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TANGENT LIFTS OF HIGHER ORDER OF MULTIPLICATIVE DIRAC STRUCTURES

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Abstract. The tangent lifts of higher order of Dirac structures and some properties have been defined in [9] and studied in [11]. By the same way, the tangent lifts of higher order of Poisson structures have been studied in [10] and some applications are given. In particular, the authors have studied the nature of the Lie algebroids and singular foliations induced by these lifting. In this paper, we study the tangent lifts of higher order of multiplicative Poisson structures, multiplicative Dirac structures and we describe the Lie bialgebroid structures and the algebroid-Dirac structures induced by these prolongations.

Introduction

We denote by $\mathcal{LG}$ and $\mathcal{LA}$ the categories of Lie groupoids and Lie algebroids, respectively. There is a natural functor $A\colon \mathcal{LG} \to \mathcal{LA}$, which maps each object $G \in \mathcal{LG}$ to the object $AG \in \mathcal{LA}$, and every morphism of Lie groupoids $\phi\colon G_1 \to G_2$ is mapped to the Lie algebroid morphism $A\phi\colon AG_1 \to AG_2$. It is called the Lie functor and preserves the product bundles. Let G be a Lie groupoid over a manifold M, we denote by $G_{(2)}$ the set of composable groupoid pairs and we recall that, a Poisson groupoid is a pair $(G;\Pi_G)$ where Π_G is a Poisson structure on G which is multiplicative in the sense that the graph of the multiplication map

$$\Lambda = \left\{(g,\, h,\, gh)\,, (g,h) \in G_{(2)}\right\}$$

is a coisotropic submanifold of $G \times G \times \overline{G}$, where $\overline{G}$ means that G is equipped with the opposite Poisson structure. We say that the bivector Π_G is a multiplicative bivector. On the other hand, it is well known that Lie bialgebroid is a

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pair of Lie algebroids in duality (E, E^*) satisfying

$$d_{E^*}\left([u,v]\right) = \left[d_{E^*}\left(u\right), v\right] + \left[u, d_{E^*}\left(v\right)\right]$$

for any $u, v \in \Gamma(E)$. Here $d_{E^*}\colon \Gamma\big(\bigwedge^k E\big) \to \Gamma\big(\bigwedge^{k+1} E\big)$ denotes the Lie algebroid differential induced by E^* and $[\cdot,\cdot]$ is the Schouten bracket on multisections of E. A classical example of a Lie bialgebroid is a Lie bialgebra. As a Lie bialgebras arise as the infinitesimal counterpart of Poisson-Lie groups, K. Mackenzie and P. Xu have shown that, the Lie bialgebroids are the infinitesimal version of Poisson groupoids (see [12]). More precisely, if $(G; \Pi_G)$ is a Poisson groupoid, then $(AG,\, A^*G)$ is a Lie bialgebroid.

Let G be a Lie groupoid over a manifold M, with Lie algebroid AG. The tangent bundle of order r T^rG has a natural Lie groupoid structure over T^rM. This structure is obtained by applying the tangent functor of order r to each of the structure maps defining G (source, target, multiplication, inversion and identity section). In the particular case where $r = 1$, we obtain the tangent Lie groupoid.

Consider now the cotangent bundle T^*G over G, we know that T^*G is a Lie groupoid over A^*G. The source and target maps are defined by:

$$s^*\left(\gamma_g\right)(u) = \gamma_g\big(TL_g\left(u - Tt\left(u\right)\right)\big) \quad \text{and} \quad t^*\left(\delta_g\right)(v) = \delta_g\big(TR_g(v)\big)$$

where $\gamma_g \in T^*_gG$, $u \in A_{s(g)}G$ and $\delta_g \in T^*_gG$, $v \in A_{t(g)}G$. The multiplication on T^*G is defined by:

$$\left(\beta_g \bullet \gamma_h\right)\left(X_g \centerdot X_h\right) = \beta_g\left(X_g\right) + \gamma_h\left(X_h\right)$$

for $(X_g,\, X_h) \in T_{(g,h)}G_{(2)}$. In [13], the author defines the cotangent Lie algebroid and proves that: There is a natural isomorphism of Lie algebroids

$$j_G\colon A\left(T^*G\right) \to T^*\left(AG\right)$$

such that the following diagram commutes

$$\begin{tikzcd}
A(T^*G) \arrow[r, "j_G"] \arrow[d, "i_{A(T^*G)}"'] & T^*(AG) \\
TT^*G \arrow[r, "\varepsilon_G"'] & T^*TG \arrow[u, "T^*i_{AG}"']
\end{tikzcd}$$

where $i_{AG}\colon AG \to TG$ is the natural injection and $\varepsilon_G\colon TT^*G \to T^*TG$ is the natural isomorphism of Tulczyjew. By this isomorphism, we identify $A\left(T^*G\right)$ with $T^*\left(AG\right)$. Let (E, M, π) be a vector bundle, we denote by $\left(x^i, y^j\right)$ an adapted coordinates system of E, it induces the local coordinates system

$$\left(x^i, \pi_j\right) \quad \text{in} \quad E^*$$

$$\left(x^i, y^j, p_i, \zeta_j\right) \quad \text{in} \quad T^*E$$

and

$$\left(x^i, \pi_j, p_i, \xi^j\right) \quad \text{in} \quad T^*E^*$$

In [13], [12] is defined the natural submersion $r_E \colon T^*E \to E^*$ such that locally

$$r_E\left(x^i, y^j, p_i, \zeta_j\right) = \left(x^i, \zeta_j\right) .$$

There exists a Legendre type map

$$R_E \colon T^*E^* \to T^*E$$

which is an anti-symplectomorphism with respect to the canonical symplectic structures on T^*E^* and T^*E respectively, and is locally defined by:

$$R_E\left(x^i, \pi_j, p_i, \xi^j\right) = \left(x^i, y^j, -p_i, \zeta_j\right)$$

with

$$\begin{cases} y^j = \xi^j \\ \pi_j = \zeta_j \end{cases}$$

In this paper, we study the tangent lifts of higher order of multiplicative Poisson structures, multiplicative Dirac structures and we study some properties. In particular, we describe the structures of Dirac-algebroids induced by the tangent lifts of higher order of multiplicative Dirac structures. Thus, the main results are Propositions 3, 8, Theorems 1 and 2.

For the prolongations of functions, vector fields and differential forms, we adopt the same notations of [14]. More precisely, for any $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$ we denote by $X^{(\alpha)}$ and $\omega^{(\alpha)}$ the α-prolongations of X and ω respectively. All geometric objects and maps are assumed to be infinitely differentiable. r will be a natural integer ($r \geq 1$).

1. Preliminaries

1.1. **Some canonical transformations.**

For each $\beta \in \{0, \ldots, r\}$, we denote by τ_β the canonical linear form on $J_0^r(\mathbb{R}, \mathbb{R})$ defined by:

$$\tau_\beta(j_0^r g) = \frac{1}{\beta!} \cdot \frac{d^\beta}{dt^\beta}\big(g(t)\big)|_{t=0} \, , \quad \text{where} \quad g \in C^\infty(\mathbb{R}, \mathbb{R}) \, .$$

For each manifold M, there is a canonical diffeomorphism (see [5], [8])

$$\kappa_M^r \colon T^rTM \to TT^rM$$

which is an isomorphism of vector bundles from

$$T^r(\pi_M)\colon T^rTM \to T^rM \quad \text{to} \quad \pi_{T^rM}\colon TT^rM \to T^rM\,.$$

It is called the canonical isomorphism of flow associated to the bundle functor T^r. Let $(x^1, \dots, x^m)$ be a local coordinates system of M, we introduce a coordinates $\left(x^i, \dot{x}^i\right)$ in TM, $(x^i, \dot{x}^i, x^i_\beta, \dot{x}^i_\beta)$ in T^rTM and $(x^i, x^i_\beta, \dot{x}^i, \dot{x^i_\beta})$ in TT^rM. The local expression of κ_M^r is given by:

$$\kappa_M^r(x^i, \dot{x}^i, x^i_\beta, \dot{x}^i_\beta) = (x^i, x^i_\beta, \dot{x}^i, \dot{x^i_\beta}) \quad \text{with} \quad \dot{x^i_\beta} = \dot{x}^i_\beta\,.$$

By the same way, there is a canonical diffeomorphism (see [1])

$$\alpha_M^r \colon T^*T^rM \to T^rT^*M$$

which is an isomorphism of vector bundles

$$\pi^*_{T^rM}\colon T^*T^rM \to T^rM \quad \text{and} \quad T^r(\pi^*_M)\colon T^rT^*M \to T^rM$$

dual of κ_M^r with respect to pairings $\langle\cdot,\cdot\rangle'_{T^rM} = \tau_r \circ T^r(\langle\cdot,\cdot\rangle_M)$ and $\langle\cdot,\cdot\rangle_{T^rM}$, i.e. for any $(u, u^*) \in T^rTM \oplus T^*T^rM$,

$$\langle \kappa_M^r(u), u^*\rangle_{T^rM} = \langle u, \alpha_M^r(u^*)\rangle'_{T^rM}\,. \tag{1.1}$$

Let $(x^1, \dots, x^m)$ be a local coordinates system of M, we introduce the coordinates (x^i, p_j) in T^*M, $(x^i, p_j, x^i_\beta, p^\beta_j)$ in T^rT^*M and $(x^i, x^i_\beta, \pi_j, \pi^\beta_j)$ in T^*T^rM. We have:

$$\alpha_M^r\left(x^i, \pi_j, x^i_\beta, \pi^\beta_j\right) = \left(x^i, x^i_\beta, p_j, p^\beta_j\right)$$

with

$$\begin{cases} p_j = \pi^r_j \\ p^\beta_j = \pi^{r-\beta}_j \end{cases}.$$

By ε_M^r we denote the map $(\alpha_M^r)^{-1}$.

Remark 1. In the particular case where $r = 1$, the canonical isomorphism $\alpha^1_M \colon T^*TM \to TT^*M$ coincides with the canonical isomorphism of Tulczyjew.

1.2. **Tangent lifts of higher order of Poisson manifolds.**
We recall that in [10], it is defined for each integer $q \geq 1$, the natural transformations

$$\kappa^{r,q} : T^r \circ (\bigwedge^q T) \to \bigwedge^q T \circ T^r \tag{1.2}$$

such that, for each manifold M of dimension m, we have locally:

$$\kappa_M^{r,q}(x_\alpha^i, \Pi_\alpha^{i_1 \dots i_q}) = (x_\alpha^i, \widetilde{\Pi}^{i_1,\alpha_1 \dots i_q,\alpha_q}) \tag{1.3}$$

with

$$\widetilde{\Pi}^{i_1,\alpha_1 \dots i_q,\alpha_q} = \sum_{\gamma_1+\cdots+\gamma_q+\beta=r} \delta_{\alpha_1}^{r-\gamma_1} \dots \delta_{\alpha_q}^{r-\gamma_q} \Pi_\beta^{i_1 \dots i_q} . \tag{1.4}$$

Let Π be a multivector field of degree q on M. We put

$$\Pi^{(c)} = \kappa_M^{r,q} \circ T^r \Pi \colon T^r M \to \bigwedge^q T T^r M . \tag{1.5}$$

$\Pi^{(c)}$ is a multivector field of degree q on $T^r M$. Therefore, if locally

$$\Pi = \sum_{1 \leq i_1 < \cdots < i_q \leq m} \Pi^{i_1 \dots i_q} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_q}}$$

then, we have

$$\Pi^{(c)} = \sum_{\alpha_1+\cdots+\alpha_q+\mu=r} (\Pi^{i_1 \cdots i_q})^{(\mu)} \frac{\partial}{\partial x_{r-\alpha_1}^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{r-\alpha_q}^{i_q}} . \tag{1.6}$$

In the particular case where $q = 2$, and $\Pi = \Pi^{ij} \dfrac{\partial}{\partial x^i} \wedge \dfrac{\partial}{\partial x^j}$ we have:

$$\Pi^{(c)} = (\Pi^{ij})^{(\alpha+\beta-r)} \frac{\partial}{\partial x_\alpha^i} \wedge \frac{\partial}{\partial x_\beta^j} . \tag{1.7}$$

For a simple k-vector field of $\Pi = X_1 \wedge \cdots \wedge X_k$ with $X_1, \dots, X_k \in \mathfrak{X}(M)$, we have

$$\Pi^{(c)} = \sum_{\beta_1+\cdots+\beta_k=r} X_1^{(r-\beta_1)} \wedge \cdots \wedge X_k^{t(r-\beta_k)} . \tag{1.8}$$

Using the equation (1.8), we prove in [10] the following equality, for any $\Phi \in \mathfrak{X}^p(M)$ and $\Psi \in \mathfrak{X}^q(M)$, we have:

$$\left[\Phi^{(c)}, \Psi^{(c)}\right] = [\Phi, \Psi]^{(c)} . \tag{1.9}$$

Thus, if (M, Π_M) is a Poisson manifold then, the pair $(T^r M, \Pi_M^{(c)})$ is also a Poisson manifold. This structure is called the tangent lift of order r of Poisson structure.

Proposition 1. *Let (M, Π_M) be a Poisson manifold. We denote by $\sharp_{\Pi_M}$ the anchor map induced by Π_M. We have the following formula*

$$\sharp_{\Pi_M^{(c)}} = \kappa_M^r \circ T^r(\sharp_{\Pi_M}) \circ \alpha_M^r \,. \tag{1.10}$$

Proof. See [10]. □

For some properties of the Poisson manifold $(T^r M, \Pi_M^{(c)})$ see [10]. however, these fundamental properties come from the formula:

$$i_{\Pi_M^{(c)}} \omega^{(r-\alpha)} = (i_{\Pi_M} \omega)^{(\alpha)}$$

where $\omega \in \Omega^1(M)$ and $\alpha \in \{0, 1, \dots, r\}$.

1.3. **Tangent lifts of higher order of Lie algebroids.**
For any vector bundle (E, M, π), we consider the natural vector bundle morphism

$$\chi_E^{(\alpha)} \colon T^r E \to T^r E$$

defined for each $j_0^r \varphi \in T^r E$ by:

$$\chi_E^{(\alpha)}(j_0^r \varphi) = j_0^r(t^\alpha \varphi) \,.$$

Let $u \in \Gamma(E)$, we define the section $u^{(\alpha)}$ of $(T^r E, T^r M, T^r \pi)$ by:

$$u^{(\alpha)} = \chi_E^{(\alpha)} \circ T^r u \,, \qquad 0 \le \alpha \le r \,.$$

$u^{(\alpha)}$ is called α-prolongation of the section u (see [5] or [16]).
We suppose that E is a Lie algebroid over M of anchor map ρ. In [10], we have shown that: it exists one and only one Lie algebroid structure on $T^r E$, of anchor map $\rho^{(r)} = \kappa_M^r \circ T^r \rho$ such that: for any u, $v \in \Gamma(E)$ and $\alpha, \beta \in \{0, \dots, r\}$ we have:

$$[u^{(\alpha)}, v^{(\beta)}] = [u, v]^{(\alpha+\beta)} \,. \tag{1.11}$$

This Lie algebroid structure is called tangent lift of order r of Lie algebroid $(E, [\cdot,\cdot], \rho)$. For some properties of the Lie algebroid $(T^r E, [\cdot,\cdot], \rho^{(r)})$, see [10]. In particular for $r = 1$, we obtain the tangent lift of Lie algebroid $(E, [\cdot,\cdot], \rho)$ defined in [4].
For $s \in \{1, \dots, r\}$, we consider the natural projection $\pi_E^{r,s} \colon T^r E \to T^s E$ defined by:

$$\pi_E^{r,s}(j_0^r \varphi) = j_0^s \varphi \,.$$

For any $u \in \Gamma(E)$ and natural number $\alpha \le r$, we have:

$$\pi_E^{r,s}(u^{(\alpha)}) = \begin{cases} u^{(\alpha)} & \text{if} \quad \alpha \le s \\ 0 & \text{if} \quad \alpha > s \end{cases} \,. \tag{1.12}$$

In this case, we have the following result:

Proposition 2. *The vector bundle projection* $\pi_E^{r,s}\colon T^rE \to T^sE$ *is a morphism of Lie algebroids over* $\pi_M^{r,s}\colon T^rM \to T^sM$.

Proof. The property of compatibility of Lie bracket between $\Gamma(T^rE)$ and $\Gamma(T^sE)$ is obtained by the formula (1.12). Since, for any $u \in \Gamma(E)$ and $\alpha \in \{0,\dots,r\}$,

$$\begin{aligned} T(\pi_M^{r,s}) \circ \rho^{(r)}(u^{(\alpha)}) &= T(\pi_M^{r,s}) \circ [\rho(u)]^{(\alpha)} \\ &= T(\pi_M^{r,s}) \circ \big(\kappa_M^r \circ \chi_{TM}^{(\alpha)} \circ T^r[\rho(u)]\big) \\ &= \kappa_M^s \circ \pi_{TM}^{r,s}\big(j_0^r(t^\alpha[\rho(u)])\big) \\ &= \rho^{(s)} \circ \pi_A^{r,s}(u^{(\alpha)}) \end{aligned}$$

we deduce that, the projection $\pi_E^{r,s}\colon T^rE \to T^sE$ is a morphism of Lie algebroids over $\pi_M^{r,s}$. □

Remark 2. In the particular case where $s = 1$, the bundle projection $\pi_E^{r,1}\colon T^rE \to TE$ is a morphism of Lie algebroids over $\pi_M^{r,1}\colon T^rM \to TM$. In [12], it is shown that, the bundle map (tangent projection) $\tau_E\colon TE \to E$ is a morphism of Lie algebroids over $\tau_M\colon TM \to M$, therefore we have:

Corollary 1. *The vector bundle projection* $\tau_E^r : T^rE \to E$ *is a morphism of Lie algebroids over* $\tau_M^r : T^rM \to M$.

Let (E, M, π) be a vector bundle. Consider the canonical pairing $E^* \times_M E \to \mathbb{R}$. Applying the tangent functor of order r and projecting onto the $(r+1)$-component, we get a non degenerate pairing $T^rE^* \times_{T^rM} T^rE \to \mathbb{R}$. We use this pairing to define an isomorphism of vector bundles

$$I_E^r\colon T^rE^* \to (T^rE)^* . \tag{1.13}$$

Theorem 1. *Let* $(E, [\cdot,\cdot], \rho)$ *be a Lie algebroid. The natural vector bundle*

$$\alpha_E^r\colon T^*T^rE \to T^rT^*E$$

is an isomorphism of Lie algebroids over the canonical isomorphism I_E^r.

Proof. As E is a Lie algebroid, it follows that E^* has a Poisson structure. Therefore, the vector bundle morphism $\alpha_{E^*}^r\colon T^*T^rE^* \to T^rT^*E^*$ is an isomorphism of Lie algebroids (see [10]). The rest of the proof comes from the commutative diagram

$$\begin{array}{ccc} T^*T^rE^* & \xrightarrow{\alpha_{E^*}^r} & T^rT^*E^* \\ \Big\downarrow{\scriptstyle \widehat{R}_{T^rE}} & & \Big\downarrow{\scriptstyle T^rR_E} \\ T^*T^rE & \xrightarrow[\alpha_E^r]{} & T^rT^*E \end{array}$$

Where $R_E\colon T^*E^* \to T^*E$ is the Legendre map and $\widehat{R}_{T^rE} = R_{T^rE} \circ T^*((I_E^r)^{-1})$. □

Let $(G \rightrightarrows M)$ be a Lie groupoid. The vector bundle morphism $\kappa_G^r\colon T^rTG \to TT^rG$ is an isomorphism of Lie groupoids. So, it induces the isomorphism of vector bundles

$$j_G^r\colon T^r(AG) \to A(T^rG) \tag{1.14}$$

such that, the following diagram commutes

$$\begin{tikzcd} T^r(AG) \arrow[r, "j_G^r"] \arrow[d, "T^r i_{AG}"'] & A(T^rG) \arrow[d, "i_{A(T^rG)}"] \\ T^rTG \arrow[r, "\kappa_G^r"'] & TT^rG \end{tikzcd}$$

In [7], it is defined the natural isomorphism (1.14) by the replacement of tangent functor of order r, by any Weil functor.

2. Tangent lifts of higher order of Lie bialgebroids and some properties

2.1. **Lifting of Lie bialgebroids.**

Proposition 3. *Let (A, A^*) be a Lie bialgebroid. The pair $(T^rA, (T^rA)^*)$ has a canonical structure of Lie bialgebroid.*

Proof. It is well-know that, if (A, A^*) is a pair of Lie algebroids and Π_A be a linear Poisson bivector on A defined by the Lie algebroid A^*, then (A, A^*) is a Lie bialgebroids if and only if

$$T\pi \circ \sharp_{\Pi_A} = \rho_{A^*} \circ r_A\,.$$

Since A and A^* are the Lie algebroids, it follows that T^rA and T^rA^* are also Lie algebroids. The structure of Lie algebroid of $(T^rA)^*$ is such that the map $I_A^r\colon T^rA^* \to (T^rA)^*$ is an isomorphism of Lie algebroids. In this case, we have two structures of Poisson manifolds $\Pi_A^{(c)}$ and Π_{T^rA} on T^rA. By calculation in local coordinates, we deduce that $\Pi_A^{(c)} = \Pi_{T^rA}$. In fact, let $(u_i)_{i=1\ldots,n}$ be a basis of sections of A^* the local expression of Lie bracket of sections and anchor map are given by:

$$[u_i, u_j] = c_{ij}^k u_k \quad \text{and} \quad \rho_{A^*}(u_j) = \rho_j^i \frac{\partial}{\partial x^i}\,.$$

So, the Poisson bivector on A induced by a Lie algebroid A^* is such that:

$$\Pi_A = \frac{1}{2} c_{ij}^k y_k \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} + \rho_j^i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_j} \, .$$

The structure of tangent lift of higher order of Lie algebroid on $T^r A^*$ is given by:

$$\left[u_i^{(\alpha)}, u_j^{(\beta)} \right] = \left(c_{ij}^k \right)^{(\gamma)} u_k^{(\alpha+\beta+\gamma)}$$

We put $u_{\cdot i}^{(\alpha)} = I_A^r (u_i^{(r-\alpha)})$. It follows that the Lie bracket of sections of $(T^r A)^*$ is given by:

$$\left[u_{\cdot i}^{(\alpha)}, u_{\cdot j}^{(\beta)} \right] = (c_{ij}^k)^{(\gamma)} u_{\cdot k}^{(\alpha+\beta-\gamma-r)} \, .$$

The Poisson structure on $T^r A$ is such that:

$$\begin{aligned} \Pi_{T^r A} &= \frac{1}{2} (c_{ij}^k)^{(\alpha+\beta-\gamma-r)} y_k^\gamma \frac{\partial}{\partial y_i^\alpha} \wedge \frac{\partial}{\partial y_j^\beta} + (\rho_j^i)^{(\alpha+\beta-r)} \frac{\partial}{\partial x_\alpha^i} \wedge \frac{\partial}{\partial y_j^\beta} \\ &= \frac{1}{2} (c_{ij}^k y_k)^{(\alpha+\beta-r)} \frac{\partial}{\partial y_i^\alpha} \wedge \frac{\partial}{\partial y_j^\beta} + (\rho_j^i)^{(\alpha+\beta-r)} \frac{\partial}{\partial x_\alpha^i} \wedge \frac{\partial}{\partial y_j^\beta} \\ &= \Pi_A^{(c)} \end{aligned}$$

Thus, $\Pi_{T^r A} = \Pi_A^{(c)}$. By the following commutative diagram

$$\begin{tikzcd} T^r T^* A \arrow[r, "T^r r_A"] & T^r A^* \arrow[d, "I_{A^*}^r"] \\ T^* T^r A \arrow[u, "\alpha_A^r"] \arrow[r, "r_{T^r A}"'] & (T^r A)^* \end{tikzcd}$$

we deduce that, the diagram

$$\begin{tikzcd} T^* T^r A \arrow[r, "\sharp_{\Pi_A^{(c)}}"] \arrow[d, "r_{T^r A}"'] & T T^r A \arrow[d, "T T^r \pi"] \\ (T^r A)^* \arrow[r, "\rho_{(T^r A)^*}"'] & T T^r M \end{tikzcd}$$

commutes. Where $\rho_{(T^r A)^*} = \rho_{A^*}^{(r)} \circ (I_A^r)^{-1}$. We deduce that the pair $(T^r A, (T^r A)^*)$ is a Lie bialgebroid (see the theorem of Mackenzie and Xu in [12]). □

Remark 3. (i) The Lie algebroids $(T^r A)^*$ and $T^r A^*$ are naturally equivalent by the isomorphism of vector bundles $(I_A^r)^{-1}$.

(ii) When $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra then, $(T^r \mathfrak{g}, (T^r \mathfrak{g})^*)$ is a Lie bialgebra.

Let (A, A^*) be a Lie bialgebroid, the composition $\rho_{A^*} \circ \rho_A^* : T^*M \to TM$ defines a Poisson structure on M of bivector Π_{A,A^*}. Therefore, we have $\sharp_{\Pi_{A,A^*}} = \rho_{A^*} \circ \rho_A^*$.

Corollary 2. *Let (A, A^*) be a Lie bialgebroid. The Poisson bivector on T^rM induced by the Lie bialgebroid $(T^rA, (T^rA)^*)$ is the complete lift of order r of the Poisson bivector Π_{A,A^*}. For further details, we have:*

$$\Pi_{T^rA,(T^rA)^*} = \Pi_{A,A^*}^{(c)} .$$

Proof. The anchor maps of T^rA and $(T^rA)^*$ are given by: $\rho_A^{(r)} = \kappa_M^r \circ T^r\rho_A$ and $\rho_{(T^rA)^*} = \kappa_M^r \circ T^r\rho_{A^*} \circ (I_A^r)^{-1}$. By the formulas

$$(I_A^r)^{-1} \circ (T^r\rho_A)^* = T^r(\rho_A^*) \circ (I_{TM}^r)^{-1}$$
$$(I_{TM}^r)^{-1} \circ (\kappa_M^r)^* = \alpha_M^r$$

we have:

$$\begin{aligned}
\sharp_{\Pi_{T^rA(T^rA)^*}} &= \kappa_M^r \circ T^r\rho_{A^*} \circ (I_A^r)^{-1} \circ (\kappa_M^r \circ T^r\rho_A)^* \\
&= \kappa_M^r \circ T^r\rho_{A^*} \circ (I_A^r)^{-1} \circ (T^r\rho_A)^* \circ (\kappa_M^r)^* \\
&= \kappa_M^r \circ T^r(\rho_{A^*} \circ \rho_A^*) \circ \alpha_M^r \\
&= \kappa_M^r \circ T^r(\sharp_{\Pi_{A,A^*}}) \circ \alpha_M^r \\
&= \sharp_{\Pi_{A,A^*}^{(c)}}
\end{aligned}$$

□

2.2. Tangent lifts of higher order of multiplicative Poisson manifolds.

Let G be a Lie groupoid over M, AG denote the Lie algebroid of Lie groupoid G and $A^*(G)$ his dual. In [12], it is known that a bivector $\Pi_G \in \Gamma(\bigwedge^2 TG)$ is a multiplicative bivector if and only if

$$\begin{tikzcd}
T^*G \arrow[r, "\sharp_{\Pi_G}"] \arrow[d, shift left] \arrow[d, shift right] & TG \arrow[d, shift left] \arrow[d, shift right] \\
A^*(G) \arrow[r, "\rho_{A^*(G)}"'] & TM
\end{tikzcd}$$

is a morphism of Lie groupoids over the vector bundle map $\rho_{A^*(G)}$.

Theorem 2. *Let (G, Π_G) be a multiplicative Poisson manifold on a Lie groupoid G over M. The pair $(T^r G, \Pi_G^{(c)})$ is a multiplicative Poisson manifold on the Lie groupoid $T^r G$ over $T^r M$.*

Proof. We put $\gamma^r_{AG} = (I^r_{A(G)})^{-1} \circ (j^r_G)^*$, we have the following commutative diagram

$$\begin{tikzcd} T^*T^rG \arrow[r, "\alpha^r_G"] \arrow[d, shift left] \arrow[d, shift right] & T^rT^*G \arrow[d, shift left] \arrow[d, shift right] \\ A^*(T^rG) \arrow[r, "\gamma^r_{AG}"'] & T^r(A^*(G)) \end{tikzcd}$$

we deduce that the diagram

$$\begin{tikzcd} T^*T^rG \arrow[r, "\sharp_{\Pi_G^{(c)}}"] \arrow[d, shift left] \arrow[d, shift right] & TT^rG \arrow[d, shift left] \arrow[d, shift right] \\ A^*(T^rG) \arrow[r, "\rho_{A^*(T^rG)}"'] & TT^rM \end{tikzcd}$$

commutes, where $\rho_{A^*(T^rG)} = (I^r_{A(G)})^{-1} \circ (j^r_G)^* \circ \rho^{(r)}_{A^*(G)}$. We deduce that $(T^r G, \Pi_G^{(c)})$ is a multiplicative Poisson manifold on the Lie groupoid $T^r G$ over $T^r M$. $\square$

Corollary 3. *Let (G, Π_G) be a multiplicative Poisson manifold. There is a natural isomorphism of Lie bialgebroids between the Lie bialgebroid $(A(T^rG), A^*(T^rG))$ of the Poisson groupoid $(T^rG, \Pi_G^{(c)})$ and the Lie bialgebroid $(T^r(AG), (T^r(AG))^*)$.*

Proof. In [12], it is shown that the diagram

$$\begin{tikzcd} A(T^*G) \arrow[r, "A(\sharp_{\Pi_G})"] \arrow[d, "\varepsilon_G"'] & A(TG) \arrow[d, "j^1_G"] \\ T^*(AG) \arrow[r, "\sharp_{\Pi_{AG}}"'] & T(AG) \end{tikzcd}$$

commutes. By the equalities

$$T^r\big((j_G^1)^{-1}\big) \circ T^r A(\sharp_{\Pi_G}) = T^r(\sharp_{\Pi_{AG}}) \circ T^r(j_G)$$

$$\kappa^r_{AG} \circ T^r(\sharp_{\Pi_{AG}}) = \sharp_{\Pi^{(c)}_{AG}} \circ \varepsilon^r_{AG}$$

$$T(j^r_G) \circ \sharp_{\Pi^{(c)}_{AG}} = \sharp_{\Pi_{A(T^rG)}} \circ T^*\big((j^r_G)^{-1}\big)$$

$$j^1_{A(T^rG)} \circ \sharp_{\Pi_{A(T^rG)}} = A(\sharp_{\Pi^{(c)}_G}) \circ j_G^{-1}\,.$$

It follows that the diagram

$$\begin{array}{ccc} A(T^*T^rG) & \xrightarrow{A(\sharp_{\Pi^{(c)}_G})} & A(TT^rG) \\ \Downarrow & & \Downarrow \\ A^*(T^rG) & \xrightarrow[\rho_{A^*(T^rG)}]{} & TT^rM \end{array}$$

commutes. So, the Lie bialgebroid $(T^r(AG), (T^r(AG))^*)$ is the Lie bialgebroid induced by the Poisson groupoid $(T^rG, \Pi^{(c)}_G)$. □

Remark 4. If (G, Π_G) is a Poisson-Lie group then, $\left(T^rG, \Pi^{(c)}_G\right)$ is a Poisson-Lie group. The Lie bialgebra defined by $(T^rG, \Pi^{(c)}_G)$, is the Lie bialgebra $(T^r\mathfrak{g}, (T^r\mathfrak{g})^*)$, where $(\mathfrak{g}, \mathfrak{g}^*)$ is the Lie bialgebra of (G, Π_G).

3. Multiplicative Dirac structures of higher order

3.1. Tangent lifts of higher order of Dirac structures.

Let (M, L_M) be an almost Dirac structure. We set

$$L_{T^rM} = (\kappa^r_M \oplus \varepsilon^r_M)(T^rL_M) \subset TT^rM \oplus T^*T^rM\,, \tag{3.1}$$

L_{T^rM} is an almost Dirac structure on T^rM.

By the formula (3.1), we deduce that: for any $S = X \oplus \omega \in \Gamma(L_M)$ and $\beta \in \{0, \dots, r\}$,

$$(\kappa^r_M \oplus \varepsilon^r_M)(S^{(\beta)}) = X^{(\beta)} \oplus \omega^{(r-\beta)} \tag{3.2}$$

is a section of vector bundle L_{T^rM}.

The integrability of an almost Dirac structure (M, L_M) is measured by the Courant 3-tensor T_{L_M} defined by:

$$\begin{aligned} T_{L_M} : \Gamma(L_M) \times \Gamma(L_M) \times \Gamma(L_M) &\to C^\infty(M) \\ (S_1, S_2, S_3) &\mapsto \langle [S_1, S_2], S_3\rangle_+ \end{aligned}$$

where $[\cdot,\cdot]$ is the Courant bracket defined on $\mathfrak{X}(M)\oplus\Omega^1(M)$. In fact, an almost Dirac structure $L_M \subset TM \oplus T^*M$ defines a Dirac structure if and only if the Courant tensor T_{L_M} vanishes.

Proposition 4. $\overline{T_{L_M}}^{(c)}$ *is a complete lift of 3-tensor* T_{L_M} *from* L_M *to* T^rL_M (*see* [2]). *We have:*

$$\overline{T_{L_M}}^{(c)} = T_{L_{T^rM}} \circ \Big(\bigoplus^{3}(\kappa^r_M \oplus \varepsilon^r_M)\Big) \tag{3.3}$$

Proof. See [11]. □

By this result, we deduce that an almost Dirac structure (M, L_M) is integrable if and only if (T^rM, L_{T^rM}) is integrable.

Let $f\colon M \to N$ be a smooth map. The elements $X \oplus \omega \in \mathfrak{X}(M)\oplus\Omega^1(M)$ and $Y \oplus \varpi \in \mathfrak{X}(N)\oplus\Omega^1(N)$ are f-related if $Y = Tf(X)$ and $T^*f(\varpi) = \omega$.

Proposition 5. *Let* (M, L_M) *and* (N, L_N) *the Dirac manifolds. If* $f : M \to N$ *is a backward* (*resp. forward*) *Dirac map then,*

$$T^rf\colon (T^rM, L_{T^rM}) \to (T^rN, L_{T^rN})$$

is a backward (*resp. forward*) *Dirac map.*

Proof. We recall that, f is a backward Dirac map if and only if, the $C^\infty(N)$-module $\Gamma(L_N)$ is the space of all f-related sections to sections of $\Gamma(L_M)$. The remainder of proof comes from the fact that, if $X \oplus \omega$ and $Y \oplus \varpi$ are f-related then, for any $\alpha = 0, \cdots, r$, $X^{(\alpha)} \oplus \omega^{(r-\alpha)}$ and $Y^{(\alpha)} \oplus \varpi^{(r-\alpha)}$ are T^rf-related and the spaces $\Gamma(L_{T^rM})$, $\Gamma(L_{T^rN})$ are generated by the sets $\big\{X^{(\alpha)} \oplus \omega^{(r-\alpha)},\, X \oplus \omega \in \Gamma(L_M)\big\}$ and $\big\{Y^{(\alpha)} \oplus \varpi^{(r-\alpha)},\, Y \oplus \varpi \in \Gamma(L_N)\big\}$ respectively. When f is a forward Dirac map, we have the same proof. □

Remark 5. We denote by $\mathcal{DM}$ the category of Dirac manifolds and Dirac maps. By the Proposition 5, we have the natural functor which send an object (M, L_M) to the tangent object of higher order (T^rM, L_{T^rM}), and a Dirac morphism $f : (M, L_M) \to (N, L_N)$ to the tangent morphism of higher order $T^rf : (T^rM, L_{T^rM}) \to (T^rN, L_{T^rN})$. This functor is denoted by $\mathcal{T}^r : \mathcal{DM} \to \mathcal{DM}$.

3.2. **Tangent lifts of higher order of multiplicative Dirac structures.**

Definition 1. Let G be a Lie groupoid over the manifold M. A Dirac structure L_G on G is said to be multiplicative if $L_G \subset TG \oplus T^*G$ is a sub groupoid over some sub bundle E of $TM \oplus A^*(G)$.

Example 1. Let (G, Π_G) be a Poisson groupoid. The multiplication of Π_G is equivalent to saying that $\sharp_{\Pi_G}\colon T^*G \to TG$ is a morphism of Lie groupoids. Therefore, the sub bundle L_{Π_G} =graph $(\sharp_{\Pi_G}) \subset TG \oplus T^*G$ defines a multiplicative Dirac structure over the sub bundle $E \subset TM \oplus A^*(G)$, where E is the graph of dual anchor map $\rho_{A^*(G)}\colon A^*(G) \to TM$.

Proposition 6. *Let $L_G \subset TG \oplus T^*G$ be a multiplicative Dirac structure on a Lie groupoid G. The tangent Dirac structure of higher order $L_{T^rG} \subset TT^rG \oplus T^*T^rG$ is also a multiplicative Dirac structure on a Lie groupoid $(T^rG \rightrightarrows T^rM)$.*

Proof. The map $\kappa_G^r\colon T^rTG \to TT^rG$ is an isomorphism of Lie groupoids over $\kappa_M^r : T^rTM \to TT^rM$. By the same way, the bundle $\varepsilon_G^r\colon T^rT^*G \to T^*T^rG$ is an isomorphism of Lie groupoids over $((j_G^r)^*)^{-1} \circ I_{AG}^r : T^r(A^*G) \to A^*(T^rG)$. Since L_G is a Lie sub groupoid of $TG \oplus T^*G$, then T^rL_G is a Lie sub groupoid of $T^rTG \oplus T^rT^*G$ over the sub bundle T^rE. By the groupoid isomorphism $\kappa_G^r \oplus \varepsilon_G^r\colon T^rTG \oplus T^rT^*G \to TT^rG \oplus T^*T^rG$, we deduce that $L_{T^rG} = (\kappa_G^r \oplus \varepsilon_G^r)(T^rL_G)$ is a Lie sub groupoid of $TT^rG \oplus T^*T^rG$ over the sub bundle $\mathcal{T}^rE = \big(\kappa_M^r \oplus \big(((j_G^r)^*)^{-1} \circ I_{AG}^r\big)\big)(T^rE) \subset TT^rM \oplus A^*(T^rG)$. Hence we conclude that L_{T^rG} is a multiplicative Dirac structure on $(T^rG \rightrightarrows T^rM)$. $\square$

Remark 6. In the particular case where G is a Lie group, the tangent Dirac structure of higher order L_G is a Dirac structure on the Lie group T^rG.

3.3. **Tangent lifts of higher order of linear Dirac structures.**
Let (E, M, π) be a vector bundle. We consider the double vector bundle structures (TE, TM, E, M), (T^*E, E^*, E, M) and $(TE \oplus T^*E, TM \oplus E^*, E, M)$

Definition 2. A Dirac structure $L_E \subset TE \oplus T^*E$ is called linear if it defines a double sub vector bundle (L, F, E, M) where F is sub bundle of $TM \oplus E^*$.

Example 2. Let Π be a linear Poisson bivector on (E, M, π). Since $\sharp_\Pi : T^*E \to TE$ is a morphism of double vector bundles, it follows that $L_\Pi = \text{graph}\,(\sharp_\Pi) \subset TE \oplus T^*E$ is a linear Dirac structure over the sub bundle $F_\Pi = \text{graph}\,(\rho_{E^*})$ where $\rho_{E^*}\colon E^* \to TM$ is the anchor map of the Lie algebroid E^*.

Proposition 7. *Let $L_E \subset TE \oplus T^*E$ be a linear Dirac structure over the sub bundle $F \subset TM \oplus E^*$. The Dirac structure $L_{T^rE} \subset TT^rE \oplus T^*T^rE$ is linear over the sub bundle $\mathcal{T}^rF = (\kappa_M^r \oplus I_E^r)(T^rF) \subset TT^rM \oplus (T^rE)^*$.*

Proof. It comes from the fact that, $\kappa_E^r\colon T^rTE \to TT^rE$ and $\varepsilon_M^r\colon T^rT^*E \to T^*T^rE$ are the isomorphism of double vector bundles. $\square$

Let $(E, [\cdot,\cdot], \rho)$ be a Lie algebroid over M, we consider the cotangent Lie algebroid $r_E\colon T^*E \to E^*$ and the Lie algebroid $TE \oplus T^*E$ over $TM \oplus E^*$.

There is a class of linear Dirac structures L_E over E, which also define a Lie subalgebroid of $TE \oplus T^*E \to TM \oplus E^*$ over some bundle $F \subset TM \oplus E^*$. It is called algebroid-Dirac structure.

Proposition 8. *Let $L_E \subset TE \oplus T^*E$ be an algebroid-Dirac structure over the sub bundle $F \subset TM \oplus E^*$. The linear Dirac structure $L_{T^rE} \subset TT^rE \oplus T^*T^rE$ is an algebroid-Dirac structure over the sub bundle $\mathcal{T}^rF = (\kappa^r_M \oplus I^r_E)\,(T^rF) \subset TT^rM \oplus (T^rE)^*$.*

Proof. Let $u \in \Gamma(E)$, for any $0 \le \alpha \le r$ and $\beta \in \{0, 1\}$ we have:

$$\kappa^r_E\big((u^{(\beta)})^{(\alpha)}\big) = (u^{(\alpha)})^{(\beta)} \quad \text{and} \quad (\rho^{(r)})^{(1)} \circ \kappa^r_E = (\rho^{(1)})^{(r)}\,.$$

By the equalities above, we deduce that $\kappa^r_E \colon T^rTE \to TT^rE$ is an isomorphism of Lie algebroids over $\kappa^r_M \colon T^rTM \to TT^rM$. As ε^r_E is an isomorphism of Lie algebroids over $I^r_E \colon T^rE^* \to (T^rE)^*$, we deduce that $L_{T^rE} \subset TT^rE \oplus T^*T^rE$ is an algebroid-Dirac structure over the sub bundle $\mathcal{T}^rF = (\kappa^r_M \oplus I^r_E)(T^rF) \subset TT^rM \oplus (T^rE)^*$. □

Remark 7. We denote by $\langle\cdot,\cdot\rangle_G$ the non degenerate symmetric pairing on $TG \oplus T^*G$. $\langle\cdot,\cdot\rangle_G$ is a morphism of Lie groupoids, where $\mathbb{R}$ is equipped with the usual abelian group structure. We apply the Lie functor, we obtain a non degenerate pairing

$$A(\langle\cdot,\cdot\rangle_G) \colon A(TG) \oplus A(T^*G) \times_{AG} A(TG) \oplus A(T^*G) \to \mathbb{R}\,.$$

By the same way, we denote by $\langle\cdot,\cdot\rangle_{AG}$ the non degenerate symmetric pairing on $T(AG) \oplus T^*(AG)$, by the canonical map,

$$(j^1_G)^{-1} \oplus j_G \colon A(TG) \oplus A(T^*G) \to T(AG) \oplus T^*(AG)$$

we deduce that:

$$\langle\cdot,\cdot\rangle_{AG} = A(\langle\cdot,\cdot\rangle_G) \circ \big((j^1_G \oplus j^{-1}_G) \oplus (j^1_G \oplus j^{-1}_G)\big)\,. \tag{3.4}$$

Let $L_G \subset TG \oplus T^*G$ be an almost multiplicative Dirac structure. We put:

$$L_{AG} = \big((j^1_G)^{-1} \oplus j_G\big)\big(A(L_G)\big) \subset T(AG) \oplus T^*(AG)\,. \tag{3.5}$$

Clearly, L_{AG} is a linear almost Dirac structure on AG. In [17], it is shown that: L_{AG} is integrable if and only if L_G is integrable.

Remark 8. We denote by $\mathrm{Dir}_{\mathrm{mult}}(G)$ (resp. $\mathrm{Dir}_{\mathrm{alg}}(E)$) the space of all multiplicative Dirac structures on G (resp. algebroid-Dirac structures on E). We have the natural map

$$\begin{gathered}\mathrm{Dir}_{\mathrm{mult}}(G) \to \mathrm{Dir}_{\mathrm{alg}}(AG)\\ L_G \mapsto L_{AG}\end{gathered}$$

where $L_{AG} = ((j^1_G)^{-1} \oplus j_G)(A(L_G))$. We have the functor which send an object (G, L_G) to the algebroid-Dirac object (AG, L_{AG}), and a multiplicative Dirac morphism $f\colon (G, L_G) \to (H, L_H)$ to the algebroid-Dirac structure $Af\colon (AG, L_{AG}) \to (AH, L_{AH})$. This functor is denoted by $\mathcal{A}$. Via the canonical natural equivalence, this functor coincides with the Lie functor A.

Corollary 4. *The natural vector bundle*

$$T(j^r_G) \oplus T^*((j^r_G)^{-1})\colon TT^r(AG) \oplus T^*T^r(AG) \to T(A(T^rG)) \oplus T^*(A(T^rG))$$

send the linear Dirac structure $L_{T^r(AG)}$ in $L_{A(T^rG)}$ and it is an isomorphism of Dirac structures.

Proof. We know that,

$$T(j^r_G) \circ \kappa^r_{AG} \circ T^r\big((j^1_G)^{-1}\big) = (j^1_{T^rG})^{-1} \circ A(\kappa^r_G) \circ j^r_{TG}$$
$$T^*\big((j^r_G)^{-1}\big) \circ \varepsilon^r_{AG} \circ T^r(j_G) = j_{T^rG} \circ A(\varepsilon^r_G) \circ j^r_{T^*G}\,.$$

In this case, we have:

$$\begin{aligned}
&\big[T(j^r_G) \oplus T^*\big((j^r_G)^{-1}\big)\big]\big((\kappa^r_{AG} \oplus \varepsilon^r_{AG})\big(T^r(L_{AG})\big)\big)\\
&= \big[\big(T(j^r_G) \circ \kappa^r_{AG}\big) \oplus \big(T^*\big((j^r_G)^{-1}\big) \circ \varepsilon^r_{AG}\big)\big]\big(T^r(L_{AG})\big)\\
&= \big[\big(T(j^r_G) \circ \kappa^r_{AG} \circ T^r\big((j^1_G)^{-1}\big)\big) \oplus \big(T^*\big((j^r_G)^{-1}\big) \circ \varepsilon^r_{AG} \circ T^r(j_G)\big)\big]\big(T^r\big(A(L_G)\big)\big)\\
&= \big[\big((j^1_{T^rG})^{-1} \circ A(\kappa^r_G) \circ j^r_{TG}\big) \oplus \big(j_{T^rG} \circ A(\varepsilon^r_G) \circ j^r_{T^*G}\big)\big]\big(T^r\big(A(L_G)\big)\big)\\
&= \big[\big((j^1_{T^rG})^{-1} \circ A(\kappa^r_G)\big) \oplus j_{T^rG} \circ A(\varepsilon^r_G)\big]\big(A(T^rL_G)\big)\\
&= \big((j^1_{T^rG})^{-1} \oplus j_{T^rG}\big)\big(A\big((\kappa^r_G \oplus \varepsilon^r_G)(T^rL_G)\big)\big)\\
&= \big((j^1_{T^rG})^{-1} \oplus j_{T^rG}\big)\big(A(L_{T^rG})\big)
\end{aligned}$$

We conclude that, $(T(j^r_G) \oplus T^*((j^r_G)^{-1}))(L_{T^r(AG)}) = L_{A(T^rG)}$. □

These results generalize the tangent lifts of higher order of multiplicative Poisson structures and multiplicative symplectic structures on the Lie groupoids.

Remark 9. By the Corollary 4, we have the natural equivalence between the functors $\mathcal{A} \circ \mathcal{T}^r$ and $\mathcal{T}^r \circ \mathcal{A}$.

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