ARCHIVUM MATHEMATICUM (BRNO)
Tomus 49 (2013), 119–124

CLO SPACES AND CENTRAL MAXIMAL OPERATORS

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Abstract. We consider central versions of the space BLO studied by Coifman and Rochberg and later by Bennett, as well as some natural relations with a central version of a maximal operator.

1. Introduction

The goal of this work is to study central versions of the space of functions with bounded lower oscillation BLO introduced by Coifman and Rochberg in [5]. It is well known that BLO $\subset$ BMO, the space of functions with bounded mean oscillation. In fact, Lin et al. [7] constructed in the setting of $\mathbb{R}^n$ an example of a non-negative function in BMO $\setminus$ BLO, which shows that the results obtained in [7] on the boundedness of Lusin area and g_λ^* functions indeed improve the known corresponding results even on $\mathbb{R}^n$. On the other hand, Bennett [3] obtained a characterization of BLO via the natural maximal operator and the classical BMO space. Recently, the theory of the space BLO has been developed greatly. The results about the classical space BLO have been extended to several settings, such as the Gauss measure metric space [9] and the non-homogeneous metric measure space [8]. More developments on the theory of the spaces BLO on these settings can be found in the monograph [12].

The results presented in this paper are the corresponding central versions of results proved by Bennett in [3]. We give a partial characterization of the space of functions of bounded central lower oscillation in terms of the space CMO^p studied in [4], [6] and [10], by using a central version of a maximal operator. Because of the lack of John-Niremberg inequality for spaces CMO^p (see [1] and [2]), unlike the non-central case, in our case we can only obtain a partial characterization in the spirit of [3]. We also provide a generalization of some of our statements to the setting of doubling measures showing in this way the boundedness of our central maximal operator from CMO^p to CLO, for $1 < p < \infty$.

All the proofs presented in this work rely on standard techniques given by classical decompositions of functions on dilated cubes and boundedness properties of the central maximal operators under consideration.

2010 *Mathematics Subject Classification*: primary 42B35; secondary 42B25.
Key words and phrases: central mean oscillation, central maximal function.
Received January 24, 2013, revised June 2013. Editor V. Müller.
DOI: 10.5817/AM2013-2-119

The notation used is standard and, as usual, we employ the letter C to denote a positive constant that could be different line by line.

2. CLO spaces

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ a real-valued function. If Q denotes a cube on $\mathbb{R}^n$ with sides parallel to the coordinate axes, let us denote by $f^{(Q)}$ the essential infimum of f on Q (not necessarily finite). The notation Q_r will always indicate that the cube is centered at 0 and has sidelenght equal to $2r$.

We say that f has bounded central lower oscilation, in short, $f \in \mathrm{CLO}$, if

$$\|f\|_{\mathrm{CLO}} := \sup_{r>0}\left(f_{Q_r} - f^{(Q_r)}\right) < \infty \tag{1}$$

where, as usual, f_{Q_r} denotes the average of f on Q_r.

Although the sum of elements in CLO belongs to CLO, only multiplication for non-negative scalars preserves the above property, thus CLO is not a linear space neither $\|\cdot\|_{\mathrm{CLO}}$ is a norm, but it is subadditive and positive homogeneous.

We have also the inclusions

$$L^\infty \subset \mathrm{BLO} \subset \mathrm{CLO} \subset \mathrm{CMO}^1$$

and the inequalities

$$\|\cdot\|_{\mathrm{CMO}^1} \leq 2\,\|\cdot\|_{\mathrm{CLO}} \leq 4\,\|\cdot\|_\infty\,,$$

where, for $1 \leq p < \infty$

$$\mathrm{CMO}^p = \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{CMO}^p} < \infty\right\},$$

$$\|f\|_{\mathrm{CMO}^p} := \sup_{r>0}\left(\frac{1}{|Q_r|}\int_{Q_r} |f(x) - f_{Q_r}|^p\, dx\right)^{1/p} \tag{2}$$

(cf. [2], [6], [10]) and BLO is the space defined in [3] and [5] by means of a condition like (1) but using general cubes. We point out that the spaces CMO^p are the dual of some Herz-type Hardy spaces and it has been proved boundedness on them of some classical operators such as Littlewood-Paley operators (see [11] for more details). It is also well known [4] that $\mathrm{CMO}^{p_2} \subsetneq \mathrm{CMO}^{p_1}$ when $p_1 < p_2$.

Remark 1. Unlike the BMO case, where John-Niremberg inequality allows to use any $p > 1$ for its definition which yields the inclusion $\mathrm{BLO} \subset \mathrm{BMO}$, in the case under consideration it can be easily seen that the space CLO is not included in CMO^p for $p > 1$.

Indeed, for $n = 1$, we can consider the function $f(x) = (x-1)^{-1/p}\mathcal{X}_{(1,\infty)}(x)$. While $\inf_{Q_r} f = 0$ for every r, we have that $f_{Q_r} = 0$ for $r < 1$ and $f_{Q_r} = \frac{p'}{2r}(r-1)^{1/p'}$ for $r \geq 1$, where p' is the conjugate exponent of p. Thus $\|f\|_{\mathrm{CLO}} = \frac{p'}{p'-1}$, however $f \notin \mathrm{CMO}^p$ since f is not even locally in $L^p(\mathbb{R})$. This example can also be adapted for general n.

We intend to give a description of the space CLO in terms of the spaces CMO^p and an appropriate maximal operator. Following the approach of [3] we state two auxiliary results.

We will be considering the following central maximal function

$$M_c f(x) := \sup_{r>0,\ x\in Q_r} \frac{1}{|Q_r|}\int_{Q_r} f(y)\, dy\,. \tag{3}$$

Since $M_c f(x) \le M f(x)$, where M is the classical Hardy-Littlewood maximal operator, it is clear that M_c is of strong type (p,p) for $1<p<\infty$, and of weak type $(1,1)$.

Proposition 2. *Let $1<p<\infty$. Then, there exists a positive constant C only depending on n and p such that for every $f\in \mathrm{CMO}^p$ and each $r>0$*

$$(M_c f)_{Q_r} \le C\,\|f\|_{\mathrm{CMO}^p} + (M_c f)^{(Q_r)}\,. \tag{4}$$

Thus, if $M_c f$ is not identically infinite, then $M_c f \in \mathrm{CLO}$ and

$$\|M_c f\|_{\mathrm{CLO}} \le C\,\|f\|_{\mathrm{CMO}^p}\,. \tag{5}$$

Proof. Decompose $f=f_1+f_2$ where

$$f_1 = \Big(f - \frac{1}{|Q_{3r}|}\int_{Q_{3r}} f\,dy\Big)\mathcal{X}_{Q_{3r}}$$

and

$$f_2 = \Big(\frac{1}{|Q_{3r}|}\int_{Q_{3r}} f\,dy\Big)\mathcal{X}_{Q_{3r}} + f\mathcal{X}_{(Q_{3r})^c}\,.$$

Notice that $f_1\in L^p$ since $f\in\mathrm{CMO}^p$ and supp $f_1\subset Q_{3r}$. Thus

$$\begin{aligned}\frac{1}{|Q_r|}\int_{Q_r} M_c f_1\,dx &\le \Big(\frac{1}{|Q_r|}\int_{Q_r} |M_c f_1|^p\,dx\Big)^{1/p}\\ &\le C\,|Q_{3r}|^{-1/p}\,\|f_1\|_p \le C\,\|f\|_{\mathrm{CMO}^p}\,. \end{aligned}\tag{6}$$

Next, we will show that for every $x\in Q_r$

$$M_c f_2(x) \le C\,\|f\|_{\mathrm{CMO}^p} + (M_c f)^{(Q_r)}\,. \tag{7}$$

The estimate (7) implies that

$$\frac{1}{|Q_r|}\int_{Q_r} M_c f_2\,dy \le C\|f\|_{\mathrm{CMO}^p} + (M_c f)^{(Q_r)} \tag{8}$$

which together with estimate (6) yields the desired result.

In order to show (7) it suffices to prove the existence of a positive constant C such that for any cube P, centered at 0 with $x\in P$

$$\frac{1}{|P|}\int_P f_2\,dy \le C\,\|f\|_{\mathrm{CMO}^p} + (M_c f)^{(Q_r)}\,. \tag{9}$$

The proof of estimate (9) is similar to the one in [3]:

When $P\subset Q_{3r}$ we can easily see that

$$\frac{1}{|P|}\int_P f_2\,dy \le (M_c f)^{(Q_r)}\,.$$

If $Q_{3r} \subset P$, then

$$\frac{1}{|P|}\int_P [f_2 - f_P]\, dy \le \frac{1}{|P|}\int_P |f - f_P|\, dy \le \|f\|_{\mathrm{CMO}^p}$$

and therefore, using the fact that $f_P \le (M_c f)^{Q_r}$ we obtain

$$\begin{aligned}\frac{1}{|P|}\int_P f_2 dy &\le \|f\|_{\mathrm{CMO}^p} + \frac{1}{|P|}\int_P f\, dy \\ &\le \|f\|_{\mathrm{CMO}^p} + (M_c f)^{Q_r}\ .\end{aligned}$$

This concludes the proof. □

We will use the central maximal operator (3) to provide a partial characterization of the space CLO in the spirit of [3]. For that purpose we require first to state a result whose proof is almost exactly the same (with the obvious modifications) of that given for Lemma 2 in [3].

Proposition 3. *For a real-valued $f \in L^1_{\mathrm{loc}}$ we have that $f \in \mathrm{CLO}$ if and only if $M_c f - f \in L^\infty$. In that case*

$$\|M_c f - f\|_\infty = \|f\|_{\mathrm{CLO}}\ .$$

Now, we can state our main result about the space CLO.

Theorem 4. *Let $f \in L^1_{\mathrm{loc}}$ and $1 < p < \infty$. Then, if $f \in \mathrm{CLO}$ there exist functions $h \in L^\infty$ and $g \in \mathrm{CMO}^1$ such that $M_c g$ is finite almost everywhere and*

$$f = M_c g + h\,. \tag{10}$$

Conversely, if f can be represented as in (10), with $h \in L^\infty$ and $g \in \mathrm{CMO}^p$, $1 < p < \infty$, then $f \in \mathrm{CLO}$. In such case, there exists a positive constant C only depending on n and p such that

$$\|f\|_{\mathrm{CLO}} \le C \inf\left(\|g\|_{\mathrm{CMO}^p} + \|h\|_\infty\right) \tag{11}$$

where the infimum is taken over all representations of f given by (10).

Proof. Assuming a representation like (10) with $h \in L^\infty$ and $g \in \mathrm{CMO}^p$, $1 < p < \infty$, by Proposition 2 we have that $M_c g \in \mathrm{CLO}$ and, since also $h \in \mathrm{CLO}$ we conclude that $f \in \mathrm{CLO}$ and, furthermore

$$\begin{aligned}\|f\|_{\mathrm{CLO}} &\le \|M_c g\|_{\mathrm{CLO}} + \|h\|_{\mathrm{CLO}} \\ &\le C\,\|g\|_{\mathrm{CMO}^p} + 2\,\|h\|_\infty \\ &\le C\left[\,\|g\|_{\mathrm{CMO}^p} + \|h\|_\infty\right],\end{aligned}$$

and taking infimum over all such representations of f we get inequality (11).

Conversely, suppose that $f \in \mathrm{CLO} \subset \mathrm{CMO}^1$, then Proposition 3 assures that $M_c f$ is finite almost everywhere and, since $f - M_c f \in L^\infty$, the decomposition

$$f = M_c f + (f - M_c f)$$

satisfies the required condition.

This concludes the proof. □

When μ is a doubling measure on $\mathbb{R}^n$, that is, there exists a positive constant C such that for any cube Q

$$\mu(2Q) \leq C\mu(Q) ,$$

it is also possible to consider generalized versions of the spaces defined above.

We define the space $\mathrm{CLO}(\mu)$ to be the set of real-valued functions $f \in L^1_{\mathrm{loc}}(\mu)$ such that

$$\|f\|_{\mathrm{CLO}(\mu)} := \sup_{r>0} \left(f_{Q_r,\mu} - f^{(Q_r)} \right) < \infty ,$$

where

$$f_{Q_r,\mu} = \frac{1}{\mu(Q_r)} \int_{Q_r} f \, d\mu ,$$

and $f^{(Q_r)}$ is the essential infimum of f in Q_r.

For $1 \leq p < \infty$, the space $\mathrm{CMO}^p(\mu)$ will be the set of real-valued functions $f \in L^p_{\mathrm{loc}}(\mu)$ such that $\|f\|_{\mathrm{CMO}^p(\mu)} < \infty$, where

$$\|f\|_{\mathrm{CMO}^p(\mu)} := \sup_{r>0} \left(\frac{1}{\mu(Q_r)} \int_{Q_r} |f - f_{Q_r,\mu}|^p \, d\mu \right)^{1/p} . \tag{12}$$

We also have the inclusions

$$L^\infty \subset \mathrm{CLO}(\mu) \subset \mathrm{CMO}^1(\mu) .$$

The corresponding central maximal function to consider is

$$M_c^\mu f(x) := \sup_{r>0,\ x \in Q_r} \frac{1}{\mu(Q_r)} \int_{Q_r} f \, d\mu , \tag{13}$$

which is of bounded on $L^p(\mu)$ for $1 < p < \infty$.

Since the measure μ is doubling, we can reproduce without any trouble the proof of Proposition 2 and state the following result.

Theorem 5. *Let $1 < p < \infty$ and μ a doubling measure on $\mathbb{R}^n$. Then, there exists a positive constant C such that for every $f \in \mathrm{CMO}^p(\mu)$ and each $r > 0$*

$$(M_c^\mu f)_{Q_r} \leq C \|f\|_{\mathrm{CMO}^p(\mu)} + (M_c^\mu f)^{(Q_r)} . \tag{14}$$

Thus, if $M_c^\mu f$ is not identically infinite, then $M_c^\mu f \in \mathrm{CLO}(\mu)$ and

$$\|M_c^\mu f\|_{\mathrm{CLO}(\mu)} \leq C \|f\|_{\mathrm{CMO}^p(\mu)} . \tag{15}$$

Acknowledgement. I would like to thank the referee for her/his valuable suggestions that improved the presentation of this paper.

References

[1] Alvarez, J., *The distribution function in the Morrey space*, Proc. Amer. Math. Soc. **83** (1981), 693–699.

[2] Alvarez, J., Guzmán-Partida, M., Lakey, J., *Spaces of bounded λ–central mean oscillation, Morrey spaces, and λ–central Carleson measures*, Collect. Math. **51** (2000), 1–47.

[3] Bennett, C., *Another characterization of BLO*, Proc. Amer. Math. Soc. **85** (1982), 552–556.

[4] Chen, Y. Z., Lau, K. S., *On some new classes of Hardy spaces*, J. Funct. Anal. **84** (1989), 255–278.

[5] Coifman, R. R., Rochberg, R., *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), 249–254.

[6] García-Cuerva, J., *Hardy spaces and Beurling algebras*, J. London Math. Soc. **39** (1989), 499–513.

[7] Lin, H., Nakai, E., Yang, D., *Boundedness of Lusin–area and g_λ^* functions on localized BMO spaces over doubling metric measure spaces*, Bull. Sci. Math. **135** (2011), 59–88.

[8] Lin, H., Yang, D., *Spaces of type BLO on non–homogeneous metric measure spaces*, Front. Math. China **6** (2011), 271–292.

[9] Liu, L., Yang, D., *BLO spaces associated with the Ornstein–Uhlenbeck operator*, Bull. Sci. Math. **132** (2008), 633–649.

[10] Lu, S., Yang, D., *The central BMO spaces and Littlewood–Paley operators*, J. Approx. Theory Appl. **11** (1995), 72–94.

[11] Lu, S., Yang, D., Hu, G., *Herz Type Spaces and Their Applications*, Science Press, Beijing, 2008.

[12] Yang, Da., Yang, Do., Hu, G., *The Hardy Space H^1 with Non–doubling Measures and Their Applications*, Lecture Notes in Math., vol. 2084, Springer–Verlag, Berlin, 2013.

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