ARCHIVUM MATHEMATICUM (BRNO)
Tomus 49 (2013), 187–197

CONTROL AFFINE SYSTEMS ON SOLVABLE THREE-DIMENSIONAL LIE GROUPS, I

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Abstract. We seek to classify the full-rank left-invariant control affine systems evolving on solvable three-dimensional Lie groups. In this paper we consider only the cases corresponding to the solvable Lie algebras of types II, IV, and V in the Bianchi-Behr classification.

1. Introduction

Left-invariant control affine systems constitute an important class of systems, extensively used in many control applications. In this paper we classify, under local detached feedback equivalence, the full-rank left-invariant control affine systems evolving on certain (real) solvable three-dimensional Lie groups. Specifically, we consider only those Lie groups with Lie algebras of types II, IV, and V, in the Bianchi-Behr classification.

We reduce the problem of classifying such systems to that of classifying affine subspaces of the associated Lie algebras. Thus, for each of the three types of Lie algebra, we need only classify their affine subspaces. A tabulation of the results is included as an appendix.

2. Invariant control systems and equivalence

A left-invariant control affine system Σ is a control system of the form

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\ , \qquad g \in \mathsf{G},\, u \in \mathbb{R}^\ell\,.$$

Here G is a (real, finite-dimensional) Lie group with Lie algebra $\mathfrak{g}$ and $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Also, the parametrisation map $\Xi(\mathbf{1}, \cdot)\colon \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i.e., $B_1, \ldots, B_\ell$ are linearly independent). The "product" $g\,\Xi\,(\mathbf{1}, u)$ is to be understood as $T_{\mathbf{1}} L_g \cdot \Xi\,(\mathbf{1}, u)$, where $L_g\colon \mathsf{G} \to \mathsf{G},\ h \mapsto gh$ is the left translation by g. Note that the dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are invariant under left translations, i.e., $\Xi\,(g, u) = g\,\Xi\,(\mathbf{1}, u)$. We shall denote such a system by $\Sigma = (\mathsf{G}, \Xi)$ (cf. [3]).

The admissible controls are piecewise continuous maps $u(\cdot)\colon [0, T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible control $u(\cdot)$ is an absolutely continuous curve

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2010 *Mathematics Subject Classification*: primary 93A10; secondary 93B27, 17B30.

Key words and phrases: left-invariant control system, (detached) feedback equivalence, affine subspace, solvable Lie algebra.

Received December 9, 2011, revised September 2013. Editor J. Slovák.

DOI: 10.5817/AM2013-3-187

$g(\cdot)\colon [0,T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0,T]$. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot)\colon [0,T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. For more details about (invariant) control systems see, e.g., [1], [10], [11], [16].

The image set $\Gamma = \operatorname{im}\Xi\,(\mathbf{1}, \cdot)$, called the *trace* of Σ, is an affine subspace of $\mathfrak{g}$. Accordingly, $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell\rangle$. A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Furthermore, Σ is said to have *full rank* if its trace generates the whole Lie algebra (i.e., the smallest Lie algebra containing Γ is $\mathfrak{g}$). Henceforth, we assume that all systems under consideration have full rank. (The full-rank condition is necessary for a system Σ to be controllable.)

A natural equivalence relation for control systems is feedback equivalence (see, e.g., [9]). We specialize feedback equivalence (in the context of left-invariant control systems) by requiring that the feedback transformations are left-invariant (i.e., constant over the state space). Such transformations are exactly those that are compatible with the Lie group structure (see, e.g., [3, 2]). More precisely, let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control affine systems. Σ and Σ' are called *locally detached feedback equivalent* (shortly DF_{loc}-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a (local) diffeomorphism $\Phi\colon N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^{\ell'}$, $(g,u) \mapsto (\phi(g), \varphi(u))$ such that $\phi(a) = a'$ and $T_g\phi \cdot \Xi\,(g,u) = \Xi'\,(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$ (i.e., the diagram

$$\begin{tikzcd} N \times \mathbb{R}^\ell \arrow[r, "\phi\times\varphi"] \arrow[d, "\Xi"'] & N' \times \mathbb{R}^{\ell'} \arrow[d, "\Xi'"] \\ TN \arrow[r, "T\phi"'] & TN' \end{tikzcd}$$

commutes).

Any DF_{loc}-equivalence between two control systems can be reduced to an equivalence between neighbourhoods of the identity (by composing the diffeomorphism ϕ with a suitable left-translation). More precisely, Σ and Σ' are DF_{loc}-equivalent at $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if and only if they are DF_{loc}-equivalent at $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1}' \in \mathsf{G}'$. Henceforth, we will assume that any DF_{loc}-equivalence is between neighbourhoods of identity. We have the following algebraic characterisation of DF_{loc}-equivalence.

Proposition 1 ([2]). *Σ and Σ' are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

Proof. Suppose Σ and Σ' are DF_{loc}-equivalent. Then $T_{\mathbf{1}}\phi\cdot\Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. As $T_{\mathbf{1}}\phi$ is a linear isomorphism, it remains only to show that it preserves the Lie bracket. Let $u, v \in \mathbb{R}^\ell$, and let $\Xi_u = \Xi(\cdot, u)$ and $\Xi_v = \Xi(\cdot, v)$ denote the corresponding vector fields. Then the push-forward $\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v]$ and so $T_{\mathbf{1}}\phi\cdot[\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] = [\Xi'_{\varphi(u)}(\mathbf{1}'), \Xi'_{\varphi(v)}(\mathbf{1}')] = [T_{\mathbf{1}}\phi\cdot\Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi\cdot \Xi_v(\mathbf{1})]$. As Σ has full rank, the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$ generate the Lie algebra $\mathfrak{g}$; hence $T_{\mathbf{1}}\phi$ is a Lie algebra isomorphism.

Conversely, suppose we have a Lie algebra isomorphism ψ such that $\psi \cdot \Gamma = \Gamma'$. Then there exists neighbourhoods N and N' of $\mathbf{1}$ and $\mathbf{1}'$, respectively, and a local group isomorphism $\phi \colon N \to N'$ such that $T_{\mathbf{1}}\phi = \psi$ (see, e.g., [12]). Furthermore, there exists a unique affine isomorphism $\varphi : \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$. Consequently, $T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}L_{\phi(g)} \cdot \psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), \varphi(u))$. Hence Σ and Σ' are DF_{loc}-equivalent. $\square$

For the purpose of classification, we may assume that Σ and Σ' have the same Lie algebra $\mathfrak{g}$. We will say that two affine subspaces Γ and Γ' are $\mathfrak{L}$-*equivalent* if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$. Then Σ and Σ' are DF_{loc}-equivalent if and only if there traces Γ and Γ' are $\mathfrak{L}$-equivalent. This reduces the problem of classifying under DF_{loc}-equivalence to that of classifying under $\mathfrak{L}$-equivalence. Suppose $\{\Gamma_i \,:\, i \in I\}$ is an exhaustive collection of (non-equivalent) class representatives (i.e., any affine subspace is $\mathfrak{L}$-equivalent to exactly one Γ_i). For each $i \in I$, we can easily find a system $\Sigma_i = (\mathsf{G}, \Xi_i)$ with trace Γ_i. Then any system Σ is DF_{loc}-equivalent to exactly one Σ_i.

3. Affine subspaces of 3D Lie algebras

The classification of three-dimensional Lie algebras is well known. The classification over $\mathbb{C}$ was done by S. Lie (1893), whereas the standard enumeration of the real cases is that of L. Bianchi (1918). In more recent times, a different (method of) classification was introduced by C. Behr (1968) and others (see [14], [13], [15] and the references therein); this is customarily referred to as the *Bianchi-Behr classification* (or even the "Bianchi-Schücking-Behr classification"). Any solvable three-dimensional Lie algebra is isomorphic to one of nine types (in fact, there are seven algebras and two parametrised infinite families of algebras). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutator operation is given by

$$\begin{aligned}
[E_2, E_3] &= n_1 E_1 - a E_2 \\
[E_3, E_1] &= a E_1 + n_2 E_2 \\
[E_1, E_2] &= n_3 E_3.
\end{aligned}$$

The (Bianchi-Behr) structure parameters a, n_1, n_2, n_3 for each type are given in Table 1.

In this paper we are only concerned with types II, IV, and V. The remaining solvable Lie algebras (i.e., those of types III, VI_h, VI_0, VII_h, and VII_0) are treated in [6]. (For the Abelian Lie algebra $3\mathfrak{g}_1$ the classification is trivial.)

An affine subspace Γ of a Lie algebra $\mathfrak{g}$ is written as

$$\Gamma = A + \Gamma^0 = A + \langle B_1, B_2, \ldots, B_\ell \rangle$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Let Γ_1 and Γ_2 be two affine subspaces of $\mathfrak{g}$. Γ_1 and Γ_2 are $\mathfrak{L}$-equivalent if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. $\mathfrak{L}$-equivalence is a genuine equivalence relation. (Note that $\Gamma_1 = A_1 + \Gamma_1^0$ and $\Gamma_2 = A_2 + \Gamma_2^0$ are $\mathfrak{L}$-equivalent if and only if there exists an automorphism ψ such that $\psi \cdot \Gamma_1^0 = \Gamma_2^0$ and $\psi \cdot A_1 \in \Gamma_2$.) An affine subspace Γ is

Type	Notation	a	n_1	n_2	n_3	Representatives
I	$3\mathfrak{g}_1$	0	0	0	0	$\mathbb{R}^3$
II	$\mathfrak{g}_{3.1}$	0	1	0	0	$\mathfrak{h}_3$
$III = VI_{-1}$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	1	1	-1	0	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$
IV	$\mathfrak{g}_{3.2}$	1	1	0	0	
V	$\mathfrak{g}_{3.3}$	1	0	0	0	
VI_0	$\mathfrak{g}_{3.4}^0$	0	1	-1	0	$\mathfrak{se}(1,1)$
$VI_h,\ {}^{h<0}_{h\neq -1}$	$\mathfrak{g}_{3.4}^h$	$\sqrt{-h}$	1	-1	0	
VII_0	$\mathfrak{g}_{3.5}^0$	0	1	1	0	$\mathfrak{se}(2)$
$VII_h,\ h>0$	$\mathfrak{g}_{3.5}^h$	$\sqrt{h}$	1	1	0	

Tab. 1: Bianchi-Behr classification (solvable)

said to have *full rank* if it generates the whole Lie algebra. The full-rank property is invariant under $\mathfrak{L}$-equivalence. Henceforth, we assume that all affine subspaces under consideration have full rank.

In this paper we classify, under $\mathfrak{L}$-equivalence, the (full-rank) affine subspaces of $\mathfrak{g}_{3.1}$, $\mathfrak{g}_{3.2}$, and $\mathfrak{g}_{3.3}$. Clearly, if Γ_1 and Γ_2 are $\mathfrak{L}$-equivalent, then they are necessarily of the same dimension. Furthermore, $0 \in \Gamma_1$ if and only if $0 \in \Gamma_2$. We shall find it convenient to refer to an ℓ-dimensional affine subspace Γ as an $(\ell, 0)$-affine subspace when $0 \in \Gamma$ (i.e., Γ is a vector subspace) and as an $(\ell, 1)$-affine subspace, otherwise. Alternatively, Γ is said to be homogeneous if $0 \in \Gamma$, and inhomogeneous otherwise.

Remark. We have the following characterization of the full-rank condition when $\dim \mathfrak{g} = 3$. No $(1,0)$-affine subspace has full rank. A $(1,1)$-affine subspace has full rank if and only if A, B_1, and $[A, B_1]$ are linearly independent. A $(2,0)$-affine subspace has full rank if and only if B_1, B_2, and $[B_1, B_2]$ are linearly independent. Any $(2,1)$-affine subspace or $(3,0)$-affine subspace has full rank.

Clearly, there is only one affine subspace whose dimension coincides with that of the Lie algebra $\mathfrak{g}$, namely the space itself. From the standpoint of classification, this case is trivial and hence will not be covered explicitly.

Let us fix a three-dimensional Lie algebra $\mathfrak{g}$ (together with an ordered basis). In order to classify the affine subspaces of $\mathfrak{g}$, we require the (group of) automorphisms of $\mathfrak{g}$. These are well known (see, e.g., [7], [8], [15]); a summary is given in Table 2. For each type of Lie algebra, we construct class representatives (by considering the action of automorphisms on a typical affine subspace). By using some classifying conditions, we explicitly construct $\mathfrak{L}$-equivalence relations relating an arbitrary affine subspace to a fixed representative. Finally, we verify that none of the representatives are equivalent.

The following result is easy to prove.

Proposition 2. *Let Γ be a $(2,0)$-affine subspace of a Lie algebra $\mathfrak{g}$. Suppose $\{\Gamma_i : i \in I\}$ is an exhaustive collection of $\mathfrak{L}$-equivalence class representatives for $(1,1)$-affine subspaces of $\mathfrak{g}$. Then Γ is $\mathfrak{L}$-equivalent to at least one element of $\{\langle \Gamma_i \rangle : i \in I\}$.*

4. Type *II* (the Heisenberg algebra)

In terms of an (appropriate) basis (E_1, E_2, E_3) for $\mathfrak{g}_{3.1}$, the commutator operation is given by

$$[E_2, E_3] = E_1\,, \quad [E_3, E_1] = 0\,, \quad [E_1, E_2] = 0\,.$$

With respect to this ordered basis, the group of automorphisms is

$$\mathsf{Aut}(\mathfrak{g}_{3.1}) = \left\{ \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix} : u, v, w, x, y, z \in \mathbb{R},\ yw \neq vz \right\}.$$

We start the classification of the affine subspace of $\mathfrak{g}_{3.1}$ with the (inhomogeneous) one-dimensional case.

Proposition 3. *Any $(1,1)$-affine subspace of $\mathfrak{g}_{3.1}$ is $\mathfrak{L}$-equivalent to $\Gamma_1 = E_2 + \langle E_3 \rangle$.*

Proof. Let Γ be a $(1,1)$-affine subspace of $\mathfrak{g}_{3.1}$. Then Γ may be written as $\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle \sum_{i=1}^{3} b_i E_i \right\rangle$. Accordingly (as Γ has full rank)

$$\psi = \begin{bmatrix} a_2 b_3 - a_3 b_2 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_3 \end{bmatrix}$$

is a Lie algebra automorphism such that $\psi \cdot \Gamma_1 = \Gamma$. □

The result for the homogeneous two-dimensional case follows from Propositions 2 and 3.

Proposition 4. *Any $(2,0)$-affine subspace of $\mathfrak{g}_{3.1}$ is $\mathfrak{L}$-equivalent to $\langle E_2, E_3 \rangle$.*

Lastly, we consider the inhomogeneous two-dimensional case.

Proposition 5. *Any $(2,1)$-affine subspace of $\mathfrak{g}_{3.1}$ is $\mathfrak{L}$-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = E_1 + \langle E_2, E_3 \rangle \qquad \Gamma_2 = E_3 + \langle E_1, E_2 \rangle\,.$$

Proof. Let $\Gamma = A + \Gamma^0$ be a $(2,1)$-affine subspace of $\mathfrak{g}_{3.1}$. First, suppose that $E_1 \in \Gamma^0$. Then $\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle E_1, \sum_{i=1}^{3} b_i E_i \right\rangle$. Consequently

$$\psi = \begin{bmatrix} a_3 b_2 - a_2 b_3 & b_1 & a_1 \\ 0 & b_2 & a_2 \\ 0 & b_3 & a_3 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_2 = \Gamma$.

On the other hand, suppose that $E_1 \notin \Gamma^0$. Again we can write Γ as $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle$. Then the equation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution for v. Moreover, a simple calculation shows that $v_1 \neq 0$. We may thus choose non-zero constants $x, y \in \mathbb{R}$ such that $xy = v_1$. Then

$$\psi = \begin{bmatrix} v_1 & v_2 & v_3 \\ 0 & x & 0 \\ 0 & 0 & y \end{bmatrix}$$

is an automorphism. A simple calculation shows that $\psi \cdot \Gamma = \Gamma_1$.

Finally, as E_1 is an eigenvector of every automorphism, it is easy to show that Γ_1 and Γ_2 cannot be $\mathfrak{L}$-equivalent. □

In summary,

Theorem 1. *Any affine subspace of $\mathfrak{g}_{3.1}$ (type II) is $\mathfrak{L}$-equivalent to exactly one of $E_2 + \langle E_3 \rangle$, $\langle E_2, E_3 \rangle$, $E_1 + \langle E_2, E_3 \rangle$, and $E_3 + \langle E_1, E_2 \rangle$.*

5. Type *IV*

The Lie algebra $\mathfrak{g}_{3.2}$ has commutator operation given by

$$[E_2, E_3] = E_1 - E_2\,, \quad [E_3, E_1] = E_1\,, \quad [E_1, E_2] = 0$$

in terms of an (appropriate) ordered basis (E_1, E_2, E_3). With respect to this basis, the group of automorphisms is

$$\mathsf{Aut}(\mathfrak{g}_{3.2}) = \left\{ \begin{bmatrix} u & x & y \\ 0 & u & z \\ 0 & 0 & 1 \end{bmatrix} \; : \; x, y, z, u \in \mathbb{R},\, u \neq 0 \right\}.$$

Again, we start with the (inhomogeneous) one-dimensional case.

Proposition 6. *Any $(1,1)$-affine subspace of $\mathfrak{g}_{3.2}$ is $\mathfrak{L}$-equivalent to exactly of the following subspaces*

$$\Gamma_1 = E_2 + \langle E_3 \rangle \qquad\qquad \Gamma_{2,\alpha} = \alpha E_3 + \langle E_2 \rangle\,.$$

Here $\alpha \neq 0$ parametrises a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Proof. Let $\Gamma = A + \Gamma^0$ be a $(1,1)$-affine subspace of $\mathfrak{g}_{3.2}$. First, suppose that $E_3^*(\Gamma^0) \neq \{0\}$. (Here E_3^* denotes the corresponding element of the dual basis.) Then $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i \right\rangle$ with $b_3 \neq 0$. Thus $\Gamma = a_1' E_1 + a_2' E_2 + \langle b_1' E_1 + b_2' E_2 + E_3 \rangle$. As Γ has full rank, a simple calculation shows that $a_2' \neq 0$. Hence

$$\psi = \begin{bmatrix} a_2' & a_1' & b_1' \\ 0 & a_2' & b_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1 = \Gamma$.

On the other hand, suppose that $E_3^*(\Gamma^0) = \{0\}$ and $E_3^*(A) = \alpha \neq 0$. (As Γ has full rank, the situation $\alpha = 0$ is impossible.) Then $\Gamma = a_1 E_1 + a_2 E_2 + \alpha E_3 + \langle b_1 E_1 + b_2 E_2 \rangle$. A simple calculation shows that $b_2 \neq 0$. Thus

$$\psi = \begin{bmatrix} b_2 & b_1 & \frac{a_1}{\alpha} \\ 0 & b_2 & \frac{a_2}{\alpha} \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_{2,\alpha} = \Gamma$.

Finally, we verify that none of representatives are $\mathfrak{L}$-equivalent. As $E_2 \in \Gamma_1$, $\alpha E_3 \in \Gamma_{2,\alpha}$, and $\langle E_1, E_2 \rangle$ is an invariant subspace of every automorphism, it follows that Γ_1 and $\Gamma_{2,\alpha}$ cannot be $\mathfrak{L}$-equivalent. Then again, as $E_3^*(\psi \cdot \alpha E_3) = \alpha$ for any automorphism ψ, it follows that $\Gamma_{2,\alpha}$ and $\Gamma_{2,\alpha'}$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. □

We obtain the result for the homogeneous two-dimensional case by use of Propositions 2 and 6.

Proposition 7. *Any* $(2,0)$*-affine subspace of* $\mathfrak{g}_{3.2}$ *is* $\mathfrak{L}$*-equivalent to* $\langle E_2, E_3 \rangle$.

Lastly, we consider the inhomogeneous two-dimensional case and then summarise the results of this section.

Proposition 8. *Any* $(2,1)$*-affine subspace of* $\mathfrak{g}_{3.2}$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = E_2 + \langle E_1, E_3 \rangle \qquad \Gamma_2 = E_1 + \langle E_2, E_3 \rangle$$
$$\Gamma_{3,\alpha} = \alpha E_3 + \langle E_1, E_2 \rangle .$$

Here $\alpha \neq 0$ *parametrises a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Proof. Let $\Gamma = A + \Gamma^0$ be a $(2,1)$-affine subspace of $\mathfrak{g}_{3.2}$. First, assume $E_3^*(\Gamma^0) \neq \{0\}$ and $E_1 \in \Gamma^0$. Then $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle E_1, \sum_{i=1}^3 b_i E_i \right\rangle$ with $b_3 \neq 0$. Hence $\Gamma = a_2' E_2 + \langle E_1, b_2' E_2 + E_3 \rangle$ with $a_2' \neq 0$. Thus

$$\psi = \begin{bmatrix} a_2' & 0 & 0 \\ 0 & a_2' & b_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1 = \Gamma$.

Next, assume $E_3^*(\Gamma^0) \neq \{0\}$ and $E_1 \notin \Gamma^0$. Then $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle$ with $c_3 \neq 0$. Hence $\Gamma = a_1' E_1 + a_2' E_2 + \langle b_1' E_1 + b_2' E_2, c_1' E_1 + c_2' E_2 + E_3 \rangle$. Now, as $E_1 \notin \Gamma^0$, it follows that $b_2' \neq 0$. Thus $\Gamma = a_1'' E_1 + \langle b_1'' E_1 + E_2, c_1'' E_1 + E_3 \rangle$ with $a_1'' \neq 0$. Therefore

$$\psi = \begin{bmatrix} a_1'' & a_1'' b_1'' & c_1'' \\ 0 & a_1'' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_2 = \Gamma$.

Lastly, assume $E_3^*(\Gamma^0) = \{0\}$ and $E_3^*(A) = \alpha \neq 0$. Then $\Gamma^0 = \langle E_1, E_2\rangle$ and so $\Gamma = \alpha E_3 + \langle E_1, E_2\rangle = \Gamma_{3,\alpha}$.

Finally, we verify that none of the representatives are $\mathfrak{L}$-equivalent. As E_1 is an eigenvector of every automorphism, it follows that Γ_2 cannot be $\mathfrak{L}$-equivalent to Γ_1 or $\Gamma_{3,\alpha}$. Then again, Γ_2 cannot be $\mathfrak{L}$-equivalent to $\Gamma_{3,\alpha}$ as $E_2 \in \Gamma_1$ and $\langle E_1, E_2\rangle$ is an invariant subspace of every automorphism. Lastly, as $E_3^*(\psi \cdot \alpha E_3) = \alpha$ for any automorphism ψ, it follows that $\Gamma_{2,\alpha}$ and $\Gamma_{2,\alpha'}$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. $\square$

Theorem 2. *Any affine subspace of* $\mathfrak{g}_{3.2}$ *(type IV) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_2+\langle E_3\rangle$, $\alpha\, E_3+\langle E_2\rangle$, $\langle E_2, E_3\rangle$, $E_1+\langle E_2, E_3\rangle$, $E_2+\langle E_3, E_1\rangle$, *and* $\alpha E_3+\langle E_1, E_2\rangle$. *Here* $\alpha \neq 0$ *parametrises two families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

6. Type V

The Lie algebra $\mathfrak{g}_{3.3}$ has commutator relations given by

$$[E_2, E_3] = -E_2\,, \quad [E_3, E_1] = E_1\,, \quad [E_1, E_2] = 0$$

in terms of an (appropriate) ordered basis (E_1, E_2, E_3). With respect to this basis, the group of automorphisms is

$$\mathsf{Aut}(\mathfrak{g}_{3.3}) = \left\{ \begin{bmatrix} x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{bmatrix} \;:\; x, y, z, u, v, w \in \mathbb{R},\; xv \neq yu \right\}.$$

Many of the affine subspaces of $\mathfrak{g}_{3.3}$ do not have full rank.

Proposition 9. *No one-dimensional or homogeneous two-dimensional affine subspace of* $\mathfrak{g}_{3.3}$ *has full rank.*

Proof. An one-dimensional affine subspace $\Gamma = A + \langle B\rangle$, or a homogeneous two-dimensional subspace $\Gamma = \langle A, B\rangle$, has full rank if and only if A, B, and $[A, B]$ are linearly independent. Let $A = \sum_{i=1}^3 a_i E_i$ and $B = \sum_{i=1}^3 b_i E_i$. Then $[A, B] = (-a_1b_3 + a_3b_1)E_1 + (-a_2b_3 + a_3b_2)E_2$. A direct computation shows that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ -a_1b_3 + a_3b_1 & -a_2b_3 + a_3b_2 & 0 \end{vmatrix} = 0\,.$$

Hence A, B, and $[A, B]$ are necessarily linearly dependent. $\square$

Accordingly, we need only consider the inhomogeneous two-dimensional case.

Theorem 3. *Any affine subspace of* $\mathfrak{g}_{3.3}$ *(type V) is* $\mathfrak{L}$*-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = E_2 + \langle E_1, E_3\rangle \qquad\qquad \Gamma_{2,\alpha} = \alpha E_3 + \langle E_1, E_2\rangle\,.$$

Here $\alpha \neq 0$ *parametrises a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Proof. Let $\Gamma = A + \Gamma^0$ be a $(2,1)$-affine subspace of $\mathfrak{g}_{3.3}$. First, assume that $E_3^*(\Gamma^0) \neq \{0\}$. (Again, E_3^* denotes the corresponding element of the dual basis.) Then $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle$ with $c_3 \neq 0$. Hence $\Gamma = a_1' E_1 + a_2' E_2 + \langle b_1' E_1 + b_2' E_2, c_1' E_1 + c_2' E_2 + E_3 \rangle$. As Γ is inhomogeneous, it follows that $a_1' b_2' - a_2' b_1' \neq 0$. Thus

$$\psi = \begin{bmatrix} b_1' & a_1' & c_1' \\ b_2' & a_2' & c_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is a automorphism such that $\psi \cdot \Gamma_1 = \Gamma$. On the other hand, assume $E_3^*(\Gamma^0) = \{0\}$ and $E_3^*(A) = \alpha \neq 0$. Then $\Gamma^0 = \langle E_1, E_2 \rangle$ and so $\Gamma = \alpha E_3 + \langle E_1, E_2 \rangle = \Gamma_{2,\alpha}$.

Lastly, we verify that none of these representatives are equivalent. As $\langle E_1, E_2 \rangle$ is an invariant subspace of every automorphism, it follows that $\Gamma_{2,\alpha}$ cannot be $\mathfrak{L}$-equivalent to Γ_1. Then again, as $E_3^*(\psi \cdot \alpha E_3) = \alpha$ for any automorphism ψ, it follows that $\Gamma_{2,\alpha}$ and $\Gamma_{2,\alpha'}$ are equivalent only if $\alpha = \alpha'$. $\square$

7. Final remark

This paper forms part of a series in which the full-rank left-invariant control affine systems, evolving on three-dimensional Lie groups, are classified. A summary of this classification can be found in [4]. The remaining solvable cases are treated in [6], whereas the semisimple cases are treated in [5].

Tabulation of results

Type	Commutators	Automorphisms
II	$[E_2, E_3] = E_1$ $[E_3, E_1] = 0$ $[E_1, E_2] = 0$	$\begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix}$; $yw \neq vz$
IV	$[E_2, E_3] = E_1 - E_2$ $[E_3, E_1] = E_1$ $[E_1, E_2] = 0$	$\begin{bmatrix} u & x & y \\ 0 & u & z \\ 0 & 0 & 1 \end{bmatrix}$; $u \neq 0$
V	$[E_2, E_3] = -E_2$ $[E_3, E_1] = E_1$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{bmatrix}$; $xv \neq yu$

Tab. 2: Lie algebra automorphisms

Type	(ℓ, ε)	Classifying conditions		Equiv. repr.
II	$(1,1)$			$E_2 + \langle E_3 \rangle$
	$(2,0)$			$\langle E_2, E_3 \rangle$
	$(2,1)$	$E_1 \notin \Gamma^0$		$E_1 + \langle E_2, E_3 \rangle$
		$E_1 \in \Gamma^0$		$E_3 + \langle E_1, E_2 \rangle$
IV	$(1,1)$	$E_3^*(\Gamma^0) \neq \{0\}$		$E_2 + \langle E_3 \rangle$
		$E_3^*(\Gamma^0) = \{0\}$, $E_3^*(A) = \alpha \neq 0$		$\alpha E_3 + \langle E_2 \rangle$
	$(2,0)$			$\langle E_2, E_3 \rangle$
	$(2,1)$	$E_3^*(\Gamma^0) \neq \{0\}$	$E_1 \notin \Gamma^0$	$E_1 + \langle E_2, E_3 \rangle$
			$E_1 \in \Gamma^0$	$E_2 + \langle E_1, E_3 \rangle$
		$E_3^*(\Gamma^0) = \{0\}$, $E_3^*(A) = \alpha \neq 0$		$\alpha E_3 + \langle E_1, E_2 \rangle$
V	$(1,1)$			$\emptyset$
	$(2,0)$			$\emptyset$
	$(2,1)$	$E_3^*(\Gamma^0) \neq \{0\}$		$E_1 + \langle E_2, E_3 \rangle$
		$E_3^*(\Gamma^0) = \{0\}$, $E_3^*(A) = \alpha \neq 0$		$\alpha E_3 + \langle E_1, E_2 \rangle$

Tab. 3: Full-rank affine subspaces of Lie algebras

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