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STABILITIES OF F-YANG-MILLS FIELDS ON SUBMANIFOLDS

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Abstract. In this paper, we define an F-Yang-Mills functional, and hence F-Yang-Mills fields. The first and the second variational formulas are calculated, and the stabilities of F-Yang-Mills fields on some submanifolds of the Euclidean spaces and the spheres are investigated, and hence the theories of Yang-Mills fields are generalized in this paper.

1. Introduction

Let $P(M,G)$ be a principal bundle over a compact Riemannian manifold M with structure group G (a Lie group), and let $E = P \times_\rho V$ be a vector bundle associated with $P(M,G)$, whose standard fibre is some vector space V, where $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G. Denote the space of E-valued p-forms by $\Omega^p(E) = \Gamma(\wedge^p T^*M \otimes E)$, and the space of connections of E by $\mathcal{C}_E$. Let $\mathfrak{g}_E = P \times_{\mathrm{Ad}_G} \mathfrak{g}$ be the adjoint vector bundle where $\mathfrak{g}$ is the Lie algebra of G. It is known that, for any $\nabla, \nabla' \in \mathcal{C}_E$, we have $\nabla - \nabla' \in \Omega^1(\mathfrak{g}_E)$. For each $\nabla \in \mathcal{C}_E$, the curvature 2-form $R^\nabla \in \Omega^2(\mathfrak{g}_E)$ is defined by $R^\nabla_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. If G is a semisimple Lie group, there is a natural invariant metric on $\mathfrak{g}_E$ which is defined by the Killing form, and this metric induces a one on $\Omega^2(\mathfrak{g}_E)$. With respect to this induced metric, the Yang-Mills functional is defined as follows:

$$\mathcal{S}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2 \,. \tag{1}$$

If a connection ∇ of E is a critical point of the Yang-Mills functional, we call it a Yang-Mills connection, the associated curvature tensor is called a Yang-Mills field.

For a connection ∇, its variation is a family ∇^t of connections with $|t| < \varepsilon$ (a small positive number) and $\nabla^0 = \nabla$. If

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathcal{S}(\nabla^t) \geq 0 \tag{2}$$

holds for any variations of a Yang-Mills connection ∇, then we call the Yang-Mills connection (and the corresponding Yang-Mills field) to be stable. Otherwise, we call it unstable.

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In the paper [2, 1], J. P. Bourguignon and H. B. Lawson obtained a well known result on stabilities of Yang-Mills fields as follows:

Theorem 1 ([1])**.** *For $n > 4$, any nonzero Yang-Mills fields on S^n are unstable.*

When $n = 4$, we have $\mathcal{S}(\nabla) \geq 4\pi^2|p_1(E)|$ for any connection ∇ (where, $p_1(E)$ is the pontryagin number of E, a topological invariant), and the equality holds if and only if the connection ∇ is self-dual or anti-self-dual (in this case, the connection is called an instanton). Hence any self-dual or anti-self-dual connection is stable. Conversely, any stable Yang-Mills connection (or field) on S^4 with $G = \mathrm{SU}_2, \mathrm{SU}_3, \mathrm{U}_2$ is either self-dual or anti-self-dual (see [1]). On the other hand, an infinite number of unstable Yang-Mills fields on S^2 with $G = SU(2)$ are constructed by L. M. Sibner, R. J. Sibner and K. Uhlenbeck in [4].

Y. L. Xin in [5] discussed the stabilities of Yang-Mills fields on submanifolds of the Euclidean space, and obtained the following

Theorem 2 ([5])**.** *Let M^n be a compact submanifold of $\mathbf{R}^{n+k}$, and satisfy the following condition:*

$$2h^\mu_{ti}h^\mu_{tj}\delta_{kl} - h^\mu_{tt}h^\mu_{ij}\delta_{kl} + 2h^\mu_{ij}h^\mu_{kl} \leq b\delta_{ij}\delta_{kl}\,, \tag{3}$$

where h^μ_{ij} is the second fundamental tensor with respect to a local orthonormal frame of M, $1 \leq i, j, k, l \leq n$, $n + 1 \leq \mu \leq n + k$, and $b < 0$. Then any nonzero Yang-Mills fields on M are unstable.

On $S^n \subseteq \mathbf{R}^{n+1}$, we can choose a local orthonormal field of frame of $\mathbf{R}^{n+1}$, such that $h^{n+1}_{ij} = \delta_{ij}$. Then the condition in Theorem 2 becomes as $n > 4$. Therefore, Theorem 2 is a generalization of Theorem 1.

Remark 3. The condition (3) means that for any tensor A_{ij}, we have

$$\left(2h^\mu_{ti}h^\mu_{tj}\delta_{kl} - h^\mu_{tt}h^\mu_{ij}\delta_{kl} + 2h^\mu_{ij}h^\mu_{kl}\right) A_{ik}A_{jl} \leq b\delta_{ij}\delta_{kl}A_{ik}A_{jl}\,.$$

If the integrand of the Yang-Mills functional is replaced by $\|R^\nabla\|^p$, then we can obtain a p-Yang-Mills functional, whose critical points are called p-Yang-Mills connections, and the associated curvature tensors are called p-Yang-Mills fields. The paper [3] investigated the stabilities of p-Yang-Mills fields of Euclidean and sphere submanifolds, and generalized the related results of [1] and [5].

Let M^n be a submanifold of $\mathbf{R}^{n+k}$ or S^{n+k}, and $h(\cdot, \cdot)$ the second fundamental form. Let $1 \leq i, j \leq n$; $n + 1 \leq \mu \leq n + k$. Choose a local orthonormal frame $\{e_i | i = 1, \cdots, n + k\}$ of $\mathbf{R}^{n+k}$ or S^{n+k}, such that, restrict to M^n, $\{e_1, \cdots, e_n\}$ are tangent to M and $\{e_\mu | \mu = n+1, \cdots, n+k\}$ are normal to M. Set $h(e_i, e_j) = h^\mu_{ij}e_\mu$ and $H^\mu = \sum h^\mu_{ii}$. Define

$$C_{ijklsr} \equiv \big(- H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml}\big)\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr} + 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\,.$$

For example, if $M^n = S^n$, as a hypersurface of $\mathbf{R}^{n+1}$, then we can choose an adapted local normal frame such that $h_{ij} = h^{n+1}_{ij} = \delta_{ij}$. In this case, $C_{ijklsr} = (2p - n)\delta_{jl}\delta_{ki}\delta_{sr}$.

The paper [3] proved the following theorems:

Theorem 4 ([3])**.** *Let M^n be a submanifold of $\mathbf{R}^{n+k}$, satisfying $C_{ijklsr} \leq b\delta_{ik}\delta_{jl}\delta_{sr}$, where $b < 0$. Then any nonzero p-Yang-Mills fields on M are unstable.*

Theorem 5 ([3])**.** *Let M^n be a submanifold of S^{n+k}, satisfying $C_{ijklsr} < (n - 2p)\delta_{ik}\delta_{jl}\delta_{sr}$. Then any nonzero p-Yang-Mills fields on M are unstable.*

When $p = 2$, the condition in Theorem 4 is the same as that in Theorem 2. So, Theorem 4 is a generalization of Theorem 2. If we consider S^n as a totally geodesic submanifold of S^{n+p}, then the condition in Theorem 5 is $n > 2p$. Therefore Theorem 5 is another generalization of Theorem 1.

Remark 6. Inequality $C_{ijklsr} \leq$ (or $<$)$a\delta_{ik}\delta_{jl}\delta_{sr}$ means that

$$\sum C_{ijklsr}A_{ij}A_{kl}B_{st}B_{rt} \leq (\text{or } <) a \sum \delta_{ik}\delta_{jl}\delta_{sr}A_{ij}A_{kl}B_{st}B_{rt}$$

for any tensor A_{ij} and B_{ij}.

Replacing the integrand of the Yang-Mills functional by $F\big(\frac{\|R^\nabla\|^2}{2}\big)$, where F is a non-negative function, we define an F-Yang-Mills functional, and hence F-Yang-Mills fields. These generalize theories of p-Yang-Mills fields. In this paper, we investigate the stabilities of F-Yang-Mills fields on submanifolds of the Euclidean space and the spheres, and our main results are in the following:

Theorem 7. *Let M^n be a submanifold of $\mathbf{R}^{n+k}$, which satisfies*

$$C_{ijklsr} \leq b\delta_{ik}\delta_{jl}\,, \tag{4}$$

where $b < 0$. Suppose that for $t > 0$, we have

$$(p-2)F'(t) \geq 2tF''(t)\,, \qquad F'(t) > 0\,,\ F(t) > 0\,. \tag{5}$$

Then any nonzero F-Yang-Mills field R^∇ on M^n is unstable.

Theorem 8. *Let M^n be a submanifold of S^{n+k}, which satisfies*

$$C_{ijklsr} < (n-2p)\delta_{ik}\delta_{jl}\,. \tag{6}$$

Suppose that for $t > 0$, we have

$$(p-2)F'(t) \geq 2tF''(t)\,, \quad F'(t) > 0\,,\ F(t) > 0\,. \tag{7}$$

Then any nonzero F-Yang-Mills field R^∇ on M^n is unstable.

Theorem 7 generalizes Theorem 4 and Theorem 8 generalizes Theorem 5.

Remark 9.

(1) The condition $(p-2)F'(t) \geq 2tF''(t)$ is equivalent to $\big(\frac{F'(t)}{t^{\frac{p-2}{2}}}\big)' \leq 0$, i.e. $\frac{F'(t)}{t^{\frac{p-2}{2}}}$ is differential and non-increasing.

(2) For $p \geq 2$, the following functions satisfy the condition 7: $\frac{1}{p}(2t)^{\frac{p}{2}}$, $\ln(1+t^{\frac{p}{2}})$, $\ln(t^{\frac{p}{2}} + \sqrt{1+t^p})$, $\frac{t^{\frac{p}{2}}}{\sqrt{1+t^p}}$, $\arctan(t^{\frac{p}{2}})$, $\int_0^{t^{\frac{p}{2}}} e^{-x^2}\,dx$, etc.

(3) In general, if $f\colon [0,\infty) \to (0,\infty)$ is differential and non-increasing, $F(t) = \int_0^{t^{\frac{p}{2}}} f(x)\,dx$, then $\frac{F'(t)}{t^{\frac{p-2}{2}}}$ is differential and non-increasing for $p \geq 2$, and hence condition (7) is satisfied by such an F.

2. Variational formulas of F-Yang-Mills fields

Definition 10. Let $F\colon [0,+\infty) \to [0,+\infty)$ be a C^∞ function. Define $\mathcal{S}_F\colon \mathcal{C}_E \to \mathbf{R}$ as following: For any $\nabla \in \mathcal{C}_E$, set

$$\mathcal{S}_F(\nabla) = \int_M F\Big(\frac{\|R^\nabla\|^2}{2}\Big)\,, \tag{8}$$

which is called an F-Yang-Mills functional. The critical points of $\mathcal{S}_F$ are called F-Yang-Mills connections, and the associated curvature tensors are called F-Yang-Mills fields.

Let $\nabla^t = \nabla + A^t$ be a variation of $\nabla \in \mathcal{C}_E$, where $A^t \in \Omega^1(\mathfrak{g}_E)$ with $A^0 = 0$. Then the curvature of ∇^t is given by

$$R^{\nabla^t} = R^\nabla + d^\nabla A^t + \frac{1}{2}[A^t \wedge A^t]\,, \tag{9}$$

where, the compound operation $[\cdot \wedge \cdot]$ is defined as follows: For $\varphi, \psi \in \Omega(\mathfrak{g}_E)$, $[\varphi \wedge \psi]_{X,Y} = [\varphi_X, \psi_Y] - [\varphi_Y, \psi_X]$. Here, d^∇ is the wedge covariant differentiation. By a straightforward calculation, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{S}_F(\nabla^t) &= \int_M \frac{d}{dt} F\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big) = \int_M F'\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big)\Big\langle \frac{d}{dt}R^{\nabla^t}, R^{\nabla^t}\Big\rangle \\ &= \int_M F'\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big)\Big\langle d^\nabla \frac{d}{dt}A^t + \Big[\frac{d}{dt}A^t \wedge A^t\Big], R^{\nabla^t}\Big\rangle\,.\end{aligned} \tag{10}$$

Let $D = \frac{d}{dt}A^t\,|_{t=0}$ and let δ^∇ be the adjoint operator of d^∇ with respect to the inner product. The above equality becomes as

$$\begin{aligned}\frac{d}{dt}\mathcal{S}_F(\nabla^t)\Big|_{t=0} &= \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\langle d^\nabla D, R^\nabla\rangle \\ &= \int_M \Big\langle D, \delta^\nabla F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)R^\nabla\Big\rangle\,.\end{aligned} \tag{11}$$

Hence the Euler-Lagrange equation of $\mathcal{S}_F(\cdot)$ is

$$\delta^\nabla F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)R^\nabla = 0\,. \tag{12}$$

In order to discuss the stabilities of F-Yang-Mills fields, we need the second variational formula. A direct calculation yields

$$\frac{d}{dt}R^{\nabla^t} = d^\nabla \frac{dA^t}{dt} + \frac{1}{2}\frac{d}{dt}\left[A^t \wedge A^t\right] \tag{13}$$

and

$$\frac{d^2}{dt^2}R^{\nabla^t} = d^\nabla\Big(\frac{d^2}{dt^2}A^t\Big) + \Big[\frac{d^2}{dt^2}A^t \wedge A^t\Big] + \Big[\frac{dA^t}{dt} \wedge \frac{dA^t}{dt}\Big]\,. \tag{14}$$

Hence we have

$$\frac{d}{dt}\Big|_{t=0} R^{\nabla^t} = d^\nabla D\,, \quad \frac{d^2}{dt^2}\Big|_{t=0} R^{\nabla^t} = d^\nabla C + [D\wedge D]\,. \tag{15}$$

where $C = \frac{d^2}{dt^2}\mid_{t=0} A^t$. Taking derivatives of (10), we have

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{S}_F(\nabla^t) &= \int_M \frac{d}{dt}\Big[F'\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big)\Big\langle \frac{d}{dt}R^{\nabla^t}, R^{\nabla^t}\Big\rangle\Big]\\ &= \int_M F''\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big)\Big\langle \frac{d}{dt}R^{\nabla^t}, R^{\nabla^t}\Big\rangle^2\\ &\quad + \int_M F'\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big)\Big\langle d^\nabla(\frac{d^2}{dt^2}A^t), R^{\nabla^t}\Big\rangle\\ &\quad + \int_M F'\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big)\Big\langle \frac{d^2}{dt^2}R^{\nabla^t}, R^{\nabla^t}\Big\rangle\\ &\quad + \int_M F'\Big(\frac{\|R^{\nabla^t}\|^2}{2}\Big)\Big\langle \frac{d}{dt}R^{\nabla^t}, \frac{d}{dt}R^{\nabla^t}\Big\rangle\,.\end{aligned} \tag{16}$$

Letting $t=0$, the above formula becomes as:

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{S}_F(\nabla^t)\Big|_{t=0} &= \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla D, R^\nabla\big\rangle^2 + \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla C, R^\nabla\big\rangle\\ &\quad + \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla C + [D\wedge D]\,, R^\nabla\big\rangle + \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|d^\nabla D\|^2\,.\end{aligned} \tag{17}$$

By (12), we have:

$$\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla C, R^\nabla\big\rangle = \int_M \Big\langle C, \delta^\nabla F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)R^\nabla\Big\rangle = 0\,. \tag{18}$$

Therefore, we obtain

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{S}_F(\nabla^t)\Big|_{t=0} &= \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla D, R^\nabla\big\rangle^2\\ &\quad + F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle [D\wedge D]\,, R^\nabla\big\rangle + F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|d^\nabla D\|^2\,.\end{aligned} \tag{19}$$

Definition 11. For $D \in \Omega^1(\mathfrak{g}_E)$, the index of an F-Yang-Mills field R^∇ is defined as

$$\begin{aligned}I(D) &= \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla D, R^\nabla\big\rangle^2\\ &\quad + \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle [D\wedge D]\,, R^\nabla\big\rangle + \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|d^\nabla D\|^2\,.\end{aligned} \tag{20}$$

If for any $D \in \Omega^1(\mathfrak{g}_E)$, there holds $I(D) \geq 0$, then we call R^∇ stable. Otherwise, it is unstable.

3. Lemmas

For $\varphi \in \Omega^2(\mathfrak{g}_E)$, $\omega \in \Omega^2(M) \otimes \mathrm{Hom}\,(\mathfrak{X}(M), \mathfrak{X}(M))$, let

$$(\varphi \circ \omega)_{X,Y} = \frac{1}{2} \sum \varphi_{e_j, \omega_{X,Y} e_j} \,. \tag{21}$$

We use R to express the Riemannian curvature tensor of M, Ric for the Ricci operator. On M, we take a local orthonormal frame field $\{e_i\}_{i=1,\cdots,n}$, and adopt the Einstein convention of summation. The range of the indices i, j, k, l, m is $\{1, \ldots, n\}$. Let

$$(\mathrm{Ric} \wedge I)_{X,Y} = \mathrm{Ric}(X) \wedge Y + X \wedge \mathrm{Ric}(Y) \tag{22}$$

and

$$\mathfrak{R}^{\nabla}(\varphi)_{X,Y} = \sum \left\{ [R^{\nabla}_{e_j,X}, \varphi_{e_j,Y}] - [R^{\nabla}_{e_j,Y}, \varphi_{e_j,X}] \right\}. \tag{23}$$

Here, $\mathrm{Ric} \wedge I \in \Omega^2(M) \otimes \mathrm{Hom}\,(\mathfrak{X}(M), \mathfrak{X}(M))$, and $X \wedge Y$ is defined as:

$$(X \wedge Y)(Z) = \langle X, Z \rangle\, Y - \langle Y, Z \rangle\, X \,. \tag{24}$$

For any $\varphi \in \Omega^2(\mathfrak{g}_E)$, we have (see [1])

$$\Delta \varphi = \nabla^* \nabla \varphi - \varphi \circ (\mathrm{Ric} \wedge I + 2R) + \mathfrak{R}^{\nabla}(\varphi)\,. \tag{25}$$

Hence we have

$$\frac{1}{2} \Delta \|\varphi\|^2 = \left\langle \Delta^{\nabla} \varphi, \varphi \right\rangle - \|\nabla \varphi\|^2 - \langle \varphi \circ (\mathrm{Ric} \wedge I + 2R), \varphi \rangle - \left\langle \mathfrak{R}^{\nabla}(\varphi), \varphi \right\rangle. \tag{26}$$

Lemma 12. *For an F-Yang-Mills field R^{∇}, we have*

$$\begin{aligned} &\int_M F''\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \|R^{\nabla}\|^2 \|\nabla \|R^{\nabla}\|\|^2 \\ &\qquad + \int_M F'\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \|R^{\nabla}\|^2 + \int_M F'\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \left\langle \mathfrak{R}^{\nabla}(R^{\nabla}), R^{\nabla} \right\rangle \\ &\qquad + \int_M F'\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \left\langle R^{\nabla} \circ (\mathrm{Ric} \wedge I + 2R), R^{\nabla} \right\rangle = 0\,. \end{aligned} \tag{27}$$

Proof. By a straightforward calculation, we get

$$\begin{aligned} \Delta F\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) &= -\sum \nabla_{e_i} \nabla_{e_i} F\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \\ &= -\sum \nabla_{e_i} \Big(F'\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \nabla_{e_i} \frac{\|R^{\nabla}\|^2}{2} \Big) \\ &= -F''\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \|R^{\nabla}\|^2 \|\nabla \|R^{\nabla}\|\,\|^2 - \frac{1}{2} F'\Big(\frac{\|R^{\nabla}\|^2}{2}\Big) \Delta \|R^{\nabla}\|^2 \,. \end{aligned} \tag{28}$$

In (26), taking $\varphi = R^\nabla$, and then substituting the result into (28), we get

$$\begin{aligned}\Delta F\Big(\frac{\|R^\nabla\|^2}{2}\Big) &= -F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\,\|^2 \\ &\quad - F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\left\langle \Re^\nabla(R^\nabla), R^\nabla\right\rangle + F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\left\langle \Delta^\nabla R^\nabla, R^\nabla\right\rangle \\ &\quad - F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|\nabla R^\nabla\|^2 - F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\left\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\right\rangle . \end{aligned} \tag{29}$$

Integrating (29) shows that it is sufficient to prove $\int_M F'\big(\frac{\|R^\nabla\|^2}{2}\big)\langle \Delta^\nabla R^\nabla, R^\nabla\rangle = 0$. By (12) and the Bianchi equality $d^\nabla R^\nabla = 0$, we have

$$\begin{aligned}&\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\langle \Delta^\nabla R^\nabla, R^\nabla\rangle \\ &= \int_M \Big\langle d^\nabla \circ \delta^\nabla R^\nabla, F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)R^\nabla\Big\rangle + \int_M \Big\langle \delta^\nabla \circ d^\nabla R^\nabla, F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)R^\nabla\Big\rangle \\ &= \int_M \Big\langle \delta^\nabla R^\nabla, \delta^\nabla F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)R^\nabla\Big\rangle + \int_M \Big\langle \delta^\nabla \circ d^\nabla R^\nabla, F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)R^\nabla\Big\rangle \\ &= 0\,. \end{aligned} \tag{30}$$

□

Let $\{X_a\}$ be an orthonormal frame of $\mathfrak{g}_E$, and $\{e_i\}$ on M. Let

$$R^\nabla_{e_i,e_j} = f^a_{ij}X_a\,, \quad \big(\nabla_{e_k}R^\nabla\big)_{e_i,e_j} = f^a_{ijk}X_a\,. \tag{31}$$

Then we have $f^a_{ij} = -f^a_{ji}$, $f^a_{ijk} = -f^a_{jik}$, $\|R^\nabla\|^2 = \frac{1}{2}f^a_{ij}f^a_{ij}$, $\|\nabla R^\nabla\|^2 = \frac{1}{2}f^a_{ijk}f^a_{ijk}$.

Lemma 13 ([3])**.** *We have*

(i) *If M^n is a submanifold of $\mathbf{R}^{n+k}$, then*

$$\left\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\right\rangle = \big[(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} - h^\mu_{ik}h^\mu_{jl}\big] f^a_{ij}f^a_{kl}; \tag{32}$$

(ii) *If M^n is a submanifold of S^{n+k}, then*

$$\begin{aligned}&\left\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\right\rangle \\ &\qquad = \big[(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} - h^\mu_{ik}h^\mu_{jl}\big] f^a_{ij}f^a_{kl} + 2(n-2)\|R^\nabla\|^2\,.\end{aligned} \tag{33}$$

4. Stabilities of F-Yang-Mills fields

Theorem 14. *Let M^n be a submanifold of $\mathbf{R}^{n+k}$, which satisfies the following condition:*

$$\big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}\big]\delta_{sr} + 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl} \le b\delta_{ik}\delta_{jl}\delta_{sr}, \tag{34}$$

where $b < 0$. If R^∇ is a nonzero F-Yang-Mills field on M^n, then it is unstable, where for $t > 0$ we have

$$(p-2)F'(t) \ge 2tF''(t)\,, \quad F'(t) > 0\,,\; F(t) > 0\,. \tag{35}$$

Proof. Let X and V be two tangent vectors to M^n, and let $D = i_V R^\nabla$, then we have $D_X = (i_V R^\nabla)_X = R^\nabla_{V,X}$ and

$$\begin{aligned}(d^\nabla D)_{e_i,e_j} &= (\nabla_{e_i} D)_{e_j} - (\nabla_{e_j} D)_{e_i}\\ &= \big(\nabla_{e_i}(i_V R^\nabla)\big)_{e_j} - \big(\nabla_{e_j}(i_V R^\nabla)\big)_{e_i}\\ &= \nabla_{e_i}(i_V R^\nabla)_{e_j} - (i_V R^\nabla)_{\nabla_{e_i} e_j} - \nabla_{e_j}(i_V R^\nabla)_{e_i} + (i_V R^\nabla)_{\nabla_{e_j} e_i}\\ &= \nabla_{e_i}(i_V R^\nabla)_{e_j} - \nabla_{e_j}(i_V R^\nabla)_{e_i} - R^\nabla_{V,\nabla_{e_i} e_j} + R^\nabla_{V,\nabla_{e_j} e_i}\,.\end{aligned}$$

Because

$$\nabla_{e_j}(i_V R^\nabla)_{e_i} = \nabla_{e_i}\big(R^\nabla_{V,e_j}\big) = (\nabla_{e_j} R^\nabla)_{V,e_i} + R^\nabla_{\nabla_{e_j} V, e_i} + R^\nabla_{V,\nabla_{e_j} e_i}\,,$$

we have

$$(d^\nabla D)_{e_i,e_j} = (\nabla_{e_i} R^\nabla)_{V,e_j} - (\nabla_{e_j} R^\nabla)_{V,e_i} + R^\nabla_{\nabla_{e_i} V, e_j} - R^\nabla_{\nabla_{e_j} V, e_i}\,. \tag{36}$$

Let $\{E_A \mid A = 1, 2, \ldots, n+k\}$ be the canonical orthonormal base of $\mathbf{R}^{n+k}$, and write $V_A = v_A^i e_i$ as the tangent part of E_A. Let the indices A, B, C run from 1 to $n+k$, the indices i, j from 1 to n, and the indice μ from $n+1$ to $n+k$. Then we have

$$v_A^B v_A^C = \delta_{BC}\,, \quad \nabla_{e_i} V_A = v_A^\mu h_{ij}^\mu e_j\,. \tag{37}$$

For $D_A = i_{V_A} R^\nabla, A = 1, 2, \ldots, n+k$, according to (20) we get

$$\begin{aligned}\sum_A I_F(D_A) = \sum_A \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla D_A, R^\nabla\big\rangle^2\\ + \sum_A \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle D_A \wedge D_A, R^\nabla\big\rangle\\ + \sum_A \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle d^\nabla D_A, d^\nabla D_A\big\rangle\,.\end{aligned} \tag{38}$$

By (36) and (37), we have

$$\begin{aligned}(d^\nabla D_A)_{e_i,e_j} &= (\nabla_{e_i} R^\nabla)_{V_A,e_j} - (\nabla_{e_j} R^\nabla)_{V_A,e_i} + R^\nabla_{\nabla_{e_i} V_A, e_j} - R^\nabla_{\nabla_{e_j} V_A, e_i}\\ &= v_A^l (\nabla_{e_i} R^\nabla)_{e_l,e_j} - v_A^l (\nabla_{e_j} R^\nabla)_{e_l,e_i} + v_A^\mu h_{il}^\mu R^\nabla_{e_l,e_j} - v_A^\mu h_{jl}^\mu R^\nabla_{e_l,e_i}\,,\end{aligned} \tag{39}$$

from which, we have

$$\begin{aligned}\langle d^\nabla D_A, R^\nabla\rangle &= \frac{1}{2}\big\langle R^\nabla_{e_i,e_j}, (d^\nabla D_A)_{e_i,e_j}\big\rangle\\ &= \frac{1}{2} v_A^l \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_i} R^\nabla)_{e_l,e_j}\big\rangle - \frac{1}{2} v_A^l \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_j} R^\nabla)_{e_l,e_i}\big\rangle\\ &\quad + \frac{1}{2} v_A^\mu h_{il}^\mu \big\langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\big\rangle - \frac{1}{2} v_A^\mu h_{jl}^\mu \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_i}\rangle\\ &= v_A^l \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_i} R^\nabla)_{e_l,e_j}\big\rangle + v_A^\mu h_{il}^\mu \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\big\rangle\,.\end{aligned}$$

According to (37) one has

$$
\begin{aligned}
\sum_A \langle d^\nabla D_A, R^\nabla\rangle^2 &= \big(v_A^l \langle R^\nabla_{e_i,e_j}, (\nabla_{e_i} R^\nabla)_{e_l,e_j}\rangle + v_A^\mu h_{il}^\mu \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\rangle\big) \\
&\quad \times \big(v_A^k \langle R^\nabla_{e_s,e_t}, (\nabla_{e_s} R^\nabla)_{e_k,e_t}\rangle + v_A^\lambda h_{sk}^\lambda \langle R^\nabla_{e_s,e_t}, R^\nabla_{e_k,e_t}\rangle\big) \\
&= \delta_{lk} \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_i} R^\nabla)_{e_l,e_j}\big\rangle \big\langle R^\nabla_{e_s,e_t}, (\nabla_{e_s} R^\nabla)_{e_k,e_t}\big\rangle \\
&\quad + \delta_{\mu\lambda} h_{il}^\mu h_{sk}^\lambda \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\rangle \langle R^\nabla_{e_s,e_t}, R^\nabla_{e_k,e_t}\rangle \\
&= \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_i} R^\nabla)_{e_l,e_j}\big\rangle \big\langle R^\nabla_{e_s,e_t}, (\nabla_{e_s} R^\nabla)_{e_l,e_t}\big\rangle \\
&\quad + h_{il}^\mu h_{sk}^\mu \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\rangle \langle R^\nabla_{e_s,e_t}, R^\nabla_{e_k,e_t}\rangle\,.
\end{aligned} \tag{40}
$$

Taking use of the Bianchi identity, we reach

$$
\begin{aligned}
\big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_i} R^\nabla)_{e_l,e_j}\big\rangle &= -\big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_l} R^\nabla)_{e_j,e_i}\big\rangle - \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_j} R^\nabla)_{e_i,e_l}\big\rangle \\
&= \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_l} R^\nabla)_{e_i,e_j}\big\rangle - \big\langle R^\nabla_{e_j,e_i}, (\nabla_{e_j} R^\nabla)_{e_l,e_i}\big\rangle\,,
\end{aligned}
$$

from which we obtain

$$
\sum_{i,j} \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_i} R^\nabla)_{e_l,e_j}\big\rangle = \frac{1}{2}\sum_{i,j} \big\langle R^\nabla_{e_i,e_j}, (\nabla_{e_l} R^\nabla)_{e_i,e_j}\big\rangle = \langle R^\nabla, \nabla_{e_l} R^\nabla\rangle\,. \tag{41}
$$

Substituting (41) into (40), we have

$$
\begin{aligned}
\sum_A \langle d^\nabla D_A, R^\nabla\rangle^2 &= \sum_l \langle R^\nabla, \nabla_{e_l} R^\nabla\rangle^2 + h_{il}^\mu h_{tm}^\mu \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\rangle \langle R^\nabla_{e_t,e_s}, R^\nabla_{e_m,e_s}\rangle \\
&= \|R^\nabla\|^2 \big\|\nabla \|R^\nabla\|\,\big\|^2 + h_{il}^\mu h_{tm}^\mu \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\rangle \langle R^\nabla_{e_t,e_s}, R^\nabla_{e_m,e_s}\rangle\,.
\end{aligned} \tag{42}
$$

Therefore

$$
\begin{aligned}
\sum_A \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) \langle d^\nabla D_A, R^\nabla\rangle^2 &= \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) \|R^\nabla\|^2 \big\|\nabla\|R^\nabla\|\,\big\|^2 \\
&\quad + \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h_{il}^\mu h_{tm}^\mu \langle R^\nabla_{e_i,e_j}, R^\nabla_{e_l,e_j}\rangle \langle R^\nabla_{e_t,e_s}, R^\nabla_{e_m,e_s}\rangle \\
&= \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) \|R^\nabla\|^2 \big\|\nabla\|R^\nabla\|\,\big\|^2 \\
&\quad + \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h_{il}^\mu h_{tm}^\mu f_{ij}^a f_{lj}^a f_{ts}^b f_{ms}^b\,,
\end{aligned} \tag{43}
$$

where f_{ij}^a's are the components of $R^\nabla_{e_i,e_j}$. Because

$$
h_{il}^\mu h_{tm}^\mu f_{ij}^a f_{lj}^a f_{ts}^b f_{ms}^b = h_{ik}^\mu h_{sr}^\mu f_{ij}^a f_{kj}^a f_{st}^b f_{rt}^b = h_{ik}^\mu h_{sr}^\mu \delta_{jl} f_{ij}^a f_{kl}^a f_{st}^b f_{rt}^b\,,
$$

we have

$$\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h^\mu_{il} h^\mu_{tm} f^a_{ij} f^a_{lj} f^b_{ts} f^b_{ms} = \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h^\mu_{ik} h^\mu_{sr} \delta_{jl} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\,. \tag{44}$$

Substituting (44) into (43), we have

$$\sum_A \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) \langle d^\nabla D_A, R^\nabla\rangle^2 = \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) \|R^\nabla\|^2 \big\|\nabla\|R^\nabla\|\,\big\|^2 + \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h^\mu_{ik} h^\mu_{sr} \delta_{jl} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\,. \tag{45}$$

Now, we calculate the third term at the right hand side of (38). By (39), we get

$$\begin{aligned}\sum_A \|d^\nabla D_A\|^2 &= \frac{1}{2}\sum_A \big\langle (d^\nabla D_A)_{e_i,e_j}, (d^\nabla D_A)_{e_i,e_j}\big\rangle \\ &= f^a_{ijk} f^a_{ijk} - f^a_{kji} f^a_{kij} + h^\mu_{ik} h^\mu_{il} f^a_{kj} f^a_{lj} - h^\mu_{ik} h^\mu_{jl} f^a_{kj} f^a_{li}\,,\end{aligned}$$

where, f^a_{ijk}'s are the components of $\nabla_{e_k} R^\nabla_{e_i,e_j}$. From the Bianchi identity, we have $f^a_{kji} f^a_{kij} = \frac{1}{2} f^a_{ijk} f^a_{ijk} = \|\nabla R^\nabla\|^2$. Hence we have

$$\sum_A \|d^\nabla D_A\|^2 = \|\nabla R^\nabla\|^2 + (h^\mu_{ik} h^\mu_{il} f^a_{kj} f^a_{lj} - h^\mu_{ik} h^\mu_{jl} f^a_{kj} f^a_{li})\,, \tag{46}$$

from which we arrive at

$$\sum_A \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big) \langle d^\nabla D_A, d^\nabla D_A\rangle = \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big) \|\nabla R^\nabla\|^2 + \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big) \big(h^\mu_{ik} h^\mu_{il} f^a_{kj} f^a_{lj} - h^\mu_{ik} h^\mu_{jl} f^a_{kj} f^a_{li}\big)\,. \tag{47}$$

Then we calculate the second term at the right hand side of (38). By a direct computation, we have

$$\begin{aligned}\langle D_A \wedge D_A, R^\nabla\rangle &= \frac{1}{2}\big\langle [D_A \wedge D_A]_{e_j,e_k}, R^\nabla_{e_j,e_k}\big\rangle \\ &= \big\langle [D_{A,e_j}, D_{A,e_k}], R^\nabla_{e_j,e_k}\big\rangle \\ &= -\big\langle [D_{A,e_k}, D_{A,e_j}], R^\nabla_{e_j,e_k}\big\rangle \\ &= -\big\langle [R_{V_A,e_k}, R_{V_A,e_j}], R^\nabla_{e_j,e_k}\big\rangle \\ &= -v^i_A v^l_A \big\langle [R_{e_i,e_k}, R_{e_l,e_j}], R^\nabla_{e_j,e_k}\big\rangle \\ &= -\big\langle [R_{e_i,e_k}, R_{e_i,e_j}], R^\nabla_{e_j,e_k}\big\rangle \\ &\equiv \langle \Re^\nabla(R^\nabla), R^\nabla\rangle\,,\end{aligned}$$

from which, we get

$$\sum_A \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\langle D_A \wedge D_A, R^\nabla\rangle = \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle \Re^\nabla(R^\nabla), R^\nabla\big\rangle\,. \tag{48}$$

Inserting (45), (47) and (48) into (38) yields

$$\begin{aligned}\sum_A I_F(D_A) = &\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\,\|^2\\ &+\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|\nabla R^\nabla\|^2+\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle \Re^\nabla(R^\nabla), R^\nabla\big\rangle\\ &+\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)h^\mu_{ik}h^\mu_{sr}\delta_{jl}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt}\\ &+\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj}-h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li}\big)\,.\end{aligned}$$

Applying Lemma 12, we can get

$$\begin{aligned}\sum_A I_F(D_A) = &-\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big\langle R^\nabla\circ(\mathrm{Ric}\wedge I+2R^\nabla), R^\nabla\big\rangle\\ &+\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)h^\mu_{ik}h^\mu_{sr}\delta_{jl}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt}\\ &+\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj}-h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li})\,.\end{aligned} \tag{49}$$

Then use Lemma 13(i) and (44), and we obtain

$$\begin{aligned}\sum_A I_F(D_A) = &\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big[-(H^\mu h^\mu_{jl}-h^\mu_{jm}h^\mu_{ml})\delta_{ki}+h^\mu_{ik}h^\mu_{jl}\big]f^a_{ij}f^a_{kl}\\ &+\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj}-h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li}\big)\\ &+\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)h^\mu_{ik}h^\mu_{sr}\delta_{jl}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt}\,.\end{aligned} \tag{50}$$

Since

$$\begin{aligned}h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} &= h^\mu_{ki}h^\mu_{il}f^a_{jk}f^a_{jl} = h^\mu_{km}h^\mu_{ml}f^a_{jk}f^a_{jl} = h^\mu_{jm}h^\mu_{ml}f^a_{kj}f^a_{kl}\\ &= h^\mu_{jm}h^\mu_{ml}\delta_{ki}f^a_{ij}f^a_{kl},\,,\end{aligned} \tag{51}$$

$$-h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li} = -h^\mu_{ik}h^\mu_{lj}f^a_{kl}f^a_{ji} = h^\mu_{ik}h^\mu_{lj}f^a_{kl}f^a_{ij}, \tag{52}$$

the inequality (50) becomes as

$$\begin{aligned}\sum_A I_F(D_A) = &\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}\big] f^a_{ij} f^a_{kl}\\ &+ \int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h^\mu_{ik}h^\mu_{sr}\delta_{jl} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\,.\end{aligned}$$

Let $M_0 = \{x \in M : R^\nabla(x) = 0\}$. Apparently, $M\backslash M_0$ is an open set of M and $M_0 = M$ is equivalent to $R^\nabla \equiv 0$. If $R^\nabla \neq 0$, and note that $2\|R^\nabla\|^2 = f^b_{rq} f^b_{rq} = \delta_{sr} f^b_{st} f^b_{rt}$, then we have

$$\begin{aligned}\sum_A I_F(D_A) = &\int_{M\backslash M_0} \frac{1}{2\|R^\nabla\|^2} F'\Big(\frac{\|R^\nabla\|^2}{2}\Big) 2\|R^\nabla\|^2\\ &\times \big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}\big] f^a_{ij} f^a_{kl}\\ &+ \int_{M\backslash M_0} F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h^\mu_{ik}h^\mu_{sr}\delta_{jl} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\\ = &\int_{M\backslash M_0} \frac{1}{2\|R^\nabla\|^2} F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\\ &\times \big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}\big] \delta_{sr} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\\ &+ \int_{M\backslash M_0} F''\Big(\frac{\|R^\nabla\|^2}{2}\Big) h^\mu_{ik}h^\mu_{sr}\delta_{jl} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\,.\end{aligned}$$

In order to make $\sum I_F(D_A)$ negative, we must assume that F' and F'' have some relations. Because $\sum h^\mu_{ik}h^\mu_{sr}\delta_{jl} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt} = \sum h^\mu_{ik} f^a_{kj} f^a_{ji} \sum h^\mu_{sr} f^b_{rt} f^b_{ts} \geq 0$, we can assume that $F''(t) \leq \frac{(p-2)F'(t)}{2t}$. In this case we have

$$\begin{aligned}\sum_A I_F(D_A) \leq &\int_{M\backslash M_0} \frac{1}{2\|R^\nabla\|^2} F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\\ &\times \big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}\big] \delta_{sr} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\\ &+ \int_{M\backslash M_0} \frac{p-2}{\|R^\nabla\|^2} F'\Big(\frac{\|R^\nabla\|^2}{2}\Big) h^\mu_{ik}h^\mu_{sr}\delta_{jl} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\\ = &\int_{M\backslash M_0} \frac{1}{2\|R^\nabla\|^2} F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\\ &\times \big\{\big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr}\big]\\ &+ 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\big\} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\,.\end{aligned}$$

Let $C_{ijklrs} = \big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr}\big] + 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}$, then by the assumption of the theorem we have

$$\begin{aligned}\sum_A I_F(D_A) &\leq \int_{M\setminus M_0} \frac{1}{2\|R^\nabla\|^2}F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)C_{ijklsr}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt} \\ &\leq \int_{M\setminus M_0} \frac{1}{2\|R^\nabla\|^2}F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)b\delta_{ik}\delta_{jl}\delta_{sr}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt} \\ &\leq \int_{M\setminus M_0} \frac{b}{2\|R^\nabla\|^2}F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)f^a_{ij}f^a_{ij}f^b_{st}f^b_{st} \\ &= \int_{M\setminus M_0} 2bF'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^2 < 0\,,\end{aligned}$$

which is a contradiction to the stability of R^∇. Therefore we have $R^\nabla \equiv 0$. □

Remark 15. We have used the condition (5) in the above proof, which is a technical assumption. This condition covers many important cases, but don't covers the exponential Yang-Mills fields. We plan to discuss the exponential Yang-Mills fields elsewhere.

Corollary 16. *Let M^n be a hypersurface of $\mathbf{R}^{n+1}$, the principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$ of which satisfies the following condition:*

$$H\lambda_i > 2\lambda_i\lambda_j + 2\lambda_j^2 + 2(p-2)\lambda_i\lambda_s\,, \tag{53}$$

where $H = \sum_i \lambda_i$. If for $t > 0$ we have

$$(p-2)F'(t) \geq 2tF''(t)\,, \qquad F'(t) > 0\,,\ F(t) > 0\,,$$

then any nonzero F-Yang-Mills field R^∇ on M must be unstable.

Especially, if $M^n = S^n \subset \mathbf{R}^{n+1}$ and $n > 2p$, then any nonzero F-Yang-Mills field R^∇ on S^n must be unstable.

Proof. Let $h^{n+1}_{ij} = \lambda_i\delta_{ij}$ and $H = H^{n+1} = \sum_i \lambda_i$, then for fixed i, j, k, s, r, t, q we have

$$C_{ijklsr} = \big[\big(-H\lambda_j + 2\lambda_j\lambda_l + 2\lambda_i\lambda_j\big) + 2(p-2)\lambda_i\lambda_s\big]\delta_{ik}\delta_{jl}\delta_{sr}\,,$$

from which we get

$$\begin{aligned}\sum_A I_F(D_A) &\leq \int_{M\setminus M_0} \frac{1}{2\|R^\nabla\|^2}F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)C_{ijklsr}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt} \\ &= \int_{M\setminus M_0} \frac{1}{2\|R^\nabla\|^2}F'\Big(\frac{\|R^\nabla\|^2}{2}\Big) \\ &\quad \times \big[\big(-H\lambda_j + 2\lambda_j\lambda_l + 2\lambda_i\lambda_j\big) + 2(p-2)\lambda_i\lambda_s\big]\delta_{ik}\delta_{jl}\delta_{sr}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt} \\ &= 2\int_{M\setminus M_0} F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big[\big(-H\lambda_j + 2\lambda_j^2 + 2\lambda_i\lambda_j\big) + 2(p-2)\lambda_i\lambda_s\big]\|R^\nabla\|^2 < 0\,.\end{aligned}$$

This is impossible because of the stability unless $M = M_0$ i.e. $R^\nabla \equiv 0$. The theorem follows.

Especially, for $M^n = S^n \subset \mathbf{R}^{n+1}$ we have $\lambda_i = 1$. The condition (53) becomes as $n > 2p$. $\square$

Theorem 17. *Let M^n be a submanifold of S^{n+k}, satisfying that*

$$\big[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}\big]\delta_{sr} + 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl} < (n-2p)\delta_{ik}\delta_{jl}\delta_{sr}\,,$$

where $p > 2$ and h^μ_{ij} is the components of the second fundamental form h of M^n in S^{n+k}. Then, any nonzero F-Yang-Mills field R^∇ on M is unstable if

$$(p-2)F'(t) \geq 2tF''(t)\,, \qquad F'(t) \geq 0\,,\ F(t) > 0\,.$$

Proof. The proof is similar to that of Theorem 14, but Lemma 13(ii) instead of (i) is used to calculate the curvature.

By Lemma 13 (ii), we can get the first term of (48) as follows:

$$\begin{aligned}
&-\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R^\nabla), R^\nabla\rangle \\
=& \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big[-(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}\big] f^a_{ij}f^a_{kl} \\
&- 2(n-2)\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^2\,.
\end{aligned}$$

Note that in the second and the third terms of (48), h^μ_{ij} is the second fundamental tensor of M in $\mathbf{R}^{n+k}$. But in Theorem 17, h^μ_{ij} is the second fundamental tensor of M in S^{n+k}. Because S^{n+k} is a hypersurface of $\mathbf{R}^{n+k+1}$, M can be view as a submanifold of $\mathbf{R}^{n+k+1}$, whose second fundamental tensor has two parts: one is that of M in S^{n+k} which is also denoted by h^μ_{ij}, another is that of S^{n+k} in $\mathbf{R}^{n+k+1}$ which is $h^{n+k+1}_{ij} = \delta_{ij}$ in an appropriate local frame field. Hence the second and the third terms of (48) become respectively as

$$\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)h^\mu_{ik}h^\mu_{sr}\delta_{jl}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt} + 4\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^4$$

and

$$\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} - h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li}\big) + 4\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^2\,.$$

Therefore we have

$$\begin{aligned}
\sum_A I_F(D_A) =& \int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\big[-(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}\big] f^a_{ij}f^a_{kl} \\
&- 2(n-2)\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^2
\end{aligned}$$

$$+\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)h^\mu_{ik}h^\mu_{sr}\delta_{jl}f^a_{ij}f^a_{kl}f^b_{st}f^b_{rt}+4\int_M F''\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^4$$

$$+\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj}-h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li})+4\int_M F'\Big(\frac{\|R^\nabla\|^2}{2}\Big)\|R^\nabla\|^2\,. \tag{54}$$

The rest discuss is similar to the proof of Theorem 14, so we omit the details. □

Similar to Corollary 16, we have

Corollary 18. *Let M^n be a hypersurface of S^{n+1}, the principal curvatures $\lambda_1,\lambda_2,\dots,\lambda_n$ of which satisfies the following condition:*

$$H\lambda_i > 2\lambda_i\lambda_j+2\lambda_j\lambda_l+2(p-2)\lambda_i\lambda_s-(n-2p), \tag{55}$$

where $H=\sum_i\lambda_i$. If for $t>0$ we have

$$(p-2)F'(t)\geq 2tF''(t)\,,\qquad F'(t)>0\,,\ F(t)>0\,,$$

then any nonzero F-Yang-Mills field R^∇ on M must be unstable.

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