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SOLVABLE EXTENSIONS OF A SPECIAL CLASS OF NILPOTENT LIE ALGEBRAS

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Abstract. A pair of sequences of nilpotent Lie algebras denoted by $N_{n,11}$ and $N_{n,19}$ are introduced. Here n denotes the dimension of the algebras that are defined for $n \geq 6$; the first term in the sequences are denoted by 6.11 and 6.19, respectively, in the standard list of six-dimensional Lie algebras. For each of $N_{n,11}$ and $N_{n,19}$ all possible solvable extensions are constructed so that $N_{n,11}$ and $N_{n,19}$ serve as the nilradical of the corresponding solvable algebras. The construction continues Winternitz' and colleagues' program of investigating solvable Lie algebras using special properties rather than trying to extend one dimension at a time.

1. Introduction

The problem of classifying all semisimple Lie algebras over the field of complex numbers was solved by Cartan in 1894, [4] and over the field of real numbers by Gantmacher (1939), [5]. For solvable indecomposable Lie algebras the problem is much more difficult. The classification of solvable Lie algebras only exists for low dimensions and was performed by, amongst others, Mubarakzyanov for solvable Lie algebras of dimension $n \leq 5$ over the field of real and partially over the field of complex numbers in [11] and [13]: the results are summarized in [15]. Mubarakzyanov also considered dimension six and classified solvable Lie algebras with a codimension one nilradical [12]. Then Turkowski classified six-dimensional solvable Lie algebras with a codimension two nilradical in [27]. Also Hindeleh and Thompson, classified seven-dimensional solvable Lie algebras with a four-dimensional nilradical [7]. Nilpotent Lie algebras in dimension six were studied as far back as Umlauf, [28] and later by Morozov, [10] and in dimension seven by Seeley and Gong, [6, 17].

Morozov's results can also be found in [15] and apparently four of them $(5, 10, 14, 18)$ contain parameters; however, it can be shown that in each case over the field of real numbers the parameter a can be reduced to ± 1 and that the case 5 for which $a = 1$ lead to a decomposable algebra and the case 10 for which $a = 1$ is isomorphic to 6.8, so that overall there are precisely 24 non-isomorphic

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six-dimensional indecomposable nilpotent Lie algebras none of which depend on parameters.

If we consider such nilpotent Lie algebras $(5, 10, 14, 18)$ over the field of complex numbers, the parameter a could be reduced to 1 and the case 10 will be removed completely because if $a = 1$ the algebra 6.10 is isomorphic to 6.8, and the parameter a could be removed to -1 in the case 5 if we want to have only indecomposable nilpotent Lie algebras. So there are exactly 21 non-isomorphic six-dimensional indecomposable nilpotent Lie algebras over the field of complex numbers.

One of the problems in attempting to classify solvable Lie algebras is that as the dimension of the algebra increases the number of parameters increases as well and the calculations become impossible to perform even with the help of a computer. In view of this fact Winternitz and colleagues Snobl, Rubin, Karasek, Tremblay, [16, 14, 22, 23, 24, 25] (see also [26, 30]) have established a program whereby they start with a particular nilpotent Lie algebra and try to find all possible solvable extensions of it. In dimension five there are up to isomorphism six nilpotent indecomposable Lie algebras that we denote by $N_{5,1}$, $N_{5,2}$, $N_{5,3}$, $N_{5,4}$, $N_{5,5}$, $N_{5,6}$. There also three decomposable Lie algebras $\mathbb{R}^5$, $N_{3,1} \oplus \mathbb{R}^2$, $N_{4,1} \oplus \mathbb{R}$. See [12]. The following references [16, 14, 23, 24, 25, 30] consider sequences of nilpotent Lie algebras that generalize $\mathbb{R}^5$, $N_{4,1} \oplus \mathbb{R}$, $N_{5,2}$, $N_{5,4}$, $N_{5,5}$, $N_{5,6}$ in the sense that the five-dimensional algebras are the first term in the sequence. Then the authors consider all possible extensions to solvable algebras so that the original nilpotent algebra is its nilradical. It is worth noting that in the cases of $N_{4,1} \oplus \mathbb{R}$ and $N_{5,1}$ there are already plethoras of possible extensions to a six-dimensional solvable Lie algebra. In the case of $N_{5,3}$ there is no obvious way to extend the algebra so to produce a nilpotent sequence.

The starting point of the present article is the six-dimensional nilpotent Lie algebras, [10, 15] of which, as we have said, there are 24 such algebras over the field of real numbers. It is interesting that up to dimension six the nilpotent Lie algebras do not contain parameters although they do enter into the classification of the seven-dimensional nilpotent algebras, [6, 17]. The present paper adds to the program of Winternitz et al. by creating a pair of sequences of nilpotent Lie algebras $N_{n,11}$ and $N_{n,19}$ for which the first terms in the sequence are the six-dimensional algebras $N_{6,11}$ and $N_{6,19}$. For each of these pairs we obtain all possible solvable indecomposable extensions. In the case of $N_{n,11}$ extensions of only dimensions one and two are possible: there are six classes of one-dimensional extensions where three of them depend on parameters and one algebra of two-dimensional extension which does not depend on parameters. In the case of $N_{n,19}$ extensions of only dimension one are possible and there is a unique such algebra up to isomorphism. The results are summarized in Theorems 4.1, 4.2 and 4.3. The complicated details of the proofs of Theorems 4.1 and 4.3 have been off-loaded into an Appendix.

It would be interesting to compare the results obtained in this article and elsewhere with the idea of naturally graded algebras introduced in [1, 2, 3]. In particular Campoamor-Stursberg and his colleagues find that in order for an algebra to be naturally graded it must be *filiform* although they do not use this terminology. In fact a nilpotent algebra is filiform if its lower central series is as long as possible, which means the dimensions drop by two at the first step and then one by at all remaining steps. Of the two algebras that we consider $N_{n,11}$ is not filiform whereas $N_{n,19}$ is.

As regards notation we use $\langle e_1, e_2, \dots, e_r\rangle$ to denote the r-dimensional subspace of $\mathfrak{g}$ generated by $e_1, e_2, \dots, e_r$, where $r \in \mathbb{N}$. Below we use DS for the derived series, LS for the lower central series and US for the upper central series of a Lie algebra and refer to then collectively as the characteristic series. Let $\mathfrak{g}$ be any Lie algebra.

The *derived series* is a sequence of ideals $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots \supseteq \mathfrak{g}^{(k)} \supseteq \dots$, where $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$, $(k \geq 1)$. We call $\mathfrak{g}$ solvable if $\mathfrak{g}^{(k)} = \{0\}$ for some $k \in \mathbb{N}$.

The *lower central series* is $\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots \supseteq \mathfrak{g}^k \supseteq \dots$, where $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$, $(k \geq 1)$. We call $\mathfrak{g}$ nilpotent if $\mathfrak{g}^k = \{0\}$ for some $k \in \mathbb{N}$.

The *upper central series* is a sequence of ideals $c_1 \subseteq c_2 \subseteq \cdots \subseteq c_k \subseteq \cdots \subseteq \mathfrak{g}$. In this series $c_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0\, \forall\, y \in \mathfrak{g}\}$ is the center of $\mathfrak{g}$, c_k is the unique ideal in $\mathfrak{g}$ such that c_k/c_{k-1} is the center of $\mathfrak{g}/c_{k-1}$. The upper central series terminates i.e. there exists a number $k \in \mathbb{N}$ such that $c_k = \mathfrak{g}$ if and only if $\mathfrak{g}$ is nilpotent, [9].

2. Constructing solvable Lie algebras with a given nilradical

Every solvable Lie algebra $\mathfrak{g}$ contains a unique maximal nilpotent ideal called the nilradical and denoted $\mathfrak{nil}(\mathfrak{g})$ such that $\dim \mathfrak{nil}(\mathfrak{g}) \geq \frac{1}{2} \dim \mathfrak{g}$, [13]. Let us consider the problem of constructing solvable indecomposable Lie algebras $\mathfrak{g}$ with a given nilradical $N = \mathfrak{nil}(\mathfrak{g})$. Suppose $\{e_1, e_2, e_3, \dots, e_n\}$ is a basis for the nilradical and $\{e_{n+1}, \dots, e_p\}$ is a basis for a subspace complementary to the nilradical.

Since the derived algebra of solvable is nilpotent, [8, 9],

$$[\mathfrak{g}, \mathfrak{g}] \subseteq N \tag{2.1}$$

we have the following structure equations

$$[e_i, e_j] = C^k_{ij} e_k\,, \quad [e_a, e_i] = A^k_{ai} e_k\,, \quad [e_a, e_b] = B^k_{ab} e_k\,, \tag{2.2}$$

where $1 \leq i,\, j,\, k,\, l,\, m \leq n$ and $n + 1 \leq a,\, b,\, c \leq p$.

Calculation shows that the Jacobi identity is equivalent to the following conditions:

$$\begin{aligned} &C^l_{[ij}C^m_{k]l} = 0\,, \qquad C^k_{ij}A^l_{ak} = C^l_{ik}A^k_{aj} - C^l_{jk}A^k_{ai}\,, \\ &B^k_{ab}C^l_{ik} = A^l_{bk}A^k_{ai} - A^l_{ak}A^k_{bi}\,, \qquad B^k_{[ab}A^i_{c]k} = 0\,. \end{aligned} \tag{2.3}$$

Then the entries of the matrices A_a must satisfy linear relations (2.3) which come from the Jacobi identity between the triples $\{e_a, e_i, e_j\}$. Also notice that the Jacobi identity (2.3) between the triples $\{e_a, e_b, e_i\}$ gives linear relations for the structure constants B^k_{ab} in terms of the commutators of the matrices A_a and A_b.

Since N is the nilradical of $\mathfrak{g}$, no nontrivial linear combination of the matrices A_a, $(n+1 \leq a \leq p)$ is nilpotent which means that the matrices A_a must be "nil-independent", that is, no non-trivial combination of them is nilpotent [13].

Let us now consider the adjoint representation of $\mathfrak{g}$, restrict it to N and find $ad|_N(e_a)$, $(n+1 \leq a \leq p)$. Notice we shall get outer derivations of the nilradical $N = \mathfrak{nil}(\mathfrak{g})$ [8, 9]. Then finding the matrices A_a is the same as finding outer derivations D^a of N. Further the commutators $[D^a, D^b]$, $(n+1 \leq a,\, b \leq p)$ due to (2.1) consist of inner derivations of N. So those commutators give the structure constants B^k_{ab} but only up to the elements in the center of the nilradical N, because if e_i, $1 \leq i \leq n$ is in the center of N then $ad|_N(e_i) = 0$, where $ad|_N(e_i)$ is an inner derivation of the nilradical N. Notice that outer derivations can be nilpotent whereas inner derivations (of N) must be nilpotent.

Once the outer derivations are found:

(i) We can carry out the technique of "absorption", which means we can simplify outer derivations by adding linear combinations of inner to outer derivations.

(ii) We can change basis such that the brackets for the nilradical in (2.2) are unchanged.

3. The nilpotent sequences $N_{n,11}$ and $N_{n,19}$

3.1. $N_{n,11}$.

In $N_{n,11}$ and $N_{n,19}$ the positive integer n denotes the dimension of the algebras. Each of the algebras contain a maximal abelian ideal of dimension $n-2$. The algebra $N_{n,11}$ can be described explicitly as follows: in the basis $\{e_1, e_2, e_3, e_4, e_5, \ldots, e_n\}$ it has only the following non-zero Lie brackets $(n \geq 6)$

$$[e_1, e_k] = e_{k+1}\,, \quad (2 \leq k \leq n-2)\,, \quad [e_2, e_3] = e_n\,. \tag{3.1}$$

The dimensions of the ideals in the characteristic series are

$$DS = [n, n-2, 0]\,,\ LS = [n, n-2, n-3, n-5, n-6, \ldots, 1, 0]\,,\ US = [2, 3, \ldots, n-2, n]\,.$$

The notation for the nilradical $N_{n,11}$ is based on the six-dimensional nilpotent Lie algebra $A_{6,11}$ [15], which is exactly this algebra for $n = 6$ and N emphasizes the fact that this nilpotent Lie algebra is at the same time the nilradical of the solvable indecomposable Lie algebras that we construct in this paper. It is shown below that solvable indecomposable Lie algebras $\mathfrak{g}$ with the nilradical $N_{n,11}$ only exist for $\dim \mathfrak{g} = n+1$ and $\dim \mathfrak{g} = n+2$.

For the solvable indecomposable Lie algebras with a codimension one nilradical we use the notation $g_{n+1,j}$ where $n+1$ indicates the dimension of the algebra $\mathfrak{g}$ and j its numbering within the list of algebras. There are six types of such algebras up to isomorphism so $1 \leq j \leq 6$. For each n where $(n \geq 6)$ the last algebra $g^i_{n+1,6}$ itself comprises i algebras where $1 \leq i \leq n-5$.

There is only one solvable indecomposable Lie algebra up to isomorphism with a codimension two nilradical denoted $g_{n+2,1}$.

3.2. $N_{n,19}$.

The nilradical $N_{n,19}$, $(n \geq 6)$ in the basis $\{e_1, e_2, e_3, e_4, e_5, \dots, e_n\}$ has the following non-zero Lie brackets

$$(3.2) \quad [e_1, e_i] = e_{i+1}\,, \quad (2 \leq i \leq n-1)\,, \qquad [e_2, e_j] = e_{j+3}\,, \quad (3 \leq j \leq n-3)\,.$$

The dimensions of the ideals in the characteristic series are

$$DS = [n, n-2, 0]\,, \;\; LS = [n, n-2, n-3, n-4, \dots, 1, 0]\,, \;\; US = [1, 2, 3, \dots, n-2, n]\,.$$

Note that $N_{n,19}$ is a *filiform* Lie algebra [29] which means that it is nilpotent and the LS is as long as possible.

The notation for the nilradical $N_{n,19}$ is also based on the six-dimensional nilpotent Lie algebra denoted $A_{6,19}$ [15], which is exactly this algebra for $n = 6$. It is shown below that solvable indecomposable extensions $\mathfrak{g}$ with the nilradical $N_{n,19}$ only exist for $\dim \mathfrak{g} = n + 1$.

It turns out that there is only one solvable indecomposable Lie algebra with a codimension one nilradical $N_{n,19}$ up to isomorphism. It is denoted $g'_{n+1,1}$.

4. Classification of solvable indecomposable Lie algebras with a given nilradical

Our goal in this section is to find all possible solvable indecomposable extensions of the nilpotent Lie algebras $N_{n,11}$ and $N_{n,19}$ which serve as the nilradical of the extended algebra. We begin by finding all possible derivations of these two algebras before classifying the extensions proper.

4.1. **Derivations of $N_{n,11}$.**

The nilpotent Lie algebra $N_{n,11}$ is defined in (3.1), where $n \geq 6$. In order to find all derivations of $N_{n,11}$ let us consider the structure of automorphisms of $N_{n,11}$. Looking at the LS we see that there are no terms in the sequence of dimension $n-1$ and $n-4$; however, we can obtain a complete flag of ideals, (where each term in the flag has codimension one in the previous one), which is invariant under any automorphism

$$\mathfrak{n} \supset (\mathfrak{n}^{n-4})_n \supset \mathfrak{n}^1 \supset \mathfrak{n}^2 \supset c_{n-5} \supset \mathfrak{n}^3 \supset \mathfrak{n}^4 \supset \cdots \supset \mathfrak{n}^{n-3} \supset \{0\}\,.$$

Here $\mathfrak{n} = \mathfrak{n}^0 = N_{n,11}$; $\mathfrak{n}^k = [\mathfrak{n}^{k-1}, \mathfrak{n}]$, $(k \geq 1)$; $(\mathfrak{n}^{n-4})_{\mathfrak{n}}$ denotes the centralizer of $\mathfrak{n}^{n-4}$ in $\mathfrak{n}$ and c_{n-5} is the $(n-5)$-th term of the upper central series.

In any basis consistent with the flag (structure introduced above) any automorphism will be represented by a lower triangular matrix and so too will any derivation. If D is any such derivation it is apparent from the Lie bracket structure that D is completely determined once its effect on e_1 and e_2 are known. In particular D depends on at most $2n-1$ parameters.

Let us write $D(e_1) = \sum_{k=1}^{n} D_{k,1} e_k$ and $D(e_2) = \sum_{k=2}^{n} D_{k,2} e_k$. Since D is a derivation we find that $D(e_3) = (D_{1,1} + D_{2,2}) e_3 + \sum_{k=3}^{n-2} D_{k,2} e_{k+1} - D_{3,1} e_n$, $D(e_4) = (2D_{1,1} + D_{2,2}) e_4 + \sum_{i=4}^{n-2} D_{i-1,2} e_{i+1} + D_{2,1} e_n$; $D(e_j) = [(j-2) D_{1,1} + D_{2,2}] e_j + \sum_{i=j+1}^{n-1} D_{i-j+2,2} e_i$, $(5 \leq j \leq n-1)$ and $D(e_n) = (2D_{2,2} + D_{1,1}) e_n$.

Therefore the vector space of derivations is represented by the $n \times n$ matrix depending on $2n-1$ parameters ($n \geq 6$) given below in which all the parameters $D_{i,j}$'s that appear are independent of each other:

$$D = \begin{bmatrix} D_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ D_{2,1} & D_{2,2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ D_{3,1} & D_{3,2} & D_{1,1}+D_{2,2} & 0 & 0 & \cdots & 0 & 0 \\ D_{4,1} & D_{4,2} & D_{3,2} & 2D_{1,1}+D_{2,2} & 0 & \cdots & 0 & 0 \\ D_{5,1} & D_{5,2} & D_{4,2} & D_{3,2} & 3D_{1,1}+D_{2,2} & & \vdots & \vdots \\ D_{6,1} & D_{6,2} & D_{5,2} & D_{4,2} & D_{3,2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ D_{n-1,1} & D_{n-1,2} & D_{n-2,2} & D_{n-3,2} & D_{n-4,2} & \cdots\ D_{3,2} & (n-3)D_{1,1}+D_{2,2} & 0 \\ D_{n,1} & D_{n,2} & -D_{3,1} & D_{2,1} & 0 & \cdots & 0 & 2D_{2,2}+D_{1,1} \end{bmatrix}$$

On the other hand the non-zero inner derivations of $N_{n,11}$ are given by

$$ad(e_1) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad ad(e_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$ad(e_3) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -1 & 0 & \cdots & 0 \end{bmatrix}, \quad ad(e_i) = -E_{i+1,1} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

for $4 \leq i \leq n-2$, where $E_{i+1,1}$ is the matrix which has 1 in the $(i+1,1)$-th entry and all other entries zero. Then considering $D - D_{3,2}ad(e_1) + D_{3,1}ad(e_2) + D_{4,1}ad(e_3) + \sum_{i=4}^{n-2} D_{i+1,1}ad(e_i)$ and setting $D_{1,1} = a^r$ and $D_{2,2} = b^r$, $(1 \leq r \leq p-n,\ n \geq 6)$

gives an equivalent but simpler space of outer derivations:

$$D^r = \begin{bmatrix} a^r & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ D^r_{2,1} & b^r & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a^r+b^r & 0 & 0 & \cdots & 0 & 0 \\ 0 & D^r_{4,2} & 0 & 2a^r+b^r & 0 & \cdots & 0 & 0 \\ 0 & D^r_{5,2} & D^r_{4,2} & 0 & 3a^r+b^r & & \vdots & \vdots \\ 0 & D^r_{6,2} & D^r_{5,2} & D^r_{4,2} & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ 0 & D^r_{n-1,2} & D^r_{n-2,2} & D^r_{n-3,2} & D^r_{n-4,2} & \cdots \; 0 & (n-3)a^r+b^r & 0 \\ D^r_{n,1} & D^r_{n,2} & 0 & D^r_{2,1} & 0 & \cdots & 0 & 2b^r+a^r \end{bmatrix},$$

where $p-n$ is the number of outer derivations, such as $D^1 = -ad|_N(e_{n+1})$, $D^2 = -ad|_N(e_{n+2}), \ldots, D^{p-n} = -ad|_N(e_p)$ and $e_{n+1}, e_{n+2}, \ldots, e_p$ (2.2) are non-nilpotent basis elements of $\mathfrak{g}$.

Remark 4.1. If we have three or more derivations they are nil-dependent. Therefore the solvable algebra that we are constructing is of codimension at most two.

Remark 4.2. It is assumed throughout the next subsection that the solvable Lie algebras have the nilradical $N_{n,11}$ and in the subsection after that $N_{n,19}$; however, for the sake of simplicity the Lie brackets of the nilradical will be omitted.

4.2. Finding solvable indecomposable Lie algebras with a codimension one nilradical $N_{n,11}$.

Recall that $\{e_1, e_2, e_3, e_4, \ldots, e_n\}$ is a basis for $N_{n,11}$ defined by (3.1), and e_{n+1} is not in $N_{n,11}$. We have $D^1 = -ad|_{N_{n,11}}(e_{n+1})$. Therefore to find solvable algebras we need to simplify D^1 as much as possible by changing basis such that the brackets of $N_{n,11}$ are unchanged. There are a number of cases depending on the possible values of a^1 and b^1. Although we state a number of conditions at the beginning of each case they only become apparent after the fact as we work through the calculation and simplify the form of the algebra. The simplifications involved are elementary and do offer much insight to the results obtained; therefore we relegate the details to the Appendix and simply summarize the results in the following theorem.

Theorem 4.1. *There are six types of solvable indecomposable Lie algebras with a codimension one nilradical $N_{n,11}$, unique up to isomorphism, which are given below:*

(i)

$g_{n+1,1} : [e_1, e_{n+1}] = e_1,\ [e_k, e_{n+1}] = (b+k-2)e_k, \quad (2 \le k \le n-1),$

$[e_n, e_{n+1}] = (2b+1)e_n, \quad (n \ge 6),$

$DS = [n+1, n, n-2, 0], \quad LS = [n+1, n, n, \ldots], \quad US = [0].$

(ii)

$$\begin{aligned}
&g_{n+1,2}:[e_1,e_{n+1}]=e_1+e_2,\ [e_k,e_{n+1}]=(k-1)e_k,\quad (k\neq 4, 2\leq k\leq n-1),\\
&[e_4,e_{n+1}]=3e_4+e_n,\ [e_n,e_{n+1}]=3e_n,\quad (n\geq 6),\\
&DS=[n+1,n,n-2,0],\quad LS=[n+1,n,n,\dots],\quad US=[0].
\end{aligned}$$

(iii)

$$\begin{aligned}
&g_{n+1,3}:[e_1,e_{n+1}]=-e_1+\epsilon e_4,\ [e_2,e_{n+1}]=e_2+de_n,\\
&[e_k,e_{n+1}]=(3-k)e_k,\quad (4\leq k\leq n-1),\\
&[e_n,e_{n+1}]=e_n,\quad (\epsilon=0,1,\ \epsilon^2+d^2\neq 0,\ d\in\mathbb{R},\ n\geq 6),\\
&DS=[n+1,n,n-2,0]\,,\quad LS=[n+1,n,n,\dots]\,,\quad US=[0].
\end{aligned}$$

(iv)

$$\begin{aligned}
&g_{n+1,4}:[e_k,e_{n+1}]=e_k+\epsilon e_{k+2}+\sum_{i=k+3}^{n-1}d_{i-k-2}e_i\,,\quad (2\leq k\leq n-3,\ d_{i-k-2}\in\mathbb{R}),\\
&[e_j,e_{n+1}]=e_j\,,\quad (n-2\leq j\leq n-1)\,,\quad [e_n,e_{n+1}]=2e_n\,,\quad (\epsilon=0,\pm 1,\ n\geq 6),\\
&DS=[n+1,n-1,1,0]\,,\quad LS=[n+1,n-1,n-1,\dots]\,,\quad US=[0].
\end{aligned}$$

(v)

$$\begin{aligned}
&g_{n+1,5}:[e_1,e_{n+1}]=e_1+\epsilon e_n,\ [e_k,e_{n+1}]=(k-2)e_k,\quad (3\leq k\leq n-1),\\
&[e_n,e_{n+1}]=e_n,\quad (\epsilon=\pm 1,\ n\geq 6),\\
&DS=[n+1,n-1,n-4,0]\,,\quad LS=[n+1,n-1,n-1,\dots]\,,\quad US=[0].
\end{aligned}$$

(vi)

$$\begin{aligned}
&g^i_{n+1,6}:[e_1,e_{n+1}]=e_1+e_{n-i},\ [e_k,e_{n+1}]=(1+i+k-n)e_k,\quad (2\leq k\leq n-1),\\
&[e_n,e_{n+1}]=(7-2(n-i))e_n,\quad (1\leq i\leq n-5,\ n\geq 6),\\
&DS=[n+1,n,n-2,0]\,,\quad LS=[n+1,n,n,\dots]\,,\quad US=[0].
\end{aligned}$$

4.3. **Finding solvable indecomposable Lie algebras with a codimension two nilradical $N_{n,11}$.**
The nilradical $N_{n,11}$ in the basis $\{e_1,e_2,e_3,\dots,e_n\}$ is defined in (3.1). The vectors complementary to the nilradical are e_{n+1} and e_{n+2}. Hence $D^1=-ad|_{N_{n,11}}(e_{n+1})$ and $D^2=-ad|_{N_{n,11}}(e_{n+2})$ are given by the matrices before Remark 4.1. By taking linear combinations of D^1 and D^2 and keeping in mind that no nontrivial linear combination of the matrices D^1 and D^2 can be a nilpotent matrix, we could set $a^1=1$, $b^1=0$ and $a^2=0$, $b^2=1$.
Applying the transformations found in Appendix in part (v) of the proof of Theorem 4.1 with $a=1$ and $b=0$, except the part of scaling $D^1_{n,1}$ to ± 1 if different from zero, we obtain

$$D^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & n-3 & 0 \\ D^1_{n,1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

These transformations affect D^2 as well introducing the entries in $(4,1)$, $(5,1)$, ..., $(n-1,1)$ positions, which we denote by $D^2_{4,1}, D^2_{5,1}, \dots, D^2_{n-1,1}$. Besides the entries in $(2,1)$ and $(n,4)$ positions need to be renamed back by $D^2_{2,1}$, the entries in $(n,1)$ and $(n,2)$ positions by $D^2_{n,1}$ and $D^2_{n,2}$, respectively. But applying $D^2 + D^2_{4,1}ad(e_3) + \sum_{i=4}^{n-2} D^2_{i+1,1}ad(e_i)$, $D^2_{4,1}, D^2_{5,1}, \dots, D^2_{n-1,1}$ could be removed. Consider

$$[D^1, D^2] = [ad|_{N_{n,11}}(e_{n+1}), ad|_{N_{n,11}}(e_{n+2})] =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & & \cdots & 0 & 0 \\ -D^2_{2,1} & 0 & 0 & 0 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & \cdots & 0 & 0 \\ 0 & 2D^2_{4,2} & 0 & 0 & 0 & & \cdots & 0 & 0 \\ 0 & 3D^2_{5,2} & 2D^2_{4,2} & 0 & 0 & & & \vdots & \vdots \\ 0 & 4D^2_{6,2} & 3D^2_{5,2} & 2D^2_{4,2} & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \\ 0 & (n-3)D^2_{n-1,2} & (n-4)D^2_{n-2,2} & (n-5)D^2_{n-3,2} & \cdots & 2D^2_{4,2} & 0 & 0 & 0 \\ -2D^1_{n,1} & D^2_{n,2} & 0 & -D^2_{2,1} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

keeping in mind that $D^2_{3,2} = 0$ and $n \geq 6$.

Since the center of $N_{n,11}$ is $\langle e_{n-1}, e_n \rangle$, therefore $[D^1, D^2] \subseteq \langle ad(e_1), ad(e_2), ad(e_3), \dots, ad(e_{n-2}) \rangle$, i.e. $[D^1, D^2] = \sum_{i=1}^{n-2} x_i ad(e_i)$. Therefore we obtain $D^2_{2,1} = D^2_{4,2} = D^2_{5,2} = \cdots = D^2_{n,2} = D^1_{n,1} = 0$. It implies $[ad|_{N_{n,11}}(e_{n+1}), ad|_{N_{n,11}}(e_{n+2})] = 0$ but $D^2_{n,1}$ could be nonzero and remains in D^2.

The transformation which fixes everything except e_1 such that $e_1' = e_1 - \frac{D^2_{n,1}}{2} e_n$ removes $D^2_{n,1}$.

We have $[e_{n+1}, e_{n+2}] = xe_{n-1} + ye_n$ because e_{n-1} and e_n are in the center of $N_{n,11}$, $(n \geq 6)$.
Applying the transformation which fixes everything but e_{n+1} and e_{n+2} such that $e'_{n+1} = e_{n+1} - xe_{n-1}$, $e'_{n+2} = e_{n+2} + ye_n$, we remove x and y and obtain $[e_{n+1}, e_{n+2}] = 0$.

Theorem 4.2. *There is the only solvable indecomposable Lie algebra up to isomorphism with a codimension two nilradical $N_{n,11}$, which is given below:*

$$g_{n+2,1} : [e_1, e_{n+1}] = e_1\,,\ [e_i, e_{n+1}] = (i-2)e_i\,,\ (3 \leq i \leq n-1)\,, \quad [e_n, e_{n+1}] = e_n\,,$$
$$[e_k, e_{n+2}] = e_k\,,\ (2 \leq k \leq n-1)\,,\ [e_n, e_{n+2}] = 2e_n\,, \quad (n \geq 6)\,,$$
$$DS = [n+2, n, n-2, 0]\,, \quad LS = [n+2, n, n, \dots]\,, \quad US = [0]\,.$$

4.4. **Derivations of the nilradical $N_{n,19}$.**
The nilpotent Lie algebra $N_{n,19}$, $(n \geq 6)$ is defined in (3.2). In order to find all non-nilpotent (outer) derivations of $N_{n,19}$ let us consider the structure of automorphisms of $N_{n,19}$. There exists a flag of ideals (where each ideal in the flag has codimension one in the previous one), which is invariant under any automorphism

$$\mathfrak{n} \supset (\mathfrak{n}^{n-3})_{\mathfrak{n}} \supset \mathfrak{n}^1 \supset \mathfrak{n}^2 \supset \mathfrak{n}^3 \supset \mathfrak{n}^4 \supset \dots \supset \mathfrak{n}^{n-2} \supset \{0\}$$

where $\mathfrak{n} = \mathfrak{n}^0 = N_{n,19}$, $\mathfrak{n}^k = [\mathfrak{n}^{k-1}, \mathfrak{n}]$, $(k \geq 1)$ and $(\mathfrak{n}^{n-3})_{\mathfrak{n}}$ denotes the centralizer of $\mathfrak{n}^{n-3}$ in $\mathfrak{n}$, [8, 9].

In any basis respecting the flag such as used in (3.2), any automorphism will be represented by a lower triangular matrix. Let us denote this automorphism by ϕ and define $\phi(e_1) = \sum_{k=1}^{n} a_k e_k$ and $\phi(e_2) = \sum_{i=2}^{n} b_i e_i$. Because of the structure of the nilradical $N_{n,19}$, knowledge of $\phi(e_1)$ and $\phi(e_2)$ enables to determine ϕ completely.

We have $[\phi(e_2), \phi(e_j)] = \phi(e_{j+3})$, $(3 \leq j \leq n-3)$ i.e.

$$\begin{aligned} ad\big(\phi(e_2)\big)\big(\phi(e_j)\big) &= [\phi(e_1), \phi(e_{j+2})] = \big[\phi(e_1), [\phi(e_1), \phi(e_{j+1})]\big] \\ &= \big[\phi(e_1), [\phi(e_1), [\phi(e_1), \phi(e_j)]\big] = \big[ad\big(\phi(e_1)\big)\big]^3\big(\phi(e_j)\big)\,, \\ &\quad (3 \leq j \leq n-3)\,. \end{aligned}$$

Hence $ad_{|_{n^1}}(\phi(e_2)) = [ad_{|_{n^1}}(\phi(e_1))]^3$, where $n^1 = \langle e_3, e_4, e_5, e_6, \dots, e_n \rangle$.

We have $[\phi(e_1), e_j] = a_1 e_{j+1} + a_2 e_{j+3}$, $(3 \leq j \leq n-3)$, $[\phi(e_1), e_k] = a_1 e_{k+1}$, $(n-2 \leq k \leq n-1)$ and $[\phi(e_1), e_n] = 0$. Therefore

$$ad_{|_{n^1}}(\phi(e_1)) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 & 0 & a_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_2 & 0 & a_1 & 0 \end{bmatrix}.$$

Then $[\phi(e_2), e_j] = [b_2e_2, e_j] = b_2e_{j+3}$, $(3 \leq j \leq n-3)$ and $[\phi(e_2), e_k] = 0$, $(n-2 \leq k \leq n)$ give

$$ad_{|_{n^1}}(\phi(e_2)) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_2 & 0 & 0 & 0 \end{bmatrix}.$$

Thus we have $b_2 = a_1^3$,$(n \geq 6)$ and $a_2 = 0$, $(n \geq 8)$. Hence $\phi(e_1) = a_1e_1 + a_2e_2 + \sum_{k=3}^{n} a_ke_k$ and $\phi(e_2) = a_1^3e_2 + \sum_{i=3}^{n} b_ie_i$, which give that the automorphism ϕ and hence all outer derivations by considering automorphism infinitesimally close to the identity depend on $2n-2$ for $n = 6, 7$ or $2n-3$ for $n \geq 8$ parameters.

Let D be an outer derivation of $N_{n,19}$. We find $D(e_1) = \sum_{k=1}^{n} D_{k,1}e_k$ and $D(e_2) = 3D_{1,1}e_2 + \sum_{i=3}^{n} D_{i,2}e_i$. Since D is a derivation, we have $D[e_i, e_j] = [De_i, e_j] + [e_i, De_j]$, $(1 \leq i, j \leq n)$ and $D(e_3) = 4D_{1,1}e_3 + D_{3,2}e_4 + D_{4,2}e_5 + \sum_{k=6}^{n}(D_{k-1,2} - D_{k-3,1})e_k$. Renaming $D_{k-1,2} - D_{k-3,1}$ by $D_{k,3}$,$(k \geq 6)$ we have $D(e_2) = 3D_{1,1}e_2 + D_{3,2}e_3 + D_{4,2}e_4 + \sum_{i=5}^{n-1}(D_{i+1,3} + D_{i-2,1})e_i + D_{n,2}e_n$ and $D(e_3) = 4D_{1,1}e_3 + D_{3,2}e_4 + D_{4,2}e_5 + \sum_{k=6}^{n} D_{k,3}e_k$. Similarly we find $D(e_j) = (j+1)D_{1,1}e_j + D_{3,2}e_{j+1} + [(j-3)D_{2,1} + D_{4,2}]e_{j+2} + \sum_{i=j+3}^{n} D_{i-j+3,3}e_i$, $(4 \leq j \leq n)$, where $e_{n+1} = e_{n+2} = 0$ by (2.1).

Therefore all outer derivations are represented by the $n \times n$ matrix given below depending on $2n-2$ for $n = 6, 7$ or $2n-3$ for $n \geq 8$ parameters $D_{i,j}$:

$D =$

$$\begin{bmatrix} D_{1,1} & 0 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ D_{2,1} & 3D_{1,1} & 0 & 0 & 0 & \cdots & & 0 & 0 \\ D_{3,1} & D_{3,2} & 4D_{1,1} & 0 & 0 & \cdots & & 0 & 0 \\ D_{4,1} & D_{4,2} & D_{3,2} & 5D_{1,1} & 0 & \cdots & & 0 & 0 \\ D_{5,1} & D_{6,3}+D_{3,1} & D_{4,2} & D_{3,2} & 6D_{1,1} & & & \vdots & \vdots \\ D_{6,1} & D_{7,3}+D_{4,1} & D_{6,3} & D_{2,1}+D_{4,2} & D_{3,2} & & & \vdots & \vdots \\ D_{7,1} & D_{8,3}+D_{5,1} & D_{7,3} & D_{6,3} & 2D_{2,1}+D_{4,2} & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ D_{n-1,1} & D_{n,3}+D_{n-3,1} & D_{n-1,3} & D_{n-2,3} & D_{n-3,3} & \cdots \ddots & D_{3,2} & nD_{1,1} & 0 \\ D_{n,1} & D_{n,2} & D_{n,3} & D_{n-1,3} & D_{n-2,3} & \cdots D_{6,3} & (n-5)D_{2,1}+D_{4,2} & D_{3,2} & (n+1)D_{1,1} \end{bmatrix},$$

such that $D_{2,1} = 0 \quad$ for $n \geq 8$.

The non-zero inner derivations of the $N_{n,19}$ are

$$ad(e_1)=\begin{bmatrix} 0&0&0&\cdots&0&0&0\\ 0&0&0&\cdots&0&0&0\\ 0&1&0&\cdots&0&0&0\\ 0&0&1&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&1&0&0\\ 0&0&0&\cdots&0&1&0 \end{bmatrix},\quad ad(e_2)=\begin{bmatrix} 0&0&0&\cdots&0&0&0&0\\ 0&0&0&\cdots&0&0&0&0\\ -1&0&0&\cdots&0&0&0&0\\ 0&0&0&\cdots&0&0&0&0\\ 0&0&0&\cdots&0&0&0&0\\ 0&0&1&\cdots&0&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&1&0&0&0 \end{bmatrix},$$

$$ad(e_i)=-E_{i+1,1}-E_{i+3,2}=\begin{bmatrix} 0&0&\cdots&0\\ 0&0&\cdots&0\\ 0&0&\cdots&0\\ -1&0&\cdots&0\\ \vdots&\vdots&&\vdots\\ 0&0&\cdots&0\\ 0&0&\cdots&0 \end{bmatrix}+\begin{bmatrix} 0&0&\cdots&0\\ 0&0&\cdots&0\\ 0&0&\cdots&0\\ 0&0&\cdots&0\\ 0&0&\cdots&0\\ 0&-1&\cdots&0\\ \vdots&\vdots&&\vdots\\ 0&.&\cdots&0 \end{bmatrix},\ (3\le i\le n-3),$$

$$ad(e_j)=-E_{j+1,1}=\begin{bmatrix} 0&0&\cdots&0\\ 0&0&\cdots&0\\ 0&0&\cdots&0\\ 0&0&\cdots&0\\ \vdots&\vdots&&\vdots\\ -1&0&\cdots&0\\ .&0&\cdots&0 \end{bmatrix},\quad (n-2\le j\le n-1),$$

where $E_{i,j}$ denote the matrices that have 1 in (i,j)th entry and zeros everywhere else.

Then considering $D-D_{3,2}ad\,(e_1)+D_{3,1}ad\,(e_2)+\sum_{i=3}^{n-3}D_{i+1,1}ad\,(e_i)+$ $+\sum_{j=n-2}^{n-1}D_{j+1,1}ad(e_j)$, setting $D_{1,1}=a$ and renaming $D_{6,3}+D_{3,1}$ by $D_{6,3}$ give the equivalent outer derivation

$D'=$

$$\begin{bmatrix} a&0&0&0&0&\cdots&&&0&0\\ D_{2,1}&3a&0&0&0&\cdots&&&0&0\\ 0&0&4a&0&0&\cdots&&&0&0\\ 0&D_{4,2}&0&5a&0&\cdots&&&0&0\\ 0&D_{6,3}&D_{4,2}&0&6a&&&&\vdots&\vdots\\ 0&D_{7,3}&D_{6,3}&D_{2,1}+D_{4,2}&0&&&&\vdots&\vdots\\ 0&D_{8,3}&D_{7,3}&D_{6,3}&2D_{2,1}+D_{4,2}&&&&\vdots&\vdots\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&&&&\\ 0&D_{n,3}&D_{n-1,3}&D_{n-2,3}&D_{n-3,3}&\cdots&\ddots&0&na&0\\ 0&D_{n,2}-D_{n-2,1}&D_{n,3}&D_{n-1,3}&D_{n-2,3}&\cdots&D_{6,3}&(n-5)D_{2,1}+D_{4,2}&0&(n+1)a \end{bmatrix},$$

such that $D_{2,1}=0$ for $n\ge 8$ and $a\ne 0$ otherwise D' is nilpotent.

Remark 4.3. If we have two or more outer derivations they are nil-dependent. Therefore the solvable algebra that we are constructing is of codimension at most one.

4.5. Finding a solvable indecomposable Lie algebra with a codimension one nilradical $N_{n,19}$.

Theorem 4.3. *There is the only solvable indecomposable Lie algebra up to isomorphism with a codimension one nilradical $N_{n,19}$ denoted $g'_{n+1,1}$, which is given below:*

$$[e_1, e_{n+1}] = e_1, [e_i, e_{n+1}] = (i+1)e_i\,, \qquad (2 \le i \le n,\ n \ge 6)\,,$$
$$DS = [n+1, n, n-2, 0]\,, \quad LS = [n+1, n, n, \dots]\,, \quad US = [0]\,.$$

Proof. Proof is given in Appendix. □

5. Appendix

5.1. Finding solvable indecomposable Lie algebras with a codimension one nilradical $N_{n,11}$.

The following paragraph constitutes the details of the proof of Theorem 4.1.

Remark 5.1. Even though it is understood that all the entries have superscript one, however, for the sake of simplicity it will be omitted.

(i) Let $a \ne 0, b \ne (i-n)a$, $(4 \le i \le n+1)$. Then applying the transformation $e'_1 = e_1$, $e'_2 = e_2 - \frac{D_{i,2}}{(i-2)a} e_i$, $(4 \le i \le n-2)$ such that i is fixed and $e'_k = e_k - \frac{D_{i-k+2,2}}{(i-k)a} e_i$, $(3 \le k \le n-3,\ k+2 \le i \le n-1)$ where i is fixed, $e'_j = e_j$, $(n-2 \le j \le n+1)$ and renaming the entries in $(i+1, 2)$ and $(i-k+3, 2)$ positions by $D_{i+1,2}$ and $D_{i-k+3,2}$, respectively, if necessary we remove $D_{4,2}, \dots, D_{n-2,2}$, but this transformation introduces the entries in $(4,1), (5,1), \dots, (n-1,1)$ positions, which we denote by $D_{4,1}, \dots, D_{n-1,1}$, respectively.

Remark 5.2. If $n = 6$ then only the entry in $(4,1)$ position is introduced, which we set to be $D_{4,1}$.

Also we rename the entries in $(n-1, 2)$ and $(n, 2)$-positions by $D_{n-1,2}$ and $D_{n,2}$, respectively, if necessary. The transformation which fixes all but e_1 and e_4 such that $e'_1 = e_1 + \frac{D_{2,1}}{a-b} e_2, e'_4 = e_4 + \frac{D_{2,1}}{a-b} e_n$ removes $D_{2,1}$. Then we rename the entries in $(n-1, 1)$ and $(n, 1)$ positions by $D_{n-1,1}$ and $D_{n,1}$, respectively. Finally applying the transformation which fixes all but e_1, e_2 and e_{n+1} such that $e'_1 = e_1 - \sum_{i=4}^{n-1} \frac{D_{i,1}}{(i-3)a+b} e_i - \frac{D_{n,1}}{2b} e_n$, $e'_2 = e_2 - \frac{D_{n-1,2}}{(n-3)a} e_{n-1} - \frac{D_{n,2}}{a+b} e_n$, $e'_{n+1} = \frac{e_{n+1}}{a}$ we obtain

$$\begin{aligned} &[e_1, e_{n+1}] = e_1, [e_k, e_{n+1}] = (b+k-2)e_k\,, \quad (2 \le k \le n-1)\,, \\ &[e_n, e_{n+1}] = (2b+1)e_n, n \ge 6, b \ne (i-n)\,, \quad (4 \le i \le n+1,\ n \ge 6)\,. \end{aligned} \tag{5.1}$$

(ii) Let $a = b, b \neq 0$. Then applying the transformation $e_1' = e_1$, $e_2' = e_2 - \frac{D_{i,2}}{(i-2)b}e_i$, $(4 \leq i \leq n-2)$ such that i is fixed and $e_k' = e_k - \frac{D_{i-k+2,2}}{(i-k)b}e_i$, $(3 \leq k \leq n-3,\ k+2 \leq i \leq n-1)$ where i is fixed, $e_j' = e_j$, $(n-2 \leq j \leq n+1)$ and renaming the entries in $(i+1,2)$ and $(i-k+3,2)$ positions by $D_{i+1,2}$ and $D_{i-k+3,2}$, respectively, if necessary we remove $D_{4,2}, \dots, D_{n-2,2}$, but the entries in $(4,1), (5,1), \dots, (n-1,1)$ positions are introduced, which we set to be $D_{4,1}, \dots, D_{n-1,1}$, respectively.

Remark 5.3. If $n = 6$ then only then entry in $(4,1)$ position is introduced.

Also we rename the entries in $(n-1,2)$ and $(n,2)$ positions by $D_{n-1,2}$ and $D_{n,2}$, respectively, if necessary. Applying the transformation which fixes all but e_1 such that $e_1' = e_1 - \sum_{i=4}^{n-2} \frac{D_{i,1}}{(i-2)b}e_i$ we remove $D_{4,1}, \dots, D_{n-2,1}$ and rename the entry in $(n,1)$ position by $D_{n,1}$. The transformation $e_1' = e_1$, $e_2' = e_2 - \frac{D_{n-1,2}}{(n-3)b}e_{n-1} - \frac{D_{n,2}}{2b}e_n, e_i' = e_i$, $(3 \leq i \leq n+1)$ removes $D_{n-1,2}$ and $D_{n,2}$. Then we rename the entries in $(n-1,1)$ and $(n,1)$ positions by $D_{n-1,1}$ and $D_{n,1}$, respectively. The transformation which fixes all but e_1 such that $e_1' = e_1 - \frac{D_{n-1,1}}{(n-3)b}e_{n-1} - \frac{D_{n,1}}{2b}e_n$ removes $D_{n-1,1}$ and $D_{n,1}$. If $D_{2,1} = 0$ then we have the limiting case of (5.1) with $b = 1$ after scaling b to unity applying the transformation which fixes everything but e_{n+1} such that $e_{n+1}' = \frac{e_{n+1}}{b}$. If $D_{2,1} \neq 0$ then applying the transformation $e_1' = e_1, e_i' = \frac{D_{2,1}}{b}e_i$, $(2 \leq i \leq n-1)$, $e_n' = \left(\frac{D_{2,1}}{b}\right)^2 e_n$, $e_{n+1}' = \frac{e_{n+1}}{b}$ we obtain

$$[e_1, e_{n+1}] = e_1 + e_2, [e_k, e_{n+1}] = (k-1)e_k\,, \quad (k \neq 4,\ 2 \leq k \leq n-1)\,,$$
$$[e_4, e_{n+1}] = 3e_4 + e_n, [e_n, e_{n+1}] = 3e_n\,, \quad (n \geq 6) \tag{5.2}$$

which is denoted $g_{n+1,2}$.

(iii) Let $a = -b$, $b \neq 0$. Then applying the transformation $e_1' = e_1$, $e_2' = e_2 + \frac{D_{i,2}}{(i-2)b}e_i$, $(4 \leq i \leq n-2)$ such that i is fixed and $e_k' = e_k + \frac{D_{i-k+2,2}}{(i-k)b}e_i$, $(3 \leq k \leq n-3, k+2 \leq i \leq n-1)$ where i is fixed, $e_j' = e_j$, $(n-2 \leq j \leq n+1)$ and renaming the entries in $(i+1,2)$ and $(i-k+3,2)$ positions by $D_{i+1,2}$ and $D_{i-k+3,2}$, respectively, if necessary we remove $D_{4,2}, \dots, D_{n-2,2}$, but the entries in $(4,1)$, $(5,1), \dots, (n-1,1)$ positions are introduced, which we set to be $D_{4,1}, \dots, D_{n-1,1}$, respectively.

Remark 5.4. If $n = 6$ then only the entry in $(4,1)$ position is introduced.

Also we rename the entries in $(n-1,2)$ and $(n,2)$ positions by $D_{n-1,2}$ and $D_{n,2}$, respectively, if necessary. The transformation which fixes all but e_1 and e_4 such that $e_1' = e_1 - \frac{D_{2,1}}{2b}e_2$, $e_4' = e_4 - \frac{D_{2,1}}{2b}e_n$ removes $D_{2,1}$. Then we rename the entries in $(n-1,1)$ and $(n,1)$ positions by $D_{n-1,1}$ and $D_{n,1}$, respectively. The transformation $e_1' = e_1 + \sum_{i=5}^{n-1} \frac{D_{i,1}}{(i-4)b}e_i - \frac{D_{n,1}}{2b}e_n$, $e_2' = e_2 + \frac{D_{n-1,2}}{(n-3)b}e_{n-1}$, $e_k' = e_k$, $(3 \leq k \leq n+1)$ removes $D_{5,1}, \dots, D_{n,1}$ and $D_{n-1,2}$.

Applying the transformation which fixes everything but e_{n+1} such that $e_{n+1}' = \frac{e_{n+1}}{b}$ we scale b to unity but need to rename the entries in $(4,1)$ and $(n,2)$ positions by $D_{4,1}$ and $D_{n,2}$, respectively. If $D_{4,1} = 0$, $D_{n,2} = 0$ then we have the limiting case of (5.1) with $b = -1$.

If $D_{4,1} \neq 0$ then applying the transformation $e_1' = e_1,\ e_i' = D_{4,1}e_i,\ (2 \leq i \leq n-1),\ e_n' = (D_{4,1})^2 e_n,\ e_{n+1}' = e_{n+1}$ we have

$$[e_1, e_{n+1}] = -e_1 + e_4,\ [e_2, e_{n+1}] = e_2 + de_n,\ [e_k, e_{n+1}] = (3-k)e_k,\ (4 \leq k \leq n-1)\,,$$

$$[e_n, e_{n+1}] = e_n\,, \quad (d \in \mathbb{R},\ n \geq 6)\,. \tag{5.3}$$

If $D_{4,1} = 0,\ D_{n,2} \neq 0$ then applying the transformation $e_1' = e_1,\ e_i' = D_{n,2}e_i,$ $(2 \leq i \leq n-1),\ e_n' = (D_{n,2})^2 e_n,\ e_{n+1}' = e_{n+1}$ we obtain

$$[e_1, e_{n+1}] = -e_1,\ [e_2, e_{n+1}] = e_2 + e_n,\ [e_k, e_{n+1}] = (3-k)e_k,\ \ (4 \leq k \leq n-1),$$

$$[e_n, e_{n+1}] = e_n\,, \quad (n \geq 6)\,. \tag{5.4}$$

Altogether (5.3) and (5.4) give

$$[e_1, e_{n+1}] = -e_1 + \epsilon e_4\,,\ [e_2, e_{n+1}] = e_2 + de_n\,,\ [e_k, e_{n+1}] = (3-k)e_k\,,\ (4 \leq k \leq n-1),$$

$$[e_n, e_{n+1}] = e_n\,,\ (\epsilon = 0, 1,\ \epsilon^2 + d^2 \neq 0,\ d \in \mathbb{R},\ n \geq 6) \tag{5.5}$$

which is denoted $\mathfrak{g}_{n+1,3}$.

(iv) Let $a = 0,\ b \neq 0$. The transformation which fixes all but e_1 and e_4 such that $e_1' = e_1 - \frac{D_{2,1}}{b}e_2,\ e_4' = e_4 - \frac{D_{2,1}}{b}e_n$ eliminates $D_{2,1}$ but introduces the entries in $(4,1),\ (5,1), \ldots, (n-1,1)$ positions, which we set to be $D_{4,1}, \ldots, D_{n-1,1}$. Then we rename the entries in $(n,2)$ and $(n,1)$ positions by $D_{n,2}$ and $D_{n,1}$, respectively. Then applying the transformation $e_1' = e_1 - \frac{D_{i,1}}{b}e_i - \frac{D_{n,1}}{2b}e_n$, $(4 \leq i \leq n-1)$ such that i is fixed, $e_2' = e_2 - \frac{D_{n,2}}{b}e_n,\ e_k' = e_k,\ (3 \leq k \leq n+1)$ we remove $D_{4,1}, \ldots, D_{n,1}$ and $D_{n,2}$.

The transformation which fixes everything but e_{n+1} such that $e_{n+1}' = \frac{e_{n+1}}{b}$ scales b to unity.

If $D_{4,2} = 0$ we have

$$[e_k, e_{n+1}] = e_k + \sum_{i=k+3}^{n-1} d_{i-k-2}e_i, (2 \leq k \leq n-3),$$

$$[e_j, e_{n+1}] = e_j,\ (n-2 \leq j \leq n-1)\,,\ [e_n, e_{n+1}] = 2e_n, \quad (d_i \in \mathbb{R},\ n \geq 6)\,. \tag{5.6}$$

If $D_{4,2} \neq 0$, precisely, greater than zero or less than zero, respectively, then applying the transformation $e_1' = \sqrt{\pm D_{4,2}}e_1, e_2' = e_2,\ e_i' = (\pm D_{4,2})^{\frac{i-2}{2}}e_i,\ (3 \leq i \leq n-1),$ $e_n' = \sqrt{\pm D_{4,2}}e_n,\ e_{n+1}' = e_{n+1}$ we have

$$[e_k, e_{n+1}] = e_k + \epsilon e_{k+2} + \sum_{i=k+3}^{n-1} d_{i-k-2}e_i\,,\ (2 \leq k \leq n-3,\ d_{i-k-2} \in \mathbb{R})\,, \tag{5.7}$$

$$[e_j, e_{n+1}] = e_j, \quad (n-2 \leq j \leq n-1), \quad [e_n, e_{n+1}] = 2e_n,\ (\epsilon = \pm 1,\ n \geq 6)\,.$$

Altogether (5.6) and (5.7) give

$$(5.8)\quad [e_k,e_{n+1}] = e_k+\epsilon e_{k+2}+\sum_{i=k+3}^{n-1} d_{i-k-2}e_i\,, \quad (2\leq k\leq n-3,\, d_{i-k-2}\in\mathbb{R}),$$
$$[e_j,e_{n+1}] = e_j\,,\ (n-2\leq j\leq n-1)\,,\ [e_n,e_{n+1}] = 2e_n\,,\ (\epsilon=0,\pm1,\, n\geq 6)\,,$$

which is denoted $\mathfrak{g}_{n+1,4}$.

(v) Let $b=0$, $a\neq 0$. Apply the transformation $e'_1=e_1$, $e'_2=e_2-\frac{D_{i,2}}{(i-2)a}e_i$, $(4\leq i\leq n-2)$ such that i is fixed, $e'_k=e_k-\frac{D_{i-k+2,2}}{(i-k)a}e_i$, $(3\leq k\leq n-3, k+2\leq i\leq n-1)$ where i is fixed, $e'_j=e_j$, $(n-2\leq j\leq n+1)$. Rename the entries in $(i+1,2)$ and $(i-k+3,2)$ positions if necessary by $D_{i+1,2}$ and $D_{i-k+3,2}$, respectively. This transformation removes $D_{4,2},\ldots,D_{n-2,2}$ but introduces the entries in $(4,1),(5,1),\ldots,(n-1,1)$ positions, which we set to be $D_{4,1},\ldots,D_{n-1,1}$, respectively.

Remark 5.5. If $n=6$ then only the entry in $(4,1)$ position is introduced.

Then we rename the $(n-1,2)$ and $(n,2)$ entries by $D_{n-1,2}$ and $D_{n,2}$, respectively, if necessary.

Applying the transformation which fixes all but e_1 and e_4 such that $e'_1=e_1+\frac{D_{2,1}}{a}e_2$, $e'_4=e_4+\frac{D_{2,1}}{a}e_n$ we remove $D_{2,1}$ and rename the entries in $(n-1,1)$ and $(n,1)$ positions by $D_{n-1,1}$ and $D_{n,1}$, respectively.

The transformation $e'_1=e_1-\sum_{i=4}^{n-1}\frac{D_{i,1}}{(i-3)a}e_i$, $e'_2=e_2-\frac{D_{n-1,2}}{(n-3)a}e_{n-1}-\frac{D_{n,2}}{a}e_n$, $e'_k=e_k$, $(3\leq k\leq n)$, $e'_{n+1}=\frac{e_{n+1}}{a}$ removes $D_{4,1},\ldots,D_{n-1,1},D_{n-1,2}$ and $D_{n,2}$ and scales a to unity. We also replace the $(n,1)$ entry by $D_{n,1}$.

If $D_{n,1}=0$ we have the limiting case of (5.1) with $b=0$.

If $D_{n,1}\neq 0$, precisely, greater than zero or less than zero, respectively, then applying the transformation $e'_1=e_1$, $e'_i=\sqrt{\pm D_{n,1}}e_i$, $(2\leq i\leq n-1)$, $e'_n=\pm D_{n,1}e_n$, $e'_{n+1}=e_{n+1}$ we obtain

$$(5.9)\quad [e_1,e_{n+1}]=e_1+\epsilon e_n\,,\quad [e_k,e_{n+1}]=(k-2)e_k\,,\quad (3\leq k\leq n-1)\,,$$
$$[e_n,e_{n+1}]=e_n\,,\quad (\epsilon=\pm1,\, n\geq 6)$$

which is denoted $\mathfrak{g}_{n+1,5}$.

(vi) Let $b=(j-n)a$, $a\neq 0$, $(4\leq j\leq n-2)$ such that j is fixed.

Apply the transformation $e'_1=e_1$, $e'_2=e_2-\frac{D_{i,2}}{(i-2)a}e_i$, $(4\leq i\leq n-2)$ such that i is fixed, $e'_k=e_k-\frac{D_{i-k+2,2}}{(i-k)a}e_i$, $(3\leq k\leq n-3, k+2\leq i\leq n-1)$ where i is fixed, $e'_j=e_j$, $(n-2\leq j\leq n+1)$. Rename the entries in $(i+1,2)$ and $(i-k+3,2)$ positions if necessary by $D_{i+1,2}$ and $D_{i-k+3,2}$, respectively. This transformation removes $D_{4,2},\ldots,D_{n-2,2}$ but introduces the entries in $(4,1)$, $(5,1),\ldots,(n-1,1)$ positions, which we set to be $D_{4,1},\ldots,D_{n-1,1}$, respectively.

Remark 5.6. If $n=6$ then only the entry in $(4,1)$ position is introduced.

Then we rename the $(n-1,2)$ and $(n,2)$ entries by $D_{n-1,2}$ and $D_{n,2}$, respectively, if necessary.

The transformation which fixes all but e_1 and e_4 such that $e_1' = e_1 + \frac{D_{2,1}}{(n+1-j)a}e_2$, $e_4' = e_4 + \frac{D_{2,1}}{(n+1-j)a}e_n$ removes $D_{2,1}$. We then rename the $(n-1,1)$ and $(n,1)$ entries by $D_{n-1,1}$ and $D_{n,1}$, respectively.

Further we apply the transformation which fixes everything but e_1 such that $e_1' = e_1 + \sum_{i=4}^{n-1} \frac{D_{i,1}}{(3+n-j-i)a}e_i$, $(i \neq 3+n-j)$ and rename the $(n,1)$ entry by $D_{n,1}$ if necessary. This transformation removes all the entries from $D_{4,1}$ to $D_{n-1,1}$ except $D_{3+n-j,1}$.

Then the transformation $e_1' = e_1 + \frac{D_{n,1}}{2(n-j)a}e_n$, $e_2' = e_2 - \frac{D_{n-1,2}}{(n-3)a}e_{n-1} + \frac{D_{n,2}}{(n-1-j)a}e_n$, $e_i' = e_i$, $(3 \le i \le n)$ and $e_{n+1}' = \frac{e_{n+1}}{a}$ removes $D_{n,1}, D_{n-1,2}, D_{n,2}$ and scales a to unity. If $D_{3+n-j,1} = 0$ then we have the limiting case of (5.1) with $b = j-n$.

Altogether (5.1) and all its limiting cases give

$$g_{n+1,1} : [e_1, e_{n+1}] = e_1, [e_k, e_{n+1}] = (b+k-2)e_k\,, \quad (2 \le k \le n-1)\,,$$

$$[e_n, e_{n+1}] = (2b+1)e_n\,, \quad (n \ge 6) \tag{5.10}$$

in full generality.

If $D_{3+n-j,1} \neq 0$ then applying the transformation $e_1' = e_1$, $e_i' = D_{3+n-j,1}e_i$, $(2 \le i \le n-1)$, $e_n' = (D_{3+n-j,1})^2 e_n$, $e_{n+1}' = e_{n+1}$ we obtain

$$g^i_{n+1,6} : [e_1, e_{n+1}] = e_1 + e_{n-i}\,,\ [e_k, e_{n+1}] = (1+i+k-n)e_k\,,\ (2 \le k \le n-1), \tag{5.11}$$
$$[e_n, e_{n+1}] = (7-2(n-i))e_n\,, \quad (1 \le i \le n-5,\ n \ge 6)\,.$$

These six cases are summarized in Theorem 4.1.

5.2. **Finding a solvable indecomposable Lie algebra with a codimension one nilradical $N_{n,19}$.** The following paragraph constitutes the details of the proof of Theorem 4.3.

1) If $n = 6$ then apply the transformation which fixes everything but e_1 and e_4 such that $e_1' = e_1 - \frac{D_{2,1}}{2a}e_2, e_4' = e_4 - \frac{D_{2,1}}{2a}e_6$.

   If $n = 7$ then apply the transformation $e_1' = e_1 - \frac{D_{2,1}}{2a}e_2 + \frac{D_{2,1}D_{4,2}}{8a^2}e_4, e_2' = e_2 + \frac{D_{2,1}D_{4,2}}{8a^2}e_6, e_3' = e_3, e_4' = e_4 - \frac{D_{2,1}}{2a}e_6, e_5' = e_5 - \frac{D_{2,1}}{a}e_7, e_6' = e_6, e_7' = e_7$.

   These transformations remove $D_{2,1}$ but introduce the entries in $(n-2,1), \dots, (n,1)$ positions, which we set to be $D_{n-2,1}, \dots, D_{n,1}$, respectively. For $n = 7$ rename the entries in $(6,2)$, $(7,2)$ and $(7,3)$ positions by $D_{7,3}$, $D_{7,2}$ and $D_{7,3}$, respectively, and for $n = 6$ rename the entry in $(6,2)$ position by $D_{6,2}$.

2) The transformation which fixes everything but e_1 such that $e_1' = e_1 - \frac{D_{i,1}}{ia}e_i$, $(n-2 \le i \le n,\ 6 \le n \le 7)$ removes $D_{n-2,1}, \dots, D_{n,1}$. Also we rename the $(i+1,1)$ entry by $D_{i+1,1}$ if needed.

3) Apply the transformation $e_k' = e_k - \frac{D_{4,2}}{2a}e_{k+2}$, $(2 \le k \le n-2)$, $e_{n-1}' = e_{n-1}$, $e_n' = e_n$, $e_{n+1}' = e_{n+1}$, which removes $D_{4,2}$. Also we rename any entries back to what they were originally if necessary.

4) Then apply the transformation $e_1' = e_1$, $e_2' = e_2 - \frac{D_{i+1,3}}{(i-2)a} e_i$, $(5 \le i \le n-1)$, such that i is fixed, $e_k' = e_k - \frac{D_{i-k+3,3}}{(i-k)a} e_i$, $(3 \le k \le n-3,\ k+3 \le i \le n)$, where i is fixed, $e_j' = e_j$, $(n-2 \le j \le n+1)$. Rename the entries in $(i+2,3)$ and $(i-k+4,3)$ positions by $D_{i+2,3}$ and $D_{i-k+4,3}$, respectively, if necessary. This transformation removes $D_{6,3}, \dots, D_{n,3}$ after which we rename the $(n,2)$ entry as $D_{n,2}$ if necessary.
5) The transformation which fixes everything but e_2 such that $e_2' = e_2 - \frac{D_{n,2}}{(n-2)a} e_n$ removes $D_{n,2}$.
6) To scale a to unity, we apply the transformation which fixes everything except e_{n+1} such that $e_{n+1}' = \frac{e_{n+1}}{a}$. Hence we have the algebra with the nilradical defined in (3.2)

$$[e_1, e_{n+1}] = e_1\,, \quad [e_i, e_{n+1}] = (i+1)e_i\,, \quad (2 \le i \le n,\ n \ge 6)\,, \tag{5.12}$$

which is denoted $g_{n+1,1}'$.

This case is summarized in Theorem 4.3.

References

[1] Ancochea, J. M., Campoamor-Stursberg, R., Garcia Vergnolle, L., *Solvable Lie algebras with naturally graded nilradicals and their invariants*, J. Phys. A **39** (6) (2006), 1339–1355.

[2] Ancochea, J. M., Campoamor-Stursberg, R., Garcia Vergnolle, L., *Classification of Lie algebras with naturally graded quasi–filiform nilradicals*, J. Geom. Phys. **61** (11) (2011), 2168–2186.

[3] Campoamor-Stursberg, R., *Solvable Lie algebras with an $\mathbb{N}$–graded nilradical of maximal nilpotency degree and their invariants*, J. Phys. A **43** (14) (2010), 18pp., 145202.

[4] Cartan, E., *Sur la structure des groupes de transformations finis et continus*, Paris: These, Nony, 1894; 2nd ed. Vuibert, 1933.

[5] Gantmacher, F., *On the classification of real simple Lie groups*, Mat. Sb. (1950), 103–112.

[6] Gong, M.-P., *Classification of nilpotent Lie algebras of dimension* 7, Ph.D. thesis, University of Waterloo, 1998.

[7] Hindeleh, F., Thompson, G., *Seven dimensional Lie algebras with a four-dimensional nilradical*, Algebras Groups Geom. **25** (3) (2008), 243–265.

[8] Humphreys, J., *Lie algebras and their representations*, Springer, 1997.

[9] Jacobson, N., *Lie algebras*, Interscience Publishers, 1962.

[10] Morozov, V. V., *Classification of nilpotent Lie algebras in dimension six*, Izv. Vyssh. Uchebn. Zaved. Mat. **4** (5) (1958), 161–171.

[11] Mubarakzyanov, G. M., *Classification of real Lie algebras in dimension five*, Izv. Vyssh. Uchebn. Zaved. Mat. **3** (34) (1963), 99–106.

[12] Mubarakzyanov, G. M., *Classification of solvable Lie algebras in dimension six with one non-nilpotent basis element*, Izv. Vyssh. Uchebn. Zaved. Mat. **4** (35) (1963), 104–116.

[13] Mubarakzyanov, G. M., *On solvable Lie algebras*, Izv. Vyssh. Uchebn. Zaved. Mat. **1** (32) (1963), 114–123.

[14] Ndogmo, J. C., Winternitz, P., *Solvable Lie algebras with abelian nilradicals*, J. Phys. A **27** (2) (1994), 405–423.

[15] Patera, J., Sharp, R. T., Winternitz, P., Zassenhaus, H., *Invariants of real low dimension Lie algebras*, J. Math. Phys. **17** (1976), 986–994.

[16] Rubin, J. L., Winternitz, P., *Solvable Lie algebras with Heisenberg ideals*, J. Phys. A **26** (1993), 1123–1138.

[17] Seeley, C., 7*–dimensional nilpotent Lie algebra*, Trans. Amer. Math. Soc. **335** (2) (1993), 479–496.

[18] Shabanskaya, A., *Classification of six dimensional solvable indecomposable Lie algebras with a codimension one nilradical over* $\mathbb{R}$, Ph.D. thesis, University of Toledo, 2011.

[19] Shabanskaya, A., Thompson, G., *Six–dimensional Lie algebras with a five–dimensional nilradical*, J. Lie Theory **23** (2) (2013), 313–355.

[20] Skjelbred, T., Sund, T., *Classification of nilpotent Lie algebras in dimension six*, University of Oslo, 1977, preprint.

[21] Snobl, L., *On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras*, J. Phys. A **43** (50) (2010), 17pp.

[22] Snobl, L., *Maximal solvable extensions of filiform algebras*, Arch. Math. (Brno **47** (5) (2011), 405–414.

[23] Snobl, L., Karasek, D., *Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras*, Linear Algebra Appl. **432** (7) (2010), 18/36–1850.

[24] Snobl, L., Winternitz, P., *A class of solvable Lie algebras and their Casimir invariants*, J. Phys. A **38** (12) (2005), 2687–2700.

[25] Snobl, L., Winternitz, P., *All solvable extensions of a class of nilpotent Lie algebras of dimension n and degree of nilpotency $n-1$*, J. Phys. A **2009** (2009), 16pp., 105201.

[26] Tremblay, S., Winternitz, P., *Solvable Lie algebras with triangular nilradicals*, J. Phys. A. **31** (2) (1998), 789–806.

[27] Turkowski, P., *Solvable Lie algebras of dimension six*, J. Math. Phys. **31** (6) (1990), 1344–1350.

[28] Umlauf, K. A., *Über die Zusammensetzung der endlichen continuierliche Transformation-gruppen insbesondere der Gruppen von Rang null*, Ph.D. thesis, University of Leipzig, 1891.

[29] Vergne, M., *Cohomologie des algèbres de Lie nilpotentes. Application a l'étude de la variété des algebres de Lie nilpotentes*, Bull. Math. Soc. France **78** (1970), 81–116.

[30] Wang, Y., Lin, J., Deng, S., *Solvable Lie algebras with quasifiliform nilradicals*, Comm. Algebra **36** (11) (2008), 4052–4067.

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