 0\,,$$

for all $\xi, \eta \in \mathbb{R}$ and $t \in I$, then for any solution $(u(t), v(t))$ of (1.1) the first component $u(t)$ has at least one zero in (t_1, t_2).

Elbert proved his result by means of the generalized Prüfer transformation. In the particular case $a \equiv d \equiv A \equiv D \equiv 0$, Elbert's criterion reduces to the half-linear generalization of the classical Sturm-Picone comparison theorem due to Mirzov [8]. For integral comparison results concerning two scalar equations of the form (1.5) the reader is referred to [2]–[3] and [7]. In particular, in [3] it was shown that the solutions u of (1.5) and x of an accompanying "test" equation

$$\bigl(p(t)\varphi_\alpha(x')\bigr)' + q(t)\varphi_\alpha(x) = 0\,, \tag{1.7}$$

are connected by an identity of the Picone type (the original version of which can be found in [6]) which says that if $u(t) \neq 0$ in the given interval I, then

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(u)}\bigl[\varphi_\alpha(u)p(t)\varphi_\alpha(x') - \varphi_\alpha(x)P(t)\varphi_\alpha(u')\bigr]\right\} \\
&\quad = [p(t) - P(t)]|x'|^{\alpha+1} + [Q(t) - q(t)]|x|^{\alpha+1} + P(t)\bigl[|x'|^{\alpha+1} \\
&\qquad - (\alpha+1)x'\varphi_\alpha(xu'/u) + \alpha|xu'/u|^{\alpha+1}\bigr]\,.
\end{aligned}
\tag{1.8}
$$

A simplified identity of the above kind can also be used to acquire information about a single equation without comparing it with another equation. It is based on a weaker assumption that $x(t)$ is any continuously differentiable function (not necessarily satisfying any equation or inequality) and states that for a zero-free solution $u(t)$ of (1.5), the following identity holds:

$$
\begin{aligned}
\frac{d}{dt}\left\{|x|^{\alpha+1}\frac{P(t)\varphi_\alpha(u')}{\varphi_\alpha(u)}\right\} &= P(t)|x'|^{\alpha+1} - Q(t)|x|^{\alpha+1} \\
&\quad - P(t)\bigl[|x'|^{\alpha+1} - (\alpha+1)x'\varphi_\alpha(xu'/u) + \alpha|xu'/u|^{\alpha+1}\bigr]\,.
\end{aligned}
\tag{1.9}
$$

A typical result that can be obtained from (1.9) says that if there exists a solution u of (1.5) such that $u(t) \neq 0$ on (t_1, t_2), then for all C^1-functions x defined on $[t_1, t_2]$ and satisfying $x(t_1) = x(t_2) = 0$, the inequality

$$\int_{t_1}^{t_2} \left[P(t)|x'|^{\alpha+1} - Q(t)|x|^{\alpha+1}\right] dt \geq 0 \tag{1.10}$$

is valid and the equality in (1.10) holds if and only if x is a constant multiple of u. (See [3] and [7].)

The purpose of this paper is to establish Picone's identity of the weaker kind for the nonlinear differential systems of the form (1.1) and to apply it to derive Wirtinger-type inequalities analogous to (1.10) formulated in terms of solutions of (1.1). We also obtain information about the existence and distribution of zeros of the first component of the solution of (1.1). Our results are formulated in terms of an arbitrary continuous function $G(t)$. Special choices of this function (such as $G(t) \equiv 0$ or $G(t) = (\alpha A(t) + D(t))/(\alpha + 1)$) yield a variety of particular integral inequalities of the Wirtinger type. On the other hand, we emphasize that our basic identity seems to be new even if we reduce our consideration to the special cases of $G(t)$ mentioned above.

The paper is organized as follows. In Section 2, the desired generalization of (1.9) to the nonlinear system (1.1) is derived and some particular cases of this formula are discussed. Section 3 contains applications of the basic identity which include the Wirtinger-type inequality and theorems about the existence of zeros for components of solutions of the nonlinear system (1.1).

2. Generalized Picone's identity

Denote by Φ_α the form defined for X, $Y \in \mathbb{R}$ and $\alpha > 0$ by

$$\Phi_\alpha(X, Y) := |X|^{\alpha+1} + \alpha|Y|^{\alpha+1} - (\alpha + 1)X\varphi_\alpha(Y)\,.$$

From the Young inequality it follows that $\Phi_\alpha(X, Y) \geq 0$ for all $X, Y \in \mathbb{R}$ and the equality holds if and only if $X = Y$.

The Picone-type identity in the following lemma may be verified directly by differentiation.

Lemma 2.1. *If (u, v) is a solution of (1.1) such that $u(t) \neq 0$ in I, then for any $x \in C^1(I)$ and any $G \in C(I)$,*

$$\begin{aligned}
\frac{d}{dt}\left\{|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\right\} &= -B(t)^{-\alpha}\Phi_\alpha\left[x' - G(t)x, B(t)x\frac{\varphi_{1/\alpha}(v)}{u}\right] \\
&\quad + B(t)^{-\alpha}\big|x' - G(t)x\big|^{\alpha+1} - C(t)|x|^{\alpha+1} \\
&\quad - \left[\alpha\big(A(t) - G(t)\big) + D(t) - G(t)\right]|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\,.
\end{aligned} \tag{2.1}$$

Remark 2.1. If $G \equiv 0$ in I, then (2.1) reduces to

$$\frac{d}{dt}\left\{|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\right\} = -B(t)^{-\alpha}\Phi_\alpha\Big[x', B(t)x\frac{\varphi_{1/\alpha}(v)}{u}\Big] + B(t)^{-\alpha}\big|x'\big|^{\alpha+1} - C(t)|x|^{\alpha+1} - \big[\alpha A(t) + D(t)\big]|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\,. \tag{2.2}$$

which is the half-linear generalization of the scalar version of the identity used (implicitly) in the linear case in [9].

If we put $G(t) = \frac{\alpha A(t)+D(t)}{\alpha+1}$, then (2.1) becomes

$$\frac{d}{dt}\left\{|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\right\} = -B(t)^{-\alpha}\Phi_\alpha\Big[x' - \frac{\alpha A(t)+D(t)}{\alpha+1}x, B(t)x\frac{\varphi_{1/\alpha}(v)}{u}\Big] + B(t)^{-\alpha}\Big|x' - \frac{\alpha A(t)+D(t)}{\alpha+1}x\Big|^{\alpha+1} - C(t)|x|^{\alpha+1}\,. \tag{2.3}$$

Under the further restriction $A(t) \equiv 0$ in I, the identity (2.3) reduces to the following one-dimensional version of Yoshida's formula for differential equations with α-gradient terms (see [10, Theorem 8.3.1]):

$$\frac{d}{dt}\left\{|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\right\} = -B(t)^{-\alpha}\Phi_\alpha\Big[x' - \frac{D(t)}{\alpha+1}x, B(t)x\frac{\varphi_{1/\alpha}(v)}{u}\Big] + B(t)^{-\alpha}\Big|x' - \frac{D(t)}{\alpha+1}x\Big|^{\alpha+1} - C(t)|x|^{\alpha+1}. \tag{2.4}$$

Remark 2.2. One of the first authors who established a Picone-type identity involving an arbitrary function, though in a different context, was apparently K. Kreith [5] (see also [6, p. 12]). However, his approach depends on the linear structure of damped equations studied in [5] and cannot be extended to the half-linear settings in the present paper.

3. Wirtinger inequality

Our results will be stated in terms of the functionals $J_{\sigma\tau}$ and $M_{\sigma\tau}$ defined for $t_1 < \sigma < \tau < t_2, x \in C^1(I)$ and solutions (u, v) of (1.1) with $u(t) \neq 0$ in I by

$$J_{\sigma\tau}[x] = \int_\sigma^\tau \big[B(t)^{-\alpha}\big|x' - G(t)x\big|^{\alpha+1} - C(t)|x|^{\alpha+1}\big]\,dt$$

and

$$M_{\sigma\tau}[x;u,v] = \int_\sigma^\tau B(t)^{-\alpha}\Phi_\alpha\big[x' - G(t)x, B(t)x\frac{\varphi_{1/\alpha}(v)}{u}\big]\,dt.$$

As an immediate consequence of (2.1) we have the following result.

Lemma 3.1. *Let (u, v) be a solution of (1.1) such that $u(t) \neq 0$ in I and let $[\sigma, \tau] \subset I$. Then, for any $x \in C^1(I)$ and $G \in C(I)$, the following inequality holds:*

$$\Big[|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\Big]_\sigma^\tau \le J_{\sigma\tau}[x] - \int_\sigma^\tau \big[\alpha\big(A(t) - G(t)\big) + D(t) - G(t)\big]|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\,dt\,. \tag{3.1}$$

Moreover, the equality holds in (3.1) *if and only if*

$$x' = \Big[G(t) + B(t)\frac{\varphi_{1/\alpha}(v)}{u}\Big]x\,. \tag{3.2}$$

Proof. Integrating (2.1) from σ to τ and using positive semi-definiteness of the form Φ_α we obtain

$$\begin{aligned}\Big[|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\Big]_\sigma^\tau &= -M_{\sigma\tau}[x;u,v] + J_{\sigma\tau}[x]\\ &\quad - \int_\sigma^\tau \big[\alpha\big(A(t)-G(t)\big) + D(t) - G(t)\big]|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\,dt\\ &\le J_{\sigma\tau}[x] - \int_\sigma^\tau \big[\alpha\big(A(t)-G(t)\big) + D(t) - G(t)\big]|x|^{\alpha+1}\frac{v}{\varphi_\alpha(u)}\,dt\,, \end{aligned}\tag{3.3}$$

which is nothing else than (3.1). The equality obviously holds in (3.1) if and only if $\Phi_\alpha[x' - Gx, Bx\varphi_{1/\alpha}(v)/u] = 0$ in $[\sigma,\tau]$ which is equivalent with the condition (3.2). □

From Lemma 3.1 we easily obtain an integral inequality of the Wirtinger type if we assume the existence of the limits

$$J[x] = \lim_{\sigma\to t_1+,\tau\to t_2-} J_{\sigma\tau}[x]\,, \qquad M[x;u,v] = \lim_{\sigma\to t_1+,\tau\to t_2-} M_{\sigma\tau}[x;u,v] \tag{3.4}$$

and define the domains $\mathcal{D}_J$ and $\mathcal{D}_M$ of J, resp. M, to be the sets of all real-valued functions $x \in C^1(I)$ such that $J[x]$, resp. $M[x;u,v]$, exist. Also, for $x \in \mathcal{D}_J \cap \mathcal{D}_M$ and the solution (u,v) of (1.1) with $u(t) \neq 0$ in $I = (t_1,t_2)$, we denote

$$S_1[x;u,v] = \lim_{t\to t_1+}\frac{|x(t)|^{\alpha+1}v(t)}{\varphi_\alpha(u(t))}, \qquad S_2[x;u,v] = \lim_{t\to t_2-}\frac{|x(t)|^{\alpha+1}v(t)}{\varphi_\alpha(u(t))}\,, \tag{3.5}$$

whenever the limits in (3.5) exist.

Theorem 3.1. *Let* (u,v) *be a solution of* (1.1) *with* $u(t)\neq 0$ *in* I *satisfying*

$$\big[\alpha\big(A(t)-G(t)\big) + D(t) - G(t)\big]\frac{v}{\varphi_\alpha(u)} \ge 0 \tag{3.6}$$

in I. *Then, for any* $x \in \mathcal{D}_J \cap \mathcal{D}_M$ *for which the limits in* (3.5) *exist, the inequality*

$$S_2[x;u,v] - S_1[x;u,v] \le J[x] \tag{3.7}$$

holds. Furthermore, if $[\alpha(A-G) + D - G]v/\varphi_\alpha(u) \equiv 0$ *in* I, *then equality in* (3.7) *occurs if and only if* $x(t)$ *is a solutions of* (3.2).

As an immediate consequence of the above theorem we get the following result.

Corollary 3.1. *Let* (u,v) *be a solution of* (1.1) *such that* $u(t)\neq 0$ *in* I *and*

$$\big[\alpha(A(t)-G(t)) + D(t) - G(t)\big]\frac{v}{\varphi_\alpha(u)} \equiv 0 \tag{3.8}$$

in I. *Then, for every* $x \in \mathcal{D}_J\cap\mathcal{D}_M$ *for which both limits in* (3.5) *exist, the inequality* (3.7) *is valid. Moreover, the equality holds in* (3.7) *if and only if*

$$x(t) = Ku(t)\exp\Big\{\int_{t_0}^t \big[G(s) - A(s)\big]ds\Big\} \tag{3.9}$$

for some constants $K \neq 0$ *and* $t_0 \in I$.

In the case where $A(t) \equiv D(t)$ in I (i.e. (1.1) reduces to the nonlinear Hamiltonian system (1.2)) and $G(t) = A(t)$ for $t \in I$, the condition (3.8) is trivially satisfied. Clearly, in this special case the equality in (3.7) occurs if and only if $x(t)$ is a constant multiple of $u(t)$.

Another way how to guarantee the satisfaction of (3.8) is to choose

$$G(t) = \frac{\alpha A(t) + D(t)}{\alpha + 1}.$$

Then, the last result specializes as follows.

Corollary 3.2. *If* (u, v) *is a solution of* (1.1) *with* $u(t) \neq 0$ *in* I *and* $x \in \mathcal{D}_J \cap \mathcal{D}_M$ *is such that the limits in* (3.5) *exist and satisfy* $S_2[x; u, v] \geq 0, S_1[x; u, v] \leq 0$, *then*

$$J[x] = \int_{t_1}^{t_2} \Big[B(t)^{-\alpha} \Big| x' - \frac{\alpha A(t) + D(t)}{\alpha + 1} x \Big|^{\alpha+1} - C(t)|x|^{\alpha+1} \Big] \, dt \geq 0 \,. \tag{3.10}$$

Furthermore, equality in (3.10) *occurs if and only if*

$$x(t) = K u(t) \exp \Big\{ \int_{t_0}^{t} \Big[\frac{D(s) - A(s)}{\alpha + 1} \Big] \, ds \Big\} \tag{3.11}$$

for some constants $K \neq 0$ *and* $t_0 \in I$.

The above result can be reformulated as the following theorem which generalizes the scalar version of Theorem 8.3.2 in [10].

Corollary 3.3. *If for some nontrivial* C^1*-function* x *defined on* $[t_1, t_2]$ *and satisfying* $x(t_1) = x(t_2) = 0$, *the condition*

$$J[x] = \int_{t_1}^{t_2} \Big[B(t)^{-\alpha} \Big| x' - \frac{\alpha A(t) + D(t)}{\alpha + 1} x \Big|^{\alpha+1} - C(t)|x|^{\alpha+1} \Big] \, dt \leq 0 \tag{3.12}$$

holds, then for any solution (u, v) *of* (1.1) *the first component* $u(t)$ *either has a zero in* (t_1, t_2) *or is a constant multiple of*

$$x(t) \exp \Big\{ \int_{t_0}^{t} \Big[\frac{A(s) - D(s)}{\alpha + 1} \Big] \, ds \Big\} \tag{3.13}$$

for some $t_0 \in I$.

As before, if $A(t) \equiv D(t)$ in I, then the last assertion says that $x(t)$ and $u(t)$ are proportional.

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